GENERALIZATIONS OF DUAL-BAER MODULES, DUAL-RICKART MODULES AND PURELY EXTENDING MODULES

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I, **Shiv Kumar**, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of **Dr. Ashok Ji Gupta** from **December, 2017** to **December, 2022** at the **Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi**. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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Contents

PREFACE

The thesis consists of five chapters. **Chapter 1** is preliminaries which are the collection of definitions and basic results used in the subsequent chapters.

In **Chapter 2**, we introduce the notion of *principally quasi-dual-Baer modules (in short PQ-dual-Baer modules)*, which dualizes the notion of *principally quasi-Baer modules*. We study some properties of PQ-dual-Baer modules. We find some conditions for which the direct sum of arbitrary copies of PQ-dual-Baer modules is PQ-dual-Baer. We also study the ring of endomorphisms of PQ-dual-Baer modules.

In **Chapter 3**, we dualize the concept of Σ-Rickart modules as Σ-dual-Rickart modules. We prove that each cohereditary module over the Noetherian ring is a Σ-dual-Rickart module. We introduce the notion of strongly cogenerated modules and characterize Σ-dual-Rickart modules in terms of strongly cogenerated modules. We show when a Σ-Rickart module is a Σ-dual-Rickart module and vice-versa. We also study some properties of Σ-dual-Rickart modules and find their connections with semisimple Artinian rings, von Neumann regular rings, semi-hereditary rings and FP-injective modules. Further, we study endomorphism rings of Σ-dual-Rickart modules.

In **Chapter 4**, we introduce the notion of finite Σ-dual-Rickart modules, which generalizes the notion of Σ-dual-Rickart modules. We characterize von Neumann regular rings, hereditary rings, semi-hereditary rings and semisimple Artinian rings in terms of finite Σ-dual-Rickart modules. We examine connections between finite Σ-Rickart modules and finite Σ-dual-Rickart modules. Also, we study endomorphism rings of finite Σ-dual-Rickart modules.

In **Chapter 5**, we study several properties of purely extending modules and introduce the notion of purely essentially Baer modules. A module M is said to be a purely essentially Baer if the right annihilator in M of any left ideal of the endomorphism ring of M is essential in a pure submodule of M . We study some properties of purely essentially Baer modules and characterize von Neumann regular rings in terms of purely essentially Baer modules.

ABBREVIATION

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Z}_n$	$\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$
$\mathbb{Z}_{p^\infty}$	The Prüfer p-group
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\subseteq$	A subset
$\leq$	A submodule
$\leq^{\oplus}$	A direct summand
$M^{(I)}$	The direct sum of I copies of M indexed by I
M^I	The direct product of I copies of M indexed by I
$\leq^p$	A pure submodule
$\leq^c$	A closed submodule
$\trianglelefteq$	A fully invariant submodule or An ideal
$\trianglelefteq^p$	A projection invariant submodule
$\leq^e$	An essential submodule
$Mat_n(X)$	n by n matrix over the set X
$T_n(X)$	n by n upper triangular matrix over the set X
$Ann^r_A(B)$	The right annihilator of a set B in the set A
$Ann^l_A(B)$	The left annihilator of a set B in the set A
$Cl_A(B)$	The closure of the set B in the set A

INTRODUCTION

The concept of ring theory was started in the 1870s at the time of Richard Dedekind and key contributions to this theory were given by Dedekind, Hilbert, Fraenkel and Noether. The ring is the first generalization of the Dedekind domain that occurs in the number theory. The ring theory has wide applications in number theory, algebraic geometry, algebraic graph theory, coding theory, etc. There are mainly two ways to study the structure of rings. The first way is to study the structure of rings by studying their left and right ideals (inner conditions) and the second way is to study the structure of rings by studying modules over them (outer conditions). In this thesis, we study the structure of rings by the second way.

In module theory, the concept of injective modules was introduced by Baer [7] in 1940. The study of injective modules became the center of attraction for many mathematicians when Eckmann and Schopf [23] proved the existence of the injective hull of a module. In literature, the injective module is generalized by many mathematicians to quasi-injective modules [29], pseudo injective modules [51], continuous modules and quasi-continuous modules [43], extending modules [22], etc.

Recall that a module M is *extending* (or CS) if every submodule of M is essential in a direct summand of M. The theory of extending modules developed by Harada and his school in Japan, Muller and his collaborators [43] in Canada, Osofsky, Smith, Huynh, Dung, Wisbauer [22] and many more people worldwide. In [15], Clark introduced the notion of purely extending modules which is a generalization of extending modules. A module M is said to be purely extending if every submodule of M is essential in a pure submodule of M.

The following implications are true

Injective Module $\Rightarrow$ Quasi-Injective Module $\Rightarrow$ Continuous Module $\Rightarrow$ Quasi-Continuous

Module ⇒ Extending Module ⇒ Purely Extending Module

But the converse of above implications need not be true (see [15], [22], [32], [43])

The concept of a projective module is dual to the concept of an injective module, although it originates at almost the same time as an injective module. However, the work on projective modules started only after the publication of the book "Homological Algebra" by Cartan and Elinberg [12]. Later on, the projective module is generalized to the quasi-projective module [42] then the quasi-projective module is generalized to the pseudo-projective module [53] which is further generalized to the lifting module [45], $D2$-module, $D3$-module [43], $D4$-module [21], etc.

The notion of Baer and quasi-Baer rings have their roots in functional analysis. According to Kaplansky [30] a ring R is called Baer if the right annihilator of any right ideal (or non-empty subset) of R is a right ideal generated by an idempotent element of R. Examples of Baer rings are right self-injective von Neumann regular rings, von Neumann algebras, W^*-algebras (i.e., *-algebras of bounded operators on a Hilbert space containing the identity operator which is closed under weak operator topology), any domain (with a unit element) and the endomorphism rings of semisimple modules (thus, endomorphism rings of all vector spaces), etc. The concept of Baer rings was generalized to quasi-Baer rings by W.E. Clark [16] in 1967 by replacing the 'left ideal' with a 'two-sided ideal' in the definition of Baer rings. In 2001, Birkenmeier [9] introduced the notion of principally quasi-Baer (or PQ-Baer) rings that generalizes the notion of quasi-Baer rings. A ring is said to be PQ-Baer if the right annihilator of any principal ideal is a right ideal generated by an idempotent element of R. Motivated by Kaplansky's work on Baer rings, the notion of Rickart rings initially appeared in Maeda [41, p. 510] and was further studied by Hattori [27, p. 147] and Berberian [8, p. 18]. A ring R is called right (left) *Rickart* (also known as p.p. ring) if an idempotent element of R generates the right (left) annihilator of any single element of R as a right (left) ideal of R.

In 2004, Rizvi and Roman [48] defined the module theoretical notion of Baer and quasi-Baer rings as Baer and quasi-Baer modules, respectively. Further, they find module theoretic analog of Chatters and Khuri's results of [14]. A module M is called a *Baer (quasi-Baer)* if the left annihilator in $S = End_R(M)$ of any submodule (fully invariant submodule) of M is a direct summand of S. In the last few years, there have been numerous generalizations of Baer modules. The module theoretic notion of principally quasi-Baer rings is defined by Ungor et al. [58], Dana and Moussavi [19], and Lee [33] in different aspects. A module M is known as a principally quasi-Baer module (simply PQ-Baer module) [58] if the left annihilator in $S = End_R(M)$ of any cyclic submodule of M is a direct summand of S. But according to Lee [33], a module M is called PQ-Baer if the right annihilator in M of every principal right ideal of S is a direct summand of M. In 2010, Lee et al. [36] introduced the notion of Rickart modules in general module theoretic setting by utilizing the endomorphism ring of a module. According to them M is called a Rickart module if the right annihilator in M of any single element of $S = End_R(M)$ is generated by an idempotent element of S or Kernel of every endomorphism of M is a direct summand of M.
In general, the following implications are true

$$\text{Baer module} \Rightarrow \text{Quasi-Baer module} \Rightarrow \text{PQ-Baer module}$$

But the converse of these implications need not be true (see [10], [19], [33], [48]). Close connections with module theory and applications of Baer modules and their generalizations attracted many researchers to discover dual notions of these structures. In 2010, Tutuncu and Tribak [56] introduced dual-Baer modules. Amouzegar and Talebi [4] generalized dual-Baer modules to quasi-dual-Baer modules. Every R-module over the semisimple ring R is dual-Baer and every dual-Baer module is quasi-dual-Baer.

Motivated by the theory of PQ-Baer modules, which is generalization of the theory of quasi-Baer modules, we generalize quasi-dual-Baer modules as *principally quasi-dual-Baer (in short PQ-dual-Baer) modules* in Chapter 2. We study properties of PQ-dual-Baer modules, like direct sum, direct summand and endomorphism ring of them.

We have the following hierarchy for our structure as

$$\text{Dual-Baer Module} \Rightarrow \text{Quasi-dual-Baer Module} \Rightarrow \text{PQ-dual-Baer Module}$$

Recall that a ring R is said to be hereditary if every right ideal of R is projective. Hereditary rings have been characterized in different ways, most common results of them are that a ring R is right hereditary if and only if every submodule of any projective right R-module is projective if and only if every factor module of any injective right R-module is injective. In [34], Lee and Barcenas established the module theoretic analog of a right hereditary ring as a Σ-Rickart module. A module M is called a Σ-Rickart module if every direct sum of copies of M is Rickart. Further, in [35], they generalized this notion in the finite case and called a module M finite Σ-Rickart if every finite direct sum of copies of M is a Rickart module. The following hierarchy holds for Rickart modules, but the converse need not be true (see [34], [35])

$$\Sigma\text{-Rickart Module} \Rightarrow \text{Finite } \Sigma\text{-Rickart Module} \Rightarrow \text{Rickart Module}$$

In 2011, Lee [37] dualized the class of Rickart modules and called a module M dual-Rickart if the image of each endomorphism of M is a direct summand of M. Rickart and dual-Rickart modules are closely connected with von Neumann regular rings. The key result which proves the connection between them is a consequence of the result given by Rangaswamy [47] in 1976. According to [47, Theorem 4], the endomorphism ring $S = End_R(M)$ of a module M is von Neumann regular if and only if Kernel and Image of every endomorphism of M are direct summand of M, i.e., M is a Rickart and as well as a dual-Rickart module.

The structure of Σ-Rickart modules and finite Σ-Rickart modules motivated us to introduce and study the dual version of these structures. So, in Chapter 3 we define Σ-dual-Rickart modules and study their properties. Further, in Chapter 4 we introduce the class of finite Σ-dual-Rickart modules, which contains the class of Σ-dual-Rickart modules and contained in the class of dual-Rickart modules. The following implications hold for modules.

Σ-dual-Rickart Module $\Rightarrow$ Finite Σ-dual-Rickart Module $\Rightarrow$ Dual-Rickart Module

In [6], Atani et al. introduced purely Baer modules and called a module M purely Baer if the right annihilator of every left ideal of $S = End_R(M)$ in M is a pure submodule of M. One of the key result of purely Baer modules is that a ring R is von Neumann regular ring if and only if every right R-module is purely Baer.

Motivated by notions of purely Baer modules and purely extending modules, we introduce the notion of *purely essentially Baer modules* in Chapter 5. We also study some more properties of purely extending modules. Purely essentially Baer module is a common generalization of purely Baer module and purely extending module.

Now, we give a brief description of the present thesis. The thesis consists of the following five chapters with conclusions and future scope. **Chapter 1** is preliminaries which are the collection of notations, definitions and basic results used in the subsequent chapters.

In **Chapter 2**, we introduce and study the notion of *principally quasi-dual-Baer modules (in PQ-dual-Baer modules)*, which dualizes the notion of *principally quasi-Baer modules* [33]. We study some properties of PQ-dual-Baer modules. We find some conditions for which the direct sum of arbitrary copies of PQ-dual-Baer modules is PQ-dual-Baer. We also study the ring of endomorphisms of PQ-dual-Baer modules.

The followings are the main results in Chapter 2:

1. The following are equivalent for an R-module M:

   (i) M is a PQ-dual-Baer module;

   (ii) For every cyclic submodule $P \leq M$, there exists a decomposition $M = P_1 \oplus P_2$ with $P_1 \leq^{\oplus} P$ and $Hom(M, P \cap P_2) = 0$.

2. Every direct summand of a PQ-dual-Baer module is PQ-dual-Baer.

3. The endomorphism ring of a PQ-dual-Baer module is a left PQ-Baer ring.

4. If the endomorphism ring of every direct sum of copies of a PQ-dual-Baer M is left PQ-dual-Baer, then the endomorphism ring of M is a quasi-dual-Baer ring.

In **Chapter 3**, we dualize the concept of Σ-Rickart modules [34] as Σ-dual Rickart modules. We introduce the notion of strongly cogenerated modules and characterize Σ-dual Rickart modules in terms of strongly cogenerated modules. We show when a Σ-Rickart module is a Σ-dual Rickart module and vice-versa. Further, we study the endomorphism ring of Σ-dual Rickart modules.

The following are the main results in Chapter 3:

1. An R-module is Σ-dual-Rickart if and only if R is a semisimple Artinian ring.

2. Every cohereditary module over a Noetherian ring is a Σ-dual-Rickart module.

3. Let R be a Noetherian ring. Then the following conditions are equivalent:

(i) Every injective R-module is a Σ-dual Rickart module;

(ii) R is a right hereditary ring.

4. The following conditions are equivalent for a ring R:

   (i) Every right R-module is a Σ-Rickart module;

   (ii) Every right R-module is a Σ-dual-Rickart module;

   (iii) R is a right semisimple Artinian ring.

5. If M is a finitely generated Σ-dual-Rickart module with endomorphism ring $S = End_R(M)$, then S is a left hereditary ring and ${}_SM$ is an FP-injective S-module.

In **Chapter 4**, we introduce the notion of finite Σ-dual-Rickart modules, which generalizes the notion of Σ-dual-Rickart modules. We study some basic properties of the finite Σ-dual-Rickart modules. We examine connections between finite Σ-Rickart [35] modules and finite Σ-dual-Rickart modules. Also, we study the endomorphism ring of finite Σ-dual-Rickart modules.
The following are the main results in Chapter 4:

1. Every cohereditary module is a finite Σ-dual-Rickart module.

2. For an R-module M, the following statements are true:

   (i) Every finite Σ-dual-Rickart module has SSP.

   (ii) Every finite Σ-dual-Rickart module with $D3$ condition has SIP.

3. The following statements are equivalent for a ring R.

   (i) Every finitely generated free (projective) R-module is finite Σ-dual-Rickart;

   (ii) The free R-module $R^{(2)}$ has SSP;

(iii) R is a von Neumann regular ring.

4. A projective and quasi-injective right R-module M over right hereditary ring R is a finite Σ-dual-Rickart module.

5. Every quasi-injective finite Σ-Rickart module is finite Σ-dual-Rickart module.

6. The endomorphism ring of a finite Σ-dual-Rickart module is left semi-hereditary. Conversely, if $S = End_R(M)$ is a left semi-hereditary ring with $C2$-condition as a left S-module, then M is a finite dual-Rickart module.

In **Chapter 5**, we study several properties of purely extending modules [15] and introduce the notion of purely essentially Baer modules.
The following are the main results in Chapter 5:

1. (i) A finitely generated flat module M over a Noetherian ring is purely extending module if and only if it is extending module.

   (ii) A module M over a pure semisimple ring is purely extending if and only if it is extending.

   (iii) A pure split module is purely extending if and only if it is extending.

2. Every finitely generated torsion-free module over a principal ideal domain is purely extending.

3. Let M be an R-module with endomorphism ring $S = End_R(M)$. Then the following statements are equivalent:

   (i) Every purely essentially Baer R-module is purely Baer;

   (ii) Every purely extending R-module is purely Baer;

   (iii) R is a von Neumann regular ring.

4. Let $M = \bigoplus_{i \in I} M_i$ (where I is an index set) be such that $\mathrm{Hom}_R(M_i, M_j) = 0$ for every $i \neq j \in I$. Then M is purely essentially Baer module if and only if each M_i is purely essentially Baer module.