CHAPTER 1

INTRODUCTION

This chapter contains a brief descriptions of the objective, approach, and organization of the thesis. Sect. 1.1 discusses a brief history about the fractional calculus and fractional partial differential equations (FPDEs). In Sect. 1.2, basic definitions of fractional integral and derivatives are given which is being used throughout the thesis. In sec. 1.3 we have discussed about the brief literature review for the fractional mathematical models that we have taken in our study. Sec. 1.4 presents some mathematical preliminaries. In Sect. 1.5, some fundamental mathematical results based on proposed numerical methods are discussed that are used in this thesis. The challenges and motivations behind the topics are explained in Sect. 1.6. Sect. 1.7 defines the lists the objectives of the thesis.

1.1 Background

1.1.1 Fractional calculus

The origin of the fractional calculus lies in a conversation between Leibniz and L'Hospital. Leibniz invented the notation $d^n y/dx^n$. In 1695, L'Hospital ask Leibniz,"What if n be 1/2". Leibniz replied, "This is an apparent paradox from which, one day, useful consequences can be drawn." In his correspondance with Johann Bernoilli, Leibniz mentions derivative of "general order" [4]. In 1772, J.L. Lagrange developed the law of exponents for differential operators of integer order and wote:

$$\frac{d^m}{dx^m}\frac{d^n}{dx^n}y = \frac{d^{m+n}}{dx^{m+n}}$$

In 1812, P.S. Laplace defined a fractional derivative by means of an integral and in 1819, for the first time, S.F.Lacroix [5], mentioned "the derivative of arbitrary order". He generalizes from a case of integer order to develop the derivative of arbitrary order as for $y = x^m$, m being a positive integer

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!}x^{m-n}, m \geq n.$$

Using the definition of the gamma function, he wrote

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}, m \geq n.$$

then he gave an example for $y = x$ and $n = 1/2$, he obtain

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}, m \geq n.$$

It is interesting to note that the result obtained by Lacroix is the same as yielded by present-day Riemann-Liouville definition of a fractional derivative.

Joseph Fourier [6] was the next to mention derivatives of arbitrary order. His definition of fractional operations was obtained from his integral representation of any function $f(x)$ as

$$\frac{d^\alpha}{dx^\alpha} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)du \int_{-\infty}^{\infty} p^\alpha \cos[p(x-u) + \frac{1}{2}\alpha\pi]dp. \tag{1.1}$$

The number α will be regarded as any quantity, whatsoever, positive or negative.

The first use of fractional operations was made by Niels Henrik Abel in 1823. He applied the fractional calculus in the solution of an integral equation that arises in the formation of the tautochrone problem. If the time of slide is a known constant 'k', then Abel's integral equation is

$$k = \int_0^x (x-t)^{-1/2} f(t)dt \tag{1.2}$$

Abel wrote the R.H.S. of (1.2) as $\sqrt{\pi}[d^{-1/2}/dx^{-1/2}]f(x)$. Then he operated on both side of the equation with $d^{1/2}/dx^{1/2}$ to obtain

$$\frac{d^{1/2}}{dx^{1/2}} k = \sqrt{\pi} f(x) \tag{1.3}$$

Thus, when the fractional derivative of order $1/2$ of a constant 'k' is computed, $f(x)$ is determined. This is a remarkable achievement of Abel in fractional calculus.

In 1832, the first major study of fractional calculus was done by Joseph Liouville.He assumed that the arbitrary derivative of a function $f(x)$ that may be expanded in a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \quad Re\ a_n > 0 \tag{1.4}$$

is

$$D^{\alpha} f(x) = \sum_{n=0}^{\infty} c_n a_n^{\alpha} e^{a_n x} \tag{1.5}$$

This is known as Liouville's first formula for a fractional derivatives. Later in 1847, G.F. Bernhard Riemann sought a generalization of a Taylor's series expansion and derived the following definition for fractional integration:

$$D^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t) dt + \Psi(x). \tag{1.6}$$

However, he saw fit to add a complementary function $\Psi(x)$ to the above definition. Today, this definition is in common use as a definition for fractional integration but with complementary function taken to be identically zero, and the lower limit of integration c is usually zero, i.e.

$${}_0D_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t) dt, \quad Re(v) > 0. \tag{1.7}$$

This form of the fractional integral often is referred to as Riemann-Liouville fractional integral. For $f(x) = x^a$ and $v > 0$, we have from (1.8) that

$${}_0D_x^{-v} x^a = \frac{\Gamma(a+1)}{\Gamma(a+v+1)} x^{a+v}, \quad a > -1. \tag{1.8}$$

Later, in the second half of nineteenth century, many mathematicians like A.K. Grunwald, A.V. Letnikov, H. Laurent, A. Krug, O. Heaviside etc contributed in the development of fractional calculus [7]. In twentieth century, M.T. Naraniengar, H.H. Hardy, J.E. Littilewood, H.T. Davis, W. Fabian, J. Caputo, M. Riesz, T.J. Osler, K.B. Oldham [8], J. Spanier, T.R. Prabhakar, I. Podlubny [9], A. Kilbas and H.M. Srivastava [10] etc. played an important role in the growth and application of fractional calculus in the field of science, engineering and technology.

1.1.2 Fractional order partial differential equations

In recent years, the theory of fractional partial differential equations have played an important role in modeling of many physical and natural phenomena. The fractional-order derivative allows the memory description and hereditary properties of various substances. Because of this reason, the fractional-order models have proved to be more accurate than integer-order models [9, 10]. In the past two decades, it has become more popular and important due to its applications in the various fields of science and engineering. Fractional differential equations (FDEs) provide a powerful and flexible tool for modeling and describing the behavior of real materials [11], signal and image processing [12], finance [1], fluid dynamics [13, 14], electromagnetic waves [15], electrochemical process, where a particle plume spreads at a rate inconsistent with the classical Brownian motion model [8], biological systems [16, 17], control theory [18, 19], graph theory [20] and so on. In many mathematical models, the most commonly used fractional derivatives are the Riemann-Liouville derivative and the Caputo derivative. Riemann-Liouville derivatives are generally used to model the space fractional PDEs, whereas Caputo derivatives are mostly used in modeling the time-fractional PDEs.

The fractional derivatives have important nonlocal property and memory effect. The main advantage of FDEs is that it provides a powerful tool for depicting the systems with memory, long-range interactions and hereditary properties of several materials as opposed to the classical differential equations in which such effects are difficult to incorporate [21, 22]. For the first time processes with memory were mathematically described by Ludwig Boltzmann in 1874 and 1876 [23, 24]. Memory means that the existence of output of any process will depend not only on current time but also on the history of change of input on a finite or infinite time interval. It can be described by functions called memory functions. These functions are the kernel of integro-differential operator known as power-law memory in fractional calculus. This can also be seen as an advantage of FDEs over integer-order differential equations as the latter has the property of being differentiable only in an infinitesimal neighbourhood of the considered point. So it cannot describe the processes with memory. Since, the non-integer order derivatives violate the Leibnitz rule, it allows us to represent memory. A wide range of functions with memory in continuous time models of physics actively uses fractional integro-differential equations. Some applications of FDEs in different fields of real-life problems are discussed in [25]. Now we discuss about some fractional order partial differential equations which we have considered in this thesis.

1.1.2.1 Riesz-space fractional partial differential equations

Space fractional derivatives are used to model anomalous diffusion or dispersion process [13, 14]. Space fractional advection-dispersion equations are used to describe the transport in a complex system which is governed by anomalous diffusion and non-exponential reaction patterns [26–28]. Benson et al. [29, 30] has modelled the transport of passive tracer carried by fluid flow in a porous medium for groundwater hydrology research.

Recently, many researchers have drawn their attention to Riesz-space fractional derivatives in their models. Baleanu [31] discussed fractional variational principle of constrained systems involving Riesz-derivative. Rabei et al. [32] presented a fractional Hamilton-Jacobi formulations for systems containing Riesz fractional derivatives. Saichev et al. [33], derived the RFADE from the kinetics of chaotic dynamics. The importance of taking Riesz derivative is that it is the linear sum of left and right Riemann-Liouville derivatives which makes it easier to model the flow regime that is impacted by both sides of the domain [34]. In this thesis, we consider the following space fractional partial differential equation with Riesz-space fractional derivative

$$\frac{\partial}{\partial t}\mathrm{u}(x,t) = k_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha}\mathrm{u}(x,t) + k_\beta \frac{\partial^\beta}{\partial |x|^\beta}\mathrm{u}(x,t), \qquad 0 \le t \le T, \qquad 0 \le x \le L, \tag{1.9}$$

with initial condition

$$\mathrm{u}(x,0) = f(x), \tag{1.10}$$

and Dirichlet boundary conditions

$$\mathrm{u}(0,t) = \mathrm{u}(L,t) = 0, \tag{1.11}$$

where, $1 < \alpha \le 2$, $0 < \beta < 1$, u is a solute concentration; k_α and k_β represent the dispersion coefficient and the average fluid velocity, respectively. Here, we take $k_\alpha > 0$ and $k_\beta \ge 0$.

To find the physical and mathematical behaviors of these models based on Riesz derivatives, analytical solution will play a vital role. But, due to non-local properties of fractional integral and fractional derivatives, it seems difficult to find the analytical solution of most of the FDEs. Thus, it is necessary and important to establish the numerical schemes with significant accuracy to solve the FDEs. Different forms of FDEs have been solved by using different numerical approachs e.g. finite difference methods (FDM)[1, 35–41], spectral methods [42–44], collocation methods [45–47], finite element methods [48–50], finite volume methods [51, 52] and operational matrix method [53, 54] etc.

1.1.2.2 Time fractional Black-Scholes model (TFBSM)

In the financial market, a derivative is a financial instrument whose pay-off depends on the underlying asset's value. An option is considered to be a popular and important financial derivative. It is categorized into two main categories. One is *call option* and other is *put option*. A call option gives his owner the right, but not an obligation, to buy the underlying asset at a fixed price (strike price) K at a specified time in the future, i.e., at maturity. The call option owner wants the stock price to rise so that he can buy the asset for less than its worth. The call option would be profitable if exercised at maturity when the stock price (S) is above the strike price (K). A put option gives his owner the right, but not an obligation, to sell the underlying asset at strike price K at maturity. The owner of the put option wants the stock price to fall to sell the asset for more than its worth. The put option would be profitable if it is exercised at maturity when the stock price is below the strike price. When an option is exercised only at maturity, it is called the European option and, when it is exercised before maturity, it is called the American option. Therefore, from both the theoretical and practical perspectives, pricing an option now becomes a significant problem. In 1973, Black and Scholes proposed a model called the Black-Scholes (B-S) model, which describes the underlying asset's behavior [55, 56]. Although the price of the options obtained from the B-S model is "very close" to the observed price, but it still has some drawbacks like the inability to capture the significant movements called "volatility smile" of financial market or jumps over small time intervals [57].

With the discovery of the fractal structure of a stochastic process, fractional calculus is now being employed in modeling the financial process [58]. In this connection, two types of fractional derivatives, namely space fractional derivative and time-fractional derivative, plays a vital role in modeling the financial process. Using space-fractional derivative, a finite moment log stable (FMLS) model was introduced by [59], where they used the Fourier transform method to compute the option values. [60] derived three popular space-fractional B-S models in which they established an essential connection between FMLS, CGMY, and KoBoL process. Using the time-fractional derivative, [61] priced a European call option by a time-fractional Black-Scholes model for pricing the European vanilla options. [62] modeled the European-style option by time-fractional partial differential equation utilizing tick-by-tick data via a non-explosive market point process. The author has used the Caputo derivative as a non-local operator in time-to-maturity. A time-space fractional B-S model is derived by [63] using fractional order Taylor formula and Ito's lemma, which later applied to Merton's optimal portfolio [64]. Based on this idea, [65] introduced a bi-fractional B-S model with the assumption that the underlying asset follows a fractional Ito's process, and the change in the option price with time is a fractional transmission system. In the same way, pricing an American option is also an important task in the finance. A greedy algorithm for partition of unity collocation method is used in pricing American option by [66]. Recently, [67, 68] have discussed the B-S model in distributive order.

1.1.2.3 Time-fractional wave equation (TFWE)

As we know, several fractional derivatives are used to model various physical phenomena such as Riemann Liouville derivative, Caputo derivative, Riesz derivative, Caputo-Fabrizio derivative, Atangana-Baleanu fractional derivative, etc. Among all of them, the Caputo derivative are widely used to model many physical processes into time-fractional partial differential equations (TFPDEs). One such important TFDE is time-fractional wave equations (TFWEs). The TFWE is obtained by replacing the second order time derivative with a fractional order derivative $\alpha \in (1, 2)$. In this chapter, we consider the following time-fractional wave equation [69, 70]:

$$ {}_0^C D_t^{\alpha} u(x,t) = u_{xx}(x,t) + f(x,t), \quad (x,t) \in \Omega = [0, L] \times [0, T], \tag{1.12} $$

with the initial conditions

$$ u(x,0) = \phi_1(x), \quad u_t(x,0) = \phi_2(x); \tag{1.13} $$

and boundary conditions.

$$ u(0,t) = \psi_1(x), \quad u(L,t) = \psi_2(x). \tag{1.14} $$

Here ${}_0^C D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative of order $1 < \alpha < 2$, and $(x,t) \in \Omega = [0, L] \times [0, T]$.

1.1.2.4 Time fractional telegraph equation (TFTE)

Telegraph equations are used within wave propagation of electrical signals in a cable transmission line as wave phenomena.

$$\begin{cases} {}_0^C\mathrm{D}_t^{\alpha}\mathrm{u}(\mathrm{x},t) + \gamma_1 {}_0^C\mathrm{D}_t^{\beta}\mathrm{u}(\mathrm{x},t) + \gamma_2\mathrm{u}(\mathrm{x},t) = \gamma_3\mathrm{u}_{\mathrm{xx}}(\mathrm{x},t) + \mathrm{f}(\mathrm{x},t), \\ I.C.: \mathrm{u}(\mathrm{x},0) = \phi_1(\mathrm{x}), \quad \partial_t\mathrm{u}(\mathrm{x},0) = \phi_2(\mathrm{x}); \\ B.C.: \mathrm{u}(\mathrm{x}_l,t) = \psi_1(\mathrm{x}), \quad \mathrm{u}(\mathrm{x}_r,t) = \psi_2(\mathrm{x}). \end{cases} \tag{1.15}$$

Where, $(\mathrm{x},t) \in \Omega = [\mathrm{x}_l, \mathrm{x}_r] \times [0,T]$, $1 < \alpha < 2$ and $0 < \beta < 1$. $\gamma_1, \gamma_2, \gamma_3$ are the real coefficients. If $\gamma_1 > 0$ and $\gamma_2 = 0$ then the above equation represents a damped wave motion.

1.2 Definition of some fractional derivatives

In this section, we present the definition of some fractional-order derivatives.

Definition 1.1. (Grünwald-Letnikov derivatives) The left and right Grünwald-Letnikov derivatives with order $\alpha > 0$ of the given function $u(t)$, $t \in (a,b)$ are defined as [10, 37]:

$${}_a^{GL}\mathrm{D}_t^{\alpha}u(t) = \lim_{\substack{h\to 0\\ h=\frac{t-a}{N}}} h^{-\alpha}\sum_{j=0}^{N}(-1)^j\binom{\alpha}{j}u(t-jh), \tag{1.16}$$

and,

$${}_t^{GL}\mathrm{D}_b^{\alpha}u(t) = \lim_{\substack{h\to 0\\ h=\frac{b-t}{N}}} h^{-\alpha}\sum_{j=0}^{N}(-1)^j\binom{\alpha}{j}u(t+jh), \tag{1.17}$$

respectively.

Definition 1.2. (Riemann-Liouville fractional derivative) The left and right Riemann-Liouville fractional derivatives with order $\alpha > 0$ of the given function $u(t)$, $t \in (a, b)$ are defined as [10, 37]:

$$ {}_{a}^{RL}\mathrm{D}_{t}^{\alpha} u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{(m-\alpha-1)} u(s) ds, \tag{1.18} $$

and,

$$ {}_{t}^{RL}\mathrm{D}_{b}^{\alpha} u(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_t^b (s-t)^{(m-\alpha-1)} u(s) ds, \tag{1.19} $$

respectively, where $m \in \mathbb{Z}^+$ satisfying $m - 1 \leq \alpha < m$.

Definition 1.3. (Caputo fractional derivative) The left and right Caputo fractional derivatives with order $\alpha > 0$ of the given function $u(t)$, $t \in (a, b)$ are defined as [10, 37]:

$$ {}_{a}^{C}\mathrm{D}_{t}^{\alpha} u(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} u^{(m)}(s) ds, \tag{1.20} $$

and,

$$ {}_{t}^{C}\mathrm{D}_{b}^{\alpha} u(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_t^b (s-t)^{m-\alpha-1} u^{(m)}(s) ds, \tag{1.21} $$

respectively, where $m \in \mathbb{Z}^+$ satisfying $m - 1 < \alpha \leq m$.

Definition 1.4. (Riesz fractional derivative) The Riesz fractional derivative with order $\alpha > 0$ of the given function $u(x)$, $x \in (a, b)$ is defined as [10, 37]:

$$ {}^{RZ}\mathrm{D}_t^{\alpha} u(t) = \frac{\partial^{\alpha} u(t)}{\partial |t|^{\alpha}} = -c_{\alpha}({}_a^{RL}\mathrm{D}_t^{\alpha} + {}_t^{RL}\mathrm{D}_b^{\alpha})u(t), \tag{1.22} $$

where $c_{\alpha} = -\dfrac{1}{2\cos(\alpha\pi/2)}$, $\alpha \neq 2k+1,\ k = 0, 1, \ldots$. ${}_0^{RL}\mathrm{D}_t^{\alpha}$ and ${}_t^{RL}\mathrm{D}_L^{\alpha}$ are defined as the left and right Riemann-Liouville derivatives, respectively.

Definition 1.5 ([71])**.** Suppose the Laplacian $(-\Delta)$ has complete set of orthonormal eigenfunctions $\{\phi_n\}$ corresponding to eigenvalue λ_n^2 on a bounded region Ω i.e. $(-\Delta)\phi_n = \lambda_n^2 \phi_n$ on Ω; $\mathscr{B}(\phi) = 0$ on $\partial\Omega$ where $\mathscr{B}(\phi)$ is one of the standard three homogeneous boundary conditions. Let

$$ F_{\gamma} = \Big\{ f = \sum_{n=1}^{\infty} f_n \phi_n,\ f_n =< f, \phi_n >: \sum_{n=1}^{\infty} |f_n|^2 |\lambda|_n^{\gamma} < \infty \}, \gamma = max(\alpha, 0) \Big\} \tag{1.23} $$

then for any $f \in F_{\gamma}$, $(-\Delta)^{\alpha/2} : F_{\gamma} \to L_2(\Omega)$ is defined by

$$ (-\Delta)^{\alpha/2} f = \sum_{n=1}^{\infty} f_n (\lambda_n^2)^{\alpha/2} \phi_n. \tag{1.24} $$

Definition 1.6 ([71])**.** Let $\{\phi_n\}$ be the complete set of orthonormal eigenfunctions corresponding to eigenvalues λ_n^2 for the Laplacian $(-\Delta)$ on a bounded region Ω with homogeneous boundary conditions on Ω. Then

$$ (-\Delta)^{\frac{\alpha}{2}} f = \begin{cases} (-\Delta)^m f, & \text{if } \alpha = 2m,\ m = 0{,}1{,}2... \\ (-\Delta)^{\frac{\alpha}{2}-m}(-\Delta)^m f, & \text{if } m - 1 < \frac{\alpha}{2} < m,\ m = 1{,}2... \\ \sum_{n=1}^{\infty} \lambda_n^{\alpha} \langle f, \phi_n \rangle \phi_n, & \text{if } \alpha < 0. \end{cases} \tag{1.25} $$

Lemma 1.2.1 ([10, 37]). For a function $u(t)$ defined on the infinite domain $-\infty < x < \infty$, the following inequality holds:

$$-(-\Delta)^{\alpha/2}u(t) = -c_\alpha({}_{-\infty}^{RL}\mathrm{D}_t^\alpha + {}_t^{RL}\mathrm{D}_\infty^\alpha)u(t) = \frac{\partial^\alpha u(t)}{\partial|t|^\alpha}. \tag{1.26}$$

Proof. See ([72]). □

1.3 Literature review

The theory of integrals and derivatives of fractional order has achieved much popularity and importance due to the applications in science and engineering. Mathematical models based on arbitrary order integrals and derivatives provide a powerful and flexible tool for describing the behavior of real materials [11], viscoelastic fluid [14, 73, 74], finance [1, 75–77], signal and image processing [12, 78–80], electrochemical process [8, 81], biological systems [16, 82], control theory [18, 19, 83], electromagnetic waves [15, 84, 85] and so on. We now present a brief literature review for some classes of fractional mathematical problems that are considered in this thesis.

1.3.1 Literature review on Riesz-fractional partial differential equation

In 2007, Lin and Xu [42] presented a finite difference/spectral approximation for the time fractional diffusion wave equation where they have applied the finite difference method in time and Legendre spectral method in space. Yang et al. [72] in 2010, gave three numerical methods based on the fractional method of lines to solve the RFDE and RFADE. They have shown that the MTM provides the best result in all the three numerical schemes but did not discuss the theoretical convergence and stability of the numerical schemes. In 2012, Celik et al. [86] used the Crank-Nicolson scheme for RFDE, where the Riesz derivative is approximated by fractional central difference schemes. In the same year, Ding et al. [87] developed a new numerical method with improved MTM for Riesz-space derivative and (2,2) Pade approximation for computing the exponential matrix to solve the RFDE and RFADE. They also used the matrix analysis method to prove the unconditional stability of the scheme. In 2016, Yuan et al. [88] proposed a meshfree point interpolation method (PIM) to solve RFADE. Recently, Saberi et al. [89] developed radial basis function collocation method to solve RFADE. Some more numerical investigations on RFDE and RFADE are available in literatures [90–92].

1.3.2 Literature review on time fractional Black-Scholes model governing European option

Due to the importance of fractional-order derivatives, the fractional B-S model has gained much interest amongst the researchers to find its solution. To find the analytical solution, some of the researchers have used the homotopy perturbation method [93], homotopy analysis method [94], integral transform method [61, 64, 65, 95] for the fractional B-S model. But the problem with these solutions lies in the fact that they are of the form of convolution of some special functions or an infinite series with integrals [1]. Therefore, developing a numerical approximation for the solution of TFBSM becomes more crucial. In the past few years, only a few reports are available on the numerical solution of TFBSM using different approaches.

- In 2016, Zhang et al., [1] proposed an implicit numerical scheme based on the finite difference approach, which is of the order (2-α) in time and second order accurate in space. However, the authors did not discuss the effect of various parameters on option pricing.

- In 2017, DeStaelen and Hendy [2] improved the spatial order of convergence of the scheme given by [1] by applying the compact difference scheme in space while maintaining the (2-α) order in time. The effects of market parameters on option pricing have not been discussed here also.

- In 2018, Cen et al. [96] used an integral discretization scheme on an adopted mesh in time and a central difference scheme on a piece-wise uniform mesh in space to overcome the non-physical oscillation caused by the degeneracy of the B-S differential operator.

- In 2019, Golbabai and Nikan [97] proposed a new approach based on the radial basis function combined with a finite difference approach (RBF-FD) to solve TFBSM for European options, which is again of (2-α) order in time.

- Soleymani and Akgul, [98] proposed a localized RBF-FD approach for the European multi-asset option pricing problem. Later, the authors [99] have applied the weights of Guassian RBF-FD scheme to solve the partial integro-differential equation arising from the Bates model in finance.

- In 2021, Roul and Goura, [100] gave a compact finite difference scheme to solve the TFBSM with the help of L1 scheme of order (2-α). An et al. [101] developed a space-time spectral method for the TFBSM.

1.3.3 Literature review on approximation of the Caputo derivatives and time fractional wave equation

The fractional derivative possesses the non-local property where the fractional derivative of any function f at any time t depends on the value of the function at all the previous time intervals $0 < s < t$. This makes the computation more costly and time consuming and requires more memory for data storage in our computer. Therefore, designing a good approximation of fractional derivative and high-order numerical methods has becomes an emerging area for the last decade. Several researchers have developed many different numerical approximations of Caputo fractional derivatives for $\alpha \in (0, 1)$ by using the idea of interpolation such as L1 approximation [8], L1-2 approximation [102], higher order approximation [103], L2-1_σ approximation [38], $\mathcal{F}$L1-2 approximation [104], $\mathcal{F}$L2-1_σ approximation [105], L1-2-3 approximation [106], and recently, a new L2 type approximation [107]. These approximations combined with the method of order reduction were generally used to solve the time-fractional partial differential equation (TFPDE) for $\alpha \in (1, 2)$. Till now, very few direct higher order approximation are available for the Caputo derivative when $\alpha \in (1, 2)$. Some of them are as follows.

- Liu et al. [27] in 2004 discussed the L2 approximation of order $(3 - \alpha)$ in which they have used quadratic interpolation to approximate the second order derivative.
- Lynch et al. [108] proposed the L2C scheme of order $(3 - \alpha)$ by using four point discretization and shows that the L2C method is more accurate than the L2 method when $1 < \alpha < 1.5$, whereas the L2 method gives more accuracy than the L2C method when $1.5 < \alpha < 2$. At $\alpha = 1.5$, both schemes give the same accuracy.

- In 2010, Du et al. [109] derived a compact difference scheme of order $\mathcal{O}(\tau^{3-\alpha}+h^4)$ for TFDWE. In 2014, Yang et al. [110] used the fractional multistep method and proposed a numerical scheme of order $\mathcal{O}(\tau^{\alpha}+h^2)$ to solve TFDWE.

- 2016, Sun et al. gave a second order difference scheme for TFWE [111] and multi-term TFWE [112] using the method of order reduction combined with L2-1_σ scheme and $\mathcal{F}$L2-1_σ, respectively.

- In 2018, Liu et al. [69] proposed a novel difference scheme of order $\mathcal{O}(\tau^{3-\alpha}+h^2)$ to solve the time-fractional diffusion wave equation (TFDWE).

- In 2019, Du et al. [113] designed a $(4-\alpha)$ order formula for the Caputo derivative $\frac{1}{2}\left({}_0^C D_t^{\alpha} f(t_k) + {}_0^C D_t^{\alpha} f(t_{k-1})\right)$ to solve the TFDWE for the time level $k \geq 3$. This approximation is an average of the fractional derivative at k^{th} and $(k-1)^{th}$ level and one have to use some other numerical method to obtain the numerical solution at the first two time level.

- In 2020, Shen et al. [114] has derived an H2N2 interpolation formula for the Caputo derivative of order $\alpha \in (1,2)$ and its application to TFWE in 1D and 2D.

- Recently, in 2021, Hengfei Ding [115] has developed two second order numerical differential formulas for the Caputo derivative of order $\alpha \in (0,1)$ and $\beta \in (1,2)$ at point $t_{k+1/2}$. The author has implemented this to solve the time-fractional mixed subdiffusion and diffusion-wave equation.

1.3.4 Literature review on time fractional telegraph equation

In the past decades, many researchers have found the analytical solution for the TFTE. Moami [116] in 2005 have found the analytical and approximate solution of TSFTE by Adomian decomposition method. In 2008, Chen et al. [117] have derived the analytical solution of TFTE with three different boundary conditions using method of separation of variables. Das et al. [118] have given approximate analytical solution of TFTE by Homotopy Analysis Method. In 2011, Jiang and Lin [119] gave representation of exact solution for the TFTE in the reproducing kernel space. In recent years, many scholars have focused on finding the numerical solution of TFTE. Some of them are as follows:

- In 2012, Li and Cao [120] presented a finite difference method for TFTE with order of convergence $\mathcal{O}(\tau^{3-\alpha}, h^2)$.
- In 2015, Chen et al. [121] gave a high order unconditionally stable difference schemes for the Riesz space-FTE. Shivanian et al. [122] presented local integration of 2-D fractional telegraph equation via moving least squares approximation. In the same year, Hosseini et al. [123] presented local integration of 2D fractional telegraph equation via local radial point interpolant approximation.
- In 2017, Wang and Mei [124] have used generalized finite difference/spectral Galerkin approximations for the TFTE.
- In 2020, Liang et al. [125] have proposed a fast high order difference schemes for the TFTE by using $\mathcal{F}$L2-1_σ approximation. In the same year, Akram et al. [126] proposed a novel numerical approach based on modified extended cubic

B-spline functions for solving non-linear TFTE. Kumar et al. [3] gave a local meshless method to approximate the TFTE.

- In 2021, Al-Smadi et al. [127] provided numerical simulation of telegraph and Cattaneo fractional type models using adaptive reproducing kernel framework. Khater et al. [128] numerically investigated for the fractional nonlinear space-time telegraph equation via the trigonometric Quintic B-spline scheme. Nikan et al. [129] gave numerical approximation of the nonlinear TFTE arising in neutron transport by using local radial basis function finite difference (LRBF-FD) approach.

1.4 Mathematical preliminaries

In this section, we discuss about the Legendre and Chebyshev polynomial which is used as a basis function to approximate the function.

1.4.1 Legendre polynomials

Legendre polynomials were discovered in 1782 by Adrien-Marie Legendre, which form a system of complete and orthogonal polynomials in the domain $[-1, 1]$, with a vast number of mathematical properties, and numerous applications with the orthogonality property as:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm},$$

where, δ_{nm} is the Kronecker delta.

Definition 1.7. The Legendre polynomials can be defined as the coefficients in a formal expansion in powers of t of the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

The coefficients of each t^n is a polynomial of degree n.

Definition 1.8. The Legendre polynomial is the series solution of Legendre's differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n(x) = 0.$$

The first few Legendre polynomials are

$$\begin{aligned}
P_0(x) &= 1, \\
p_1(x) &= x, \\
P_2(x) &= \frac{1}{2}(3x^2-1), \\
P_3(x) &= \frac{1}{2}(5x^3-3x), \\
P_4(x) &= \frac{1}{2}(35x^4-30x^2+3).
\end{aligned}$$

1.4.2 Chebyshev polynomials of second kind

The Chebyshev polynomial of second kind of order j defined on $[-1,1]$ are given by $U_j(x), \quad j = 0,1,2,3,...$ satisfy the following recursive formula

$$U_{j+1}(x) = 2tU_j(x) - U_{j-1}(x), \quad j = 1,2,3,... \tag{1.27}$$

where the first few Chebyshev polynomials are

$$U_0(x) = 1,$$

$$U_1(x) = 2x,$$

$$U_2(x) = 4x^2 - 1,$$

$$U_3(x) = 8x^3 - 4x,$$

$$U_4(x) = 16x^4 - 12x^2 + 1.$$

The Chebyshev polynomials of second kind are orthogonal with respect to the weight function $w(x) = \sqrt{1 - x^2}$ such that

$$\int_0^1 w(x)\phi_i(x)\phi_j(x)dt = \begin{cases} \pi/2 & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.28}$$

1.4.3 Function approximation

Let $f(t) \in L_2[0, 1]$ and $\Psi(t) = [\psi_0(t), \psi_1(t), ..., \psi_M(t)]^T$ be the basis of $L_2[0, 1]$, then the function $f(t)$ can be approximated by

$$f(t) \approx \sum_{j=0}^{M} c_j\psi_j(t) = \mathrm{C}^T\Psi(t), \tag{1.29}$$

where,

$$\mathrm{C} = [c_0, c_1, ..., c_M]^T, \tag{1.30}$$

and the coefficients c_j are calculated by

$$c_j = \frac{\langle f(t), \psi_j(t)\rangle}{\langle \psi_j(t), \psi_j(t)\rangle} = \frac{\int_0^1 w(t) f(t)\psi_j(t)dt}{\int_0^1 w(t)\psi_j(t)\psi_j(t)dt}. \tag{1.31}$$

1.5 Numerical methods

In this thesis, we have used two numerical methods to develop the numerical scheme for solving the fractional mathematical models. One is operational matrix method and other is finite difference method.

1.5.1 Operational matrices

Operational matrices are those matrices which are produced by approximating a derivative or integration of a function in terms of orthogonal functions. Orthogonal functions and polynomials play one of the important roles in theory of operational matrix. In the numerical analysis, operational matrix technique is a powerful technique for approximating solutions of integral and fractional differential equations (see [130–135]). The proposed operational matrix techniques are not only simplifies the singularity based problems but also speed up the computation as well as minimize the error. The operational matrices with respect to orthogonal polynomials are sparse in nature. This helps the original problem in transforming into system of algebraic equations.

The theory of the operational matrices mainly depends on two operators, differentiation and integration and the corresponding operational matrices can be constructed in the following manner:

$$\begin{aligned}\frac{d\Psi(t)}{dt} &\approx D\Psi(t),\\ \int_a^t \Psi(x)dx &\approx I\Psi(t),\end{aligned}$$

where, D and I are the operational matrices of differentiation and integration, respectively of dimension $N+1$ and $\Psi(t) = [\psi_1(t), \psi_2(t), \ldots, \psi_{N+1}(t)]$ is the orthonormal basis which is orthonormal in the certain interval. More general, mathematical representation of operational matrices are given below as:

$$\begin{aligned}\frac{d^k\Psi(t)}{dt^k} &\approx D^k\Psi(t),\\ \int_a^t \cdots \int_a^t \Psi(x)\,(dx)^k &\approx I^k\Psi(t).\end{aligned}$$

The operational matrix method becomes more popular among the researcher due to its smooth implementation, high order convergence, and easy to extend in higher dimensions. Some of the advantages are listed below:

- It can be easily extended into higher dimensions using Kronecker product [136].
- It reduces the given equation (PDEs, FPDEs etc.) into a system of algebraic equations which can be solved by well-known methods.
- Solution is convergent even though the size of increment is large.
- Removes the singularities in the equation.

Because of these advantages, the operational matrices of differentiation and integration have been used by many researchers in the past two decades. Many orthogonal basis functions have been used to generate operational matrix of differentiation and integration such as Legendre polynomial [137–139], Berstein polynomials [135, 140], Bessel functions [141], Fourier series [142, 143], Legendre wavelets [144, 145], Bernoulli wavelets [146], Chebyshev wavelets [147–149], Haar wavelets [150] etc..

1.5.1.1 Kronecker product

Definition 1.9. Let P and Q be two matrices of orders $p_1 \times p_2$ and $q_1 \times q_2$, respectively. Then the Kronecker product $P \otimes Q$ of matrices P and Q is defined as the following $p_1q_1 \times p_2q_2$ order block structure [151]:

$$P \otimes Q = \begin{bmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1n}Q \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}Q & p_{m2}Q & \cdots & p_{mn}Q \end{bmatrix},$$

where

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{bmatrix}, \qquad Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{m1} & q_{m2} & \cdots & q_{mn} \end{bmatrix}.$$

1.5.2 Finite difference method

In finite difference approximation, we approximate the derivative of a known function by finite difference formulas based only on the values of the function itself at discrete points. A finite difference method proceeds by replacing the derivatives in the differential equation with finite difference approximations. This gives a large but finite algebraic system of equations to be solved in place of the differential equation [152]. This method requires use of a regular grid and to facilitate explanation of the approach, and it will be considered that it is uniform, although this is not essential. The grid must be constructed such that the nodal points ar located at the intersection of either curved lines or rectilinear. Some of the applications of FDM in FDEs are described in details in the articles [1, 39, 41, 69]. Among various difference approximations method, Taylor's series expansion is one of the most popular one to derive difference approximations of DEs. The numerical solutions, based on FDM, provide the values of dependent variables at discrete nodal points in the domain.

Consider the space-time domain such that the space variable $x \in [x_0, x_l]$ and time variable $t \in [t_0, t_f]$. The discretization in spatial and temporal direction is denoted by h and τ respectively. It may be uniform and non-uniform. In order to achieve higher accuracy in FDM, one has to refine the mesh by taking more number of grid points. Let N_x & N_t be denotes the number of grid point in space and time direction, respectively. Let $h = \dfrac{x_l - x_0}{N_x}$ and $\tau = \dfrac{t_f - t_0}{N_t}$ be the uniform discretization parameters in spatial and temporal direction, respectively. Now, discretize the spacial and temporal domain as $\Omega_x = \{x_i : x_i = x_0 + ih,\ i = 0, 1, \dots, N_x\}$ and $\Omega_t = \{t_j : t_j = t_0 + k\tau,\ j = 0, 1, \dots, N_t\}$. Let u_i^j denote the approximate value of $u(x_i, t_j)$ at the nodal points (x_i, t_j), then the forward difference schemes for space and time are

$$\left.\frac{\partial u}{\partial x}\right|_{(x_i,t_j)} \approx \frac{u_{i+1}^j - u_i^j}{h} + \mathcal{O}(h), \tag{1.32}$$

$$\left.\frac{\partial u}{\partial t}\right|_{(x_i,t_j)} \approx \frac{u_i^{j+1} - u_i^j}{k} + \mathcal{O}(\tau), \tag{1.33}$$

and the backward space and time difference schemes are given by

$$\left.\frac{\partial u}{\partial x}\right|_{(x_i,t_j)} \approx \frac{u_i^j - u_{i-1}^j}{h} + \mathcal{O}(h), \tag{1.34}$$

$$\left.\frac{\partial u}{\partial t}\right|_{(x_i,t_j)} \approx \frac{u_i^j - u_i^{j-1}}{k} + \mathcal{O}(\tau),. \tag{1.35}$$

The difference approximations given in (1.32)-(1.35) are of first order accuracy in space and time direction. Second order central difference schemes in space direction are given by the relations:

$$\left.\frac{\partial u}{\partial x}\right|_{(x_i,t_j)} \approx \frac{u_{i+1}^j - u_{i-1}^j}{2h} + \mathcal{O}(h),$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i,t_j)} \approx \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \mathcal{O}(h^2),.$$

In the same way, one can generate the finite difference approximation of higher order derivative (see [153]).

1.6 Challenges and motivations

The fractional derivatives posses non-local property and have memory effect. Therefore, it is very challenging to find the analytical solution of the problems with FDs. Several researchers have developed different numerical methods to solve the FDEs like finite difference method (FDM) [1, 39–41, 69, 154], spectral methods [42–44, 155–157], collocation methods [45, 47, 158], finite element methods [48–50, 159–161], finite volume methods [51, 52], operational matrix method [135, 162]. But the main challenge in these numerical methods is to get a stable numerical solution having higher accuracy with higher order of convergence. This motivates us to develop numerical methods which are numerically stable with good rate of convergence. In this thesis, we consider different space and time fractional mathematical models governed by the Riesz and the Caputo fractional derivative. We have also designed a semi discrete scheme which is combination of finite difference method with operational matrix method. Later, two fully discrete scheme is designed for two time-fractional mathematical models with newly developed L3 approximation of the Caputo derivative.

1.7 Objective of the thesis

The objectives of the thesis are:

1. To develop the efficient numerical schemes for Riesz-space fractional partial differential equations with the help of matrix transform method and operational matrix method.
2. To develop a computational algorithm for the financial mathematical model governing European options with the help of L-12 approximation of the Caputo derivatives and operational matrix method.
3. To develop two numerical approximation of the Caputo derivative of order $\alpha \in (1, 2)$ (namely L3 and ML3 approximation) and its application to time-fractional wave equations.
4. To design a difference scheme for time-fractional telegraph equation with the help of L3 and L-123 approximation of the Caputo derivatives.

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