Chapter 3

An Inexact Newton Method to Solve Generalized Nash Equilibrium Problems

3.1 Introduction

Generalized Nash Equilibrium Problem (GNEP) is a noncooperative Nash equilibrium problem in which the strategy set of each player may depend on the strategies of the rival player. It was first formally introduced by Debreu [2] as a social equilibrium in 1952, and later as an abstract economy [3]. GNEPs have been an interesting area of research during the last two decades, and it has several real-world applications in the areas of computer science, economics and engineering, for example, Arrow and Debreu [3] proposed an abstract economy model, a power allocation problem in telecommunications [4], a competition among countries that arises from the Kyoto protocol to reduce air pollution [5], etc. A few other application areas of GNEPs include wireless communication [6,7], cloud computing [8], electricity generation [9], etc. As an application of GNEPs, Robinson [10, 11] discussed several mathematical formulations to solve the problem of measuring the effectiveness in optimization-based combat models.

3.2 Motivation

In most of the methods, researchers have analyzed the case of player convex GNEPs or jointly convex GNEPs. Facchinei et al. [1] considered the GNEPs with shared constraints and proposed Newton-type methods—semismooth Newton methods I and II and Levenberg-Marquardt method—to solve it. The numerical methods in [1] converge Q-quadratically, but they have local convergence properties. In this chapter, we develop an algorithm by using a Newton-type method to find a solution for GNEPs that converges globally. The choice of applying Newton-type method is due to the fact that it converges much faster than other well-known optimization methods. However, the conventional Newton method is not applicable to a nonsmooth system and does not converge globally. In this study, we attempt to apply an inexact Newton method that has a similar convergence rate as of Newton method and converges globally. Newton's methods are very attractive because they converge rapidly from any good initial guess. However, solving a system of linear equations at each stage can be expensive if the number of unknowns is large and may not be justified when the initial guess is far from a solution. Therefore, we consider the class of inexact Newton methods which solve the Newton equations approximately.

3.3 Contributions

The contribution of this study and the precise approach for applying the inexact Newton method on GNEPs are as follows.

- We reformulate GNEPs into a nonsmooth system with the help of a semismooth complementarity function. There are some Newtonian methods like semismooth Newton method, Levenberg-Marquardt method [1], etc., to solve GNEPs, but for the large-scale GNEPs, these methods can be expensive. Therefore, we use an inexact Newton method and use some line search techniques which makes this method advantageous as compared to the previous Newtonian methods in the sense of faster local and global convergence.

- We use the Armijo-Goldstein condition to find the step length, which has a better convergence property as compared to Armijo condition and Wolfe condition (see [66], pp. 36–41).

- We consider both types of GNEPs—player convex GNEP and jointly convex GNEP—and solve them by using inexact Newton method. The convergence analysis of the proposed algorithms is also given. In addition, we provide some numerical examples to verify the convergence of Algorithms 3 and 4.

3.4 Inexact Newton method

Using the inexact Newton method, we will solve GNEP considering the two cases of GNEP: PLayer convex GNEP and jointly convex GNEP.

3.4.1 Inexact Newton method of GNEP: player convex case

Theorem 3.1 [41] *If the GNEP is player convex, then for each solution $(\overline{x}, \overline{\lambda})$ to the system (1.5), the vector $\bar{x}$ is a generalized Nash equilibrium point.*

To solve the system (1.5), we reformulate it into a nonsmooth system of equations using a complementarity function.

Definition 3.1 [42] *A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called a complementarity function if*

$$\phi(x, y) = 0 \quad \Leftrightarrow \quad (x, y) \geq 0,\ xy = 0. \tag{3.1}$$

With the help of a complementarity function $\phi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, defined by

$$\phi(x, y) = \begin{pmatrix} \phi(x_1, y_1) \\ \phi(x_2, y_2) \\ \vdots \\ \phi(x_m, y_m) \end{pmatrix},$$

the system (1.5) can be reformulated as

$$\begin{pmatrix} L(x, \lambda) \\ \phi(-g(x), \lambda) \end{pmatrix} = 0. \tag{3.2}$$

With the 'min' complementarity function, i.e.,

$$\phi(x, y) = \min(x, y) = \begin{pmatrix} \min\{x_1, y_1\} \\ \min\{x_2, y_2\} \\ \vdots \\ \min\{x_m, y_m\} \end{pmatrix}, \tag{3.3}$$

the reformulated system (3.2) becomes

$$F(x, \lambda) = 0, \tag{3.4}$$

where

$$F(x, \lambda) = \begin{pmatrix} L(x, \lambda) \\ \min(-g(x), \lambda) \end{pmatrix}. \tag{3.5}$$

Since 'min' function is not everywhere differentiable, the system (3.4) is a system of nonsmooth equations. We can use differentiable complementarity functions with the help of Fischer-Burmeister C-function [42], defined by

$$\Phi(x, y) = \sqrt{x^2 + y^2} - (x + y),$$

which is convex and differentiable everywhere for $(x, y) \neq (0, 0)$. If we take the complementarity function

$$\phi(x, y) = \Phi(x, y)^2,$$

then the function ϕ is differentiable everywhere and the reformulated system (3.2) becomes a system of smooth equations. However, we know from several point of view that the use of nondifferentiable function ϕ is practically important [42]. So, in this chapter, we consider a semismooth (Definition 3.3) complementarity function '$\min(x, y)$'.

Definition 3.2 [67] *Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitzian function and $\Omega_H \subseteq \mathbb{R}^n$ be the set of points where H is differentiable. The limiting Jacobian of H at $y \in \mathbb{R}^n$ is defined by*

$$\text{Jac } H(y) = \left\{ V : V = \lim_{k \to \infty} JH(y^k) \textit{ for some } \{y^k\} \subseteq \Omega_H \textit{ such that } \lim_{k \to \infty} y^k = y \right\},$$

where $JH(y^k)$ is the Jacobian of H at y^k. For a given $y \in \mathbb{R}^n$, the convex hull of the set Jac $H(y)$ *is named as Clarke generalized Jacobian and is denoted by $\partial H(y)$.*

It is worthy to mention that every locally Lipschitzian function $H : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable almost everywhere (Rademacher's Theorem [68]). Hence, the set Ω_H in Definition 3.2 is nonempty.

Definition 3.3 [69] *Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz continuous mapping. If H is directionally differentiable, and for any $\delta y \in \mathbb{R}^n$ and $V \in \partial H(y + \delta y)$ with $\delta y \to 0$,*

$$H(y + \delta y) - H(y) - V(\delta y) = o(\|\delta y\|),$$

then H is called semismooth at the point[1] $y \in \mathbb{R}^n$.

We note that for $\phi(x, y) = \min\{x, y\}$, the function $F : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ as defined in (3.3) is a semismooth function (see [70]).

Definition 3.4 [69] *Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be semismooth at $y \in \mathbb{R}^n$. If the elements of the set* $\mathrm{Jac}\, H(y)$ *are nonsingular, then H is called strongly BD regular at $y \in \mathbb{R}^n$. In addition, if $\overline{y} \in \mathbb{R}^n$ satisfies the system $H(y) = 0$, then $\overline{y}$ is called a strongly BD regular solution of the system $H(y) = 0$.*

We note that identification of a solution to the GNEP (1.1)–(1.2) is equivalent to calculate the x-component of a zero of the function $F(x, \lambda)$. In the next section, we propose an algorithm to calculate the solution of the system

$$F(x, \lambda) = 0.$$

An important observation is that the mapping defined by (3.5) is semismooth, and the square of norm of F in (3.5) is not differentiable. Define a merit function:

$$\Psi(z) = \frac{1}{2}\|F(z)\|^2. \tag{3.6}$$

Since, we have $F : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ in (3.5), a locally Lipschitz continuous mapping, therefore $\Psi : \mathbb{R}^{n+m} \to \mathbb{R}$ in (3.6) is also a locally Lipschitz continuous mapping. From Clarke [71,72], Ψ is differentiable almost everywhere in $\mathbb{R}^{n+m}$. Let B be any nonempty, open subset of R^{n+m}, and Ψ be Lipschitz of B. Then, for $z \in B$, define a set

$$Jac\ \Psi(z) = \{\lim_{j\to\infty} \nabla\Psi(z^j)|\ \{z^j\} \to z,\ \text{and } \Psi \text{ is differentiable at } z^j\}. \tag{3.7}$$

[1] Throughut the chapter, we use the notations $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to represent the usual Euclidean norm and inner product in $\mathbb{R}^n$, respectively.

The convex hull of $Jac\ \Psi(z)$ is called subdifferential of Ψ at z, and is denoted by $\partial\Psi(z)$. Thus,

$$\partial\Psi(z) = Conv\{g_{\Psi(z)} \in \mathbb{R}^{n+m} \mid g_{\Psi(z)} = H(z)^\top F(z),\ H(z) \in Jac\ F(z)\}.$$

3.4.2 Algorithm: Inexact Newton method for player convex GNEPs

Since the mapping in (3.5) is strongly semismooth, the reformulated system (3.4) is nonsmooth. To find a solution of the system (3.4), we provide an inexact Newton method [69,73] which is described below (Algorithm 3). The choices of the parameters and notations that are used in the algorithm are as follows:

(1) We use the notations $z = (x, \lambda)$, $\Psi(z) = \frac{1}{2}\|F(z)\|^2$.

(2) We choose a forcing constant $\eta_0 \in (0, 1)$ and a bounded forcing sequence $\{\eta_k\}$ such that $0 < \eta_k \leq \eta_{max} < 1$, where $\eta_{max} = \max\{\eta_k : k = 0, 1, 2, \ldots\}$.

(3) For the residual vector r^k, we take $r^k = \eta_k F(z^k)$ or $r^k = \frac{\eta_k}{2} F(z^k)$, so that r^k satisfies the inexactness condition

$$\|r^k\| = \|H^k d + F(z^k)\| \leq \eta_k \|F(z^k)\|.$$

The initial residual vector will be calculated by $r^0 = \eta_0 F(z^0)$.

(4) We choose an arbitrary constant $c \in (0, 1)$, which we need for Armijo-Goldstein condition to prevent the step length from being too small or too large. Precisely, we choose the constant c as 10^{-4} or smaller.

Algorithm 3 Computing z_k such that $\|\Psi(z_k)\| < \epsilon$ to solve (3.6)

Step 0 (Initialization step). Choose positive constants $\rho > 0$ and $\kappa > 2$.

Given precision scalar $\epsilon > 0$.

Start with the initial point $z^0 \in \mathbb{R}^n \times \mathbb{R}^m$ and set the iteration counter $k = 0$.

Step 1 (Terminating condition). If $\|F(z^k)\| < \epsilon$, then stop, and give the output as z^k, an ϵ-precision solution.

Step 2 (Main steps).

Substep 2.1: (Descent direction choice). Choose $H^k \in \text{Jac } F(z^k)$.

Find a vector $d^k \in \mathbb{R}^{n+m}$ that satisfies the following systems (3.8) and (3.9):

$$H^k d = -F(z^k) + r^k \text{ with } \|r^k\| \leq \eta_k \|F(z^k)\| \tag{3.8}$$

and

$$\langle (H^k)^\top F(z^k), d^k \rangle \leq -\rho \|d^k\|^\kappa. \tag{3.9}$$

If no such d^k exists, then set $d^k = -(H^k)^\top F(z^k)$.

Substep 2.2: (Step length choice). Choose i_k, the smallest nonnegative integer i that satisfies the following pair of inequalities

$$\left.\begin{aligned} &\Psi(z^k + 2^{-i} d^k) \leq \Psi(z^k) + c 2^{-i} \langle (H^k)^\top F(z^k), d^k \rangle \text{ and} \\ &\Psi(z^k + 2^{-i} d^k) \geq \Psi(z^k) + (1-c) 2^{-i} \langle (H^k)^\top F(z^k), d^k \rangle. \end{aligned}\right\} \tag{3.10}$$

Substep 2.3: (Increase the iteration counter k and update z^k). Set $z^{k+1} = z^k + 2^{-i_k} d^k$.

Update $k = k + 1$.

Choose an $\eta_{k+1} \geq 0$ such that $\eta_{k+1} \leq \eta_k$ and go to Step 1.

Definition 3.5 *[66] Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$ that converges to $\bar{x}$. We say that the convergence is Q-superlinear if $\lim_{k\to\infty} \frac{\|x_{k+1}-\bar{x}\|}{\|x_k-\bar{x}\|} = 0$. The convergence is called Q-quadratic if there exists a positive constant M and a positive integer p such that*

$\frac{\|x_{k+1}-\bar{x}\|}{\|x_k-\bar{x}\|^2} \leq M$, *for all* $k \geq p$.

Definition 3.6 *[74] Suppose* $\Psi : \mathbb{R}^{n+m} \to \mathbb{R}$ *is locally Lipschitzian. Then, a direction* $d \in \mathbb{R}^{n+m}$ *is descent direction for* Ψ *at a point* $z \in \mathbb{R}^{n+m}$, *if* $d \in \mathbb{R}^{n+m}$ *satisfies*

$$\{\langle g_\Psi, d\rangle \mid g_\Psi \in \partial\Psi(z)\} < 0.$$

Since, we have $\Psi(z^k) = \frac{1}{2}\|F(z^k)\|^2$, *and* $\Psi(z^k)$ *is locally Lipschitzian. Here,* Ψ *and* $F(z^k)$ *are not differentiable functions, therefore, we have taken* $H^k \in Jac\ F(z^k)$ *such that* $g_{\Psi(z^k)} = {H^k}^\top F(z^k) \in Jac\ \Psi(z^k)$.
If d^k *satisfies* (13) *and* (14), *then,*

$$\langle g_{\Psi(z^k)}, d^k\rangle = \langle {H^k}^\top F(z^k), d^k\rangle \leq \rho\|d^k\|^\kappa < 0 \ (\text{ for } \rho > 0, \kappa > 2).$$

If we take $d^k = -{H^k}^\top F(z^k)$,

$$\langle g_{\Psi(z^k)}, d^k\rangle = \langle {H^k}^\top F(z^k), -{H^k}^\top F(z^k)\rangle < -\|{H^k}^\top F(z^k)\|^2 < 0.$$

Therefore, in both cases, d^k *is a descent direction.*

The following theorem proves the global quadratic convergence for Algorithm 3.

Theorem 3.2 *Let* $\{z^k\}$ *be the sequence generated by Algorithm 3. Assume that* $\{\eta_k\}$ *is a sequence such that* $0 < \eta_k \leq \eta_0 < 1$ *for every* k *with an arbitrary* $\eta_0 \in (0, 1)$. *Then, the following assertions are valid.*

(i) *For every accumulation point* $\overline{z}$ *of* $\{z^k\}$, $0 \in \partial\Psi(\overline{z})$.

(ii) *For any strongly BD-regular solution* $\overline{z}$ *of the system* $F(z) = 0$, *if* $z^k \to \overline{z}$, *then the rate of convergence of the sequence* $\{z^k\}$ *is Q-superlinear provided* $\eta_k \to 0$, *i.e.,* $\|r^k\| = o(\|F(z^k)\|)$. *Furthermore, if* $\eta_k = O(\|F(z^k)\|)$, *i.e., if* $\|r^k\| = O(\|F(z^k)\|^2)$, *then the rate of convergence of the sequence* $\{z^k\}$ *is Q-quadratic.*

Proof: (i) Assume to the contrary that $\overline{z}$ is an accumulation point of $\{z^k\}$ and $0 \notin \partial\Psi(\overline{z})$.

Let $H^k \in \text{Jac } F(z^k)$. Then, $g_{\Psi(z^k)} = (H^k)^\top F(z^k) \in \text{Jac } \Psi(z^k)$. Hence, by Definition 3.2, the assumption $0 \notin \partial\Psi(\overline{z})$ implies $g_{\Psi(\overline{z})} \neq 0$.

Since $\overline{z}$ is a limit point of $\{z^k\}$, there exists a set of indices K such that

$$\{z^k\}_{k\in K} \to \overline{z} \implies \{\Psi(z^k)\}_{k\in K} \to \Psi(\overline{z}) \implies \{\Psi(z^k) - \Psi(z^{k+1})\}_{k\in K} \to 0.$$

We note from Algorithm 3 that the direction d^k is given either by $d^k = -(H^k)^\top F(z^k)$ or by (3.8) and (3.9). In either case, as d^k is a descent direction, we have

$$\Psi(z^k) \geq \Psi(z^{k+1}), \quad \text{for all } k = 0, 1, 2, \ldots.$$

i.e., $\{\Psi(z^k)\}$ is a monotonic decreasing sequence. Thus, $\{\Psi(z^k)\}$ is a bounded sequence since $\Psi(z^k) \geq 0$ for all k.

If $d^k = -(H^k)^\top F(z^k) = -g_{\Psi(z^k)}$, then $g^T_{\Psi(z^k)} d^k = -\|g_{\Psi(z^k)}\|^2$, and hence

$$\lim_{k\to\infty} g^T_{\Psi(z^k)} d^k = -\|g_{\Psi(\overline{z})}\|^2. \tag{3.11}$$

Also, we have $\{g_{\Psi(z^k)}\}_K \to g_{\Psi(\overline{z})} \neq 0$, i.e., $\{d^k\}_K \to \overline{d} = -g_{\Psi(\overline{z})} \neq 0$. From Armijo-Goldstein condition (3.10), we get

$$\frac{\Psi(z^k) - \Psi(z^k + 2^{-i_k} d^k)}{1-c} \leq -2^{-i_k} g^\top_{\Psi(z^k)} d^k \leq \frac{\Psi(z^k) - \Psi(z^k + 2^{-i_k} d^k)}{c}.$$

Therefore, by sandwich theorem, we obtain

$$\{\Psi(z^k) - \Psi(z^{k+1})\}_{k\in K} \to 0 \implies \{2^{-i_k} g^\top_{\Psi(z^k)} d^k\}_{k\in K} \to 0. \tag{3.12}$$

We will show that $\{2^{-i_k}\}_K$ is bounded away from 0. On the contrary, by subsequencing, if necessary, we have $\{2^{-i_k}\} \to 0$ so that at each step the step size is reduced at least once. Therefore, from Armijo-Goldstein condition, there exists a number $k_0 \in K$ such that

$$\Psi(z^k + (2^{-(i_k-1)}/\beta)d^k) > \Psi(z^k) + c(2^{-(i_k-1)}/\beta)g_{\Psi(z^k)}^\top d^k,$$

for every $k \geq k_0, k \in K$, where $\beta \in (0,1)$.

Let $\alpha_k = \frac{2^{-(i_k-1)}}{\beta}\|d^k\|$ and $p_k = \frac{d^k}{\|d^k\|}$. Then,

$$\frac{\Psi(z^k + \alpha_k p_k) - \Psi(z^k)}{\alpha_k} > c g_{\Psi(z^k)}^\top p_k. \tag{3.13}$$

As $k \to \infty$, we have $\{\alpha_k\}_{k\in K} \to 0$ and $\{p_k\}_{k\in K} \to \overline{p}$ with $\|\overline{p}\| = 1$ and $\{z^k\}_K \to \overline{z}$. Thus, (3.13) gives

$$g_{\Psi(\overline{z})}^\top \overline{p} > c g_{\Psi(\overline{z})}^\top \overline{p} \implies (1-c) g_{\Psi(\overline{z})}^\top \overline{p} > 0 \implies g_{\Psi(\overline{z})}^\top \overline{p} > 0. \tag{3.14}$$

However,

$$g_{\Psi(z^k)}^\top d^k = -\|g_{\Psi(z^k)}\|^2 \leq 0, \tag{3.15}$$

and

$$\frac{g_{\Psi(z^k)}^\top d^k}{\|d^k\|} = g_{\Psi(z^k)}^\top p_k = -\frac{\|g_{\Psi(z^k)}\|^2}{\|d^k\|}. \tag{3.16}$$

Thus, as $k \to \infty$, we obtain from (3.16) that

$$g_{\Psi(\overline{z})}^\top \overline{p} = -\frac{\|g_{\Psi(\overline{z})}\|^2}{\|\overline{d}\|} \leq 0,$$

which contradicts (3.14). Therefore, $\{2^{-i_k}\}$ is bounded away from 0, i.e., $\{2^{-i_k}\}_K \to \overline{\alpha} \neq 0$. Therefore, as $k \to \infty$, (3.11) and (3.12) implies $g_{\Psi(\overline{z})} = 0$ Thus, our assumption that $\overline{z}$ is a limit point of $\{z^k\}$ and $0 \notin \partial\Psi(\overline{z})$ was wrong. Hence, $0 \in \partial\Psi(\overline{z})$ must be valid.

Now we prove that if the direction d^k is given by (3.8) and (3.9), then for an accumulation point $\overline{z}$ of $\{z^k\}$, $0 \in \partial\Psi(\overline{z})$.

If the direction d^k is given by (3.8) and (3.9), then

$$\|F(z^k) - r^k\| = \|H^k d^k\| \leq \|H^k\|\|d^k\|, \tag{3.17}$$

which implies that

$$\implies \|d^k\| \geq \frac{\|F(z^k) - r^k\|}{\|H^k\|}. \tag{3.18}$$

Here, $\|H^k\| \neq 0$. Otherwise, (3.17) implies

$$F(z^k) - r^k = 0. \tag{3.19}$$

Since $\eta_k \leq \overline{\eta} < 1$, we have

$$\|r^k\| \leq \eta_k \|F(z^k)\| < \|F(z^k)\|.$$

Hence, (3.19) is possible only if $F(z^k) = 0$. Thus, z^k will be a stationary point, which leads to a contradiction to our assumption.

Further, we will show that the direction d^k satisfies

$$0 < \gamma_1 \leq \|d^k\| \leq \gamma_2, \text{ with } 0 < \gamma_1 \leq \gamma_2 < \infty.$$

If for some subsequence $\bar{K}$, $\{\|d^k\|\}_{k \in \bar{K}} \to 0$, then (3.18) implies $\|F(z^k) - r^k\| \to 0$ because $\|H^k\|$ is bounded on the bounded sequence $\{z^k\}$ by the known properties of generalized Jacobian [75]. But we have

$$\|F(z^k) - r^k\| \geq \|F(z^k)\| - \|r^k\| \geq \|F(z^k)\| - \eta_k \|F(z^k\| = (1 - \eta_k)\|F(z^k)\|.$$

Thus, $\{\|F(z^k) - r^k\|\} \to 0 \implies \|F(z^k)\| = 0$, which contradicts that $\overline{z}$ is a stationary point.

On the other hand, $\|d^k\|$ cannot be unbounded because $g_{\Psi(z^k)}$ is bounded and by using (3.9), we have the following for $\kappa > 2$:

$$\rho\|d^k\|^\kappa \leq -g_{\Psi(z^k)}^\top d^k < \infty,$$

which implies that there exist γ_1, γ_2 with $0 < \gamma_1 \leq \gamma_2$ such that

$$0 < \gamma_1 \leq \|d^k\| \leq \gamma_2. \tag{3.20}$$

Since (3.10) holds at each iteration and $\{\Psi(z^k)\}$ is a bounded and monotonic decreasing sequence, we have $\Psi(z^k) - \Psi(z^{k+1}) \to 0$. Therefore, from Armijo-Goldstein condition (3.10), we have

$$\frac{\Psi(z^k) - \Psi(z^k + 2^{-i_k}d^k)}{1-c} \leq -2^{-i_k} g_{\Psi(z^k)}^\top d^k \leq \frac{\Psi(z^k) - \Psi(z^k + 2^{-i_k}d^k)}{c}.$$

By using sandwich theorem, we obtain

$$\{2^{-i_k} g_{\Psi(z^k)}^\top d^k\}_{k\in K} \to 0. \tag{3.21}$$

We will show that $\{2^{-i_k}\}_K$ is bounded away from 0. On the contrary, by subsequencing, if necessary, we have $\{2^{-i_k}\} \to 0$, so that at each step the step size is reduced at least once. Therefore, from Armijo-Goldstein condition, there exists $k_0 \in K$ such that

$$\Psi(z^k + (2^{-(i_k-1)}/\beta)d^k) > \Psi(z^k) + c(2^{-(i_k-1)}/\beta) g_{\Psi(z^k)}^\top d^k,$$

for every $k \geq k_0$, $k \in K$, where $\beta \in (0, 1)$.

Let $\alpha_k = \frac{2^{-(i_k-1)}}{\beta}\|d^k\|$ and $p_k = \frac{d^k}{\|d^k\|}$. Therefore,

$$\frac{\Psi(z^k + \alpha_k p_k) - \Psi(z^k)}{\alpha_k} > c g_{\Psi(z^k)}^\top p_k. \tag{3.22}$$

Taking $k \to \infty$, we have $\{\alpha_k\}_{k\in K} \to 0$ and $\{p_k\}_{k\in K} \to \overline{p}$ with $\|\overline{p}\| = 1$. Thus, from (3.22), we get

$$g_{\Psi(\overline{z})}^\top \overline{p} > c g_{\Psi(\overline{z})}^\top \overline{p} \implies (1-c) g_{\Psi(\overline{z})}^\top \overline{p} > 0 \implies g_{\Psi(\overline{z})}^\top \overline{p} > 0. \tag{3.23}$$

But from (3.9), for $\kappa > 2$, we have

$$g_{\Psi(z^k)}^\top d^k \leq -\rho\|d^k\|^\kappa$$

which implies that

$$g_{\Psi(z^k)}^\top \frac{d^k}{\|d^k\|} \leq -\rho\frac{\|d^k\|^\kappa}{\|d^k\|},$$

that is,

$$g_{\Psi(z^k)}^\top p_k \leq -\rho\|d^k\|^{\kappa-1}, \quad \text{for every } k.$$

As $k \to \infty$, we have $\{p_k\}_{k\in K} \to \overline{p}$ with $\|\overline{p}\| = 1$ and $\{z^k\} \to \overline{z}$. Therefore,

$$\nabla\Psi(\overline{z})^\top \overline{p} \leq -\rho\|\overline{d}\|^{\kappa-1} \leq 0,$$

which contradicts (3.23). Thus, our assumption that $\{2^{-i_k}\}_{k\in K} \to 0$ was wrong. Therefore, $\{2^{-i_k}\}_K$ is bounded away from 0, i.e., for some $\alpha > 0$,

$$\{2^{-i_k}\} \to \alpha, \text{ for every } k = 1, 2, \ldots. \tag{3.24}$$

Thus, with the help of (3.20) and (3.24), for $k \to \infty$, we get from (3.21) that $g_{\Psi(\overline{z})} = 0$. Thus, for every accumulation point $\overline{z}$ of $\{z^k\}$, $0 \in \partial\Psi(\overline{z})$.

(ii) Under the hypothesis, we first prove the existence of d^k that satisfies the system (3.8)–(3.9).

By strong BD-regularity assumption on $\bar{z}$, we notice that H^k, being an element of Jac $F(z^k)$, nonsingular for every $k \in \mathbb{N}$ (by Proposition 2.6 in [75]). Therefore, the system (3.8) admits a solution in the sense that $\|r^k\| \leq \eta_k \|F(z^k)\|$.

We show that a solution d^k of (3.8) satisfies the following condition for some positive real ρ and $\kappa > 2$:

$$g_{\Psi(z^k)}^{\top} d^k \leq -\rho \|d^k\|^{\kappa}. \tag{3.25}$$

Since z^k converges to a BD-regular solution $\overline{z}$ of $F(z) = 0$, from the boundedness property of the generalized Jacobian on bounded sets, there exist $m, M > 0$ such that

$$m\|v\| \leq \|H^k v\| \leq M\|v\|, \quad \text{for every } k \in \mathbb{N},\, v \in \mathbb{R}^{n+m}. \tag{3.26}$$

By using (3.26), we can write

$$m\|d^k\| \leq \|H^k d^k\| = \|F(z^k) - r^k\| \leq M\|d^k\|, \quad \text{for every } k \in \mathbb{N},\, d^k \in \mathbb{R}^{n+m}. \tag{3.27}$$

Since $\eta_k \to 0$, we have $\|r^k\| = o(\|F(z^k)\|)$; which gives

$$\frac{m}{2}\|d^k\| \leq \|F(z^k)\| \leq 2M\|d^k\|, \quad \text{for every sufficiently large } k \in \mathbb{N},\, v \in \mathbb{R}^{n}. \tag{3.28}$$

As $g_{\Psi(z^k)} = {H^k}^\top F(z^k)$, from (3.8) and (3.28), we get

$$\begin{aligned}
g_{\Psi(z^k)}^\top d^k &= ({H^k}^\top F(z^k))^\top d^k \\
&= F(z^k)^\top(-F(z^k) + r^k) \\
&= -\|F(z^k)\|^2 + F(z^k)^\top r^k \\
&\leq -\frac{m^2}{4}\|d^k\|^2 + o(\|F(z^k)\|^2) \\
&= -\frac{m^2}{4}\|d^k\|^2 + o(\|d^k\|^2) \\
&\leq -\frac{m^2}{8}\|d^k\|^2, \text{ for all } k \text{ sufficiently large.}
\end{aligned}$$

Thus, d^k satisfies (3.25) for any positive ρ and $\kappa = 2$. However, by (3.8) and the assumption $\eta_k \to 0$, we obtain $\|d^k\| \to 0$. Therefore, (3.25), and hence (3.9) is true for any $\kappa > 2$ and any positive real ρ.

To complete the proof, it only remains to show that the step size determined by Armijo-Goldstein conditions is eventually 1, i.e., eventually $i_k = 0$. Then, the rate of convergence follows immediately from Theorem 3.2 [73]. We will show that $i_k = 0$ is eventually accepted by the Armijo-Goldstein condition (3.10). For this, we will show that there exist constants β and γ with $0 < \beta \leq \gamma$ such that

$$\begin{aligned}
\Psi(z^k) + (1-c)g_{\Psi(z^k)}^\top d^k &\leq \beta\Psi(z^k) \\
\text{and } \gamma\Psi(z^k) &\leq \Psi(z^k) + c g_{\Psi(z^k)}^\top d^k \\
\text{with } \beta\Psi(z^k) \leq \Psi(z^k + d^k) &\leq \gamma\Psi(z^k).
\end{aligned}$$

Since we have $g_{\Psi(z^k)} = {H^k}^\top F(z^k)$, H^k being matrix from (3.8), d^k satisfies (3.8) with $\|r^k\| \leq \eta_k\|F(z^k)\|$. Using Cauchy-Schwarz-inequality, we have

$$\begin{aligned}
\Psi(z^k) + c g_{\Psi(z^k)}^\top d^k &= \Psi(z^k) + c({H^k}^\top F(z^k))^\top d^k \\
&= \Psi(z^k) + cF(z^k)^\top(-F(z^k) + r^k)
\end{aligned}$$

$$
\begin{aligned}
&= \Psi(z^k) - c\|F(z^k)\|^2 + cF(z^k)^\top r^k \\
&\geq \Psi(z^k) - 2c\Psi(z^k) - c\|F(z^k)\|\|r^k\| \\
&\geq \Psi(z^k) - 2c\Psi(z^k) - c\eta_k\|F(z^k)\|^2 \\
&= \Psi(z^k) - 2c\Psi(z^k) - 2c\eta_k\Psi(z^k) \\
&= (1 - 2c - 2c\eta_k)\Psi(z^k).
\end{aligned}
$$

Thus, there exists a constant $\gamma > 0$ such that

$$
\Psi(z^k) + cg_{\Psi(z^k)}^\top d^k \geq \gamma\Psi(z^k), \tag{3.29}
$$

for all k sufficiently large, as $c \in (0, 0.5)$ and $\eta_k \to 0$. Also, by using Proposition 2.6 [73], we have

$$
\gamma\Psi(z^k) \geq \Psi(z^k + d^k) \tag{3.30}
$$

for all sufficiently large k. We further have

$$
\begin{aligned}
\Psi(z^k) + (1-c)g_{\Psi(z^k)}^\top d^k &= \Psi(z^k) - (1-c)\|F(z^k)\|^2 + (1-c)F(z^k)^\top r^k \\
&\leq \Psi(z^k) - (1-c)\|F(z^k)\|^2 + (1-c)\|F(z^k)\|\|r^k\| \\
&\leq \Psi(z^k) - (1-c)2\Psi(z^k) + 2\eta_k(1-c)\Psi(z^k) \\
&= (2c - 1 + 2\eta_k - 2\eta_k c)\Psi(z^k).
\end{aligned}
$$

For sufficiently large k, $\eta_k \to 0$ and $(2c - 1 + 2\eta_k - 2c\eta_k) \to (2c-1) = \beta \in (-1, 0)$. Thus, for sufficiently large k, we have $\beta \in (-1, 0)$ and

$$
\Psi(z^k) + (1-c)g_{\Psi(z^k)}^\top d^k \leq \beta\Psi(z^k). \tag{3.31}
$$

However, we already have $\Psi(z^k + d^k) = \frac{1}{2}\|F(z^k + d^k)\|^2 \geq 0$. Therefore, for sufficiently large k, we have

$$\beta\Psi(z^k) \leq \Psi(z^k + d^k). \tag{3.32}$$

By using (3.30) and (3.32), we obtain

$$\beta\Psi(z^k) \leq \Psi(z^k + d^k) \leq \gamma\Psi(z^k). \tag{3.33}$$

Hence,

$$\Psi(z^k) + (1-c)g_{\Psi(z^k)}^\top d^k \leq \Psi(z^k + d^k) \leq \Psi(z^k) + c g_{\Psi(z^k)}^\top d^k.$$

Thus, we can see that Armijo-Goldstein condition is eventually satisfied by $i_k = 0$, i.e., $\alpha_k = 1$. By using (3.28), we get

$$\frac{\|r^k\|}{\|d^k\|} \leq 2M \frac{\|r^k\|}{\|F(z^k)\|} \to 0, \quad \text{as } \|r^k\| = o(\|F(z^k)\|).$$

Hence,

$$\frac{\|r^k\|}{\|d^k\|} = \frac{\|H^k d^k + F(z^k)\|}{\|d^k\|} \to 0, \quad \text{for } k \text{ sufficiently large.}$$

Therefore, by using Theorem 2 and Corollary 2 in [76], the sequence $\{z^k\}$ converges to $\overline{z}$ Q-superlinearly.

Furthermore, if $\eta_k = O(\|F(z^k)\|)$, i.e., $\|r^k\| = O(\|F(z^k)\|^2)$, then we can write

$$\frac{\|r^k\|}{\|d^k\|^2} \leq 4M^2 \frac{\|r^k\|}{\|F(z^k)\|^2} < \infty, \quad \text{as } \|r^k\| = O(\|F(z^k)\|^2).$$

Thus,

$$\frac{\|r^k\|}{\|d^k\|^2} = \frac{\|H^k d^k + F(z^k)\|}{\|d^k\|^2} < \infty, \quad \text{for sufficiently large } k.$$

Therefore, by Proposition 2.3 and Theorem 2.5 in [73], the sequence $\{z^k\}$ converges to $\overline{z}$ Q-quadratically, which completes the proof. $\square$

3.4.3 Inexact Newton method of GNEP: jointly convex case

In this section, we consider the jointly convex GNEP defined in Definition 1.3 with feasible set (**??**).

Therefore, we have objective function $\theta_v(x^v, \boldsymbol{x}^{-v})$ of (1.1) is convex in x^v, and the set $X_v(\boldsymbol{x}^{-v})$ is closed and convex for every v, $v = 1, 2, \ldots, N$. Accumulating the strategy sets of all the players, we get the strategy set for the GNEP as

$$X := \prod_{v=1}^{N} X_v(\boldsymbol{x}^{-v}).$$

With the help of a complementarity function ϕ as defined in (1.6), the system (1.5) for the GNEP in Definition 1.3 can be reformulated into the following system

$$G(x, \lambda, \mu) = \begin{pmatrix} L(x, \lambda, \mu) \\ \phi(-s(x), \lambda) \\ \phi(-h^1(x^1), \mu^1) \\ \vdots \\ \phi(-h^v(x^v), \mu^v) \\ \vdots \\ \phi(-h^N(x^N), \mu^N) \end{pmatrix} = 0, \tag{3.34}$$

where $\mu = \begin{pmatrix} \mu^1 \\ \mu^2 \\ \vdots \\ \mu^N \end{pmatrix}$, $L(x, \lambda, \mu) = \begin{pmatrix} \nabla_{x^1} L_1(x^1, \boldsymbol{x}^{-1}, \lambda, \mu) \\ \nabla_{x^2} L_1(x^2, \boldsymbol{x}^{-2}, \lambda, \ldots, \mu) \\ \vdots \\ \nabla_{x^N} L_N(x^N, \boldsymbol{x}^{-N}, \lambda, \mu) \end{pmatrix}$ and

$$L_v(x^v, x^{-v}, \lambda, \mu) = \theta_v(x^v, \overline{\boldsymbol{x}}^{-v}) + \lambda s(x^v, \overline{\boldsymbol{x}}^{-v}) + h(x^v)\mu^v, \quad v = 1, 2, \ldots, N.$$

In this chapter, we use the 'min' function as the complementarity function ϕ.

To find a solution to the GNEP with the strategy set $X_v(\boldsymbol{x}^{-v})$ as defined in (**??**) under the jointly convex case is equivalent to calculating the x-component of a zero of the function $G(z)$ with $z = (x, \lambda, \mu)$. As the function $G : \mathbb{R}^n \times \mathbb{R}^{m_0} \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^{m_0} \times \mathbb{R}^m$ in (3.34) is a semismooth function (see [70]), in the following, we solve the nonsmooth system (3.34) by an inexact Newton method.

Define a merit function:

$$\Psi(z) = \frac{1}{2}\|G(z)\|^2. \tag{3.35}$$

Since, we have $G : \mathbb{R}^{n+m_0+m} \to \mathbb{R}^{n+m_0+m}$ in (3.5), a locally Lipschitz continuous mapping, therefore $\Psi : \mathbb{R}^{n+m_0+m} \to \mathbb{R}$ in (3.6) is also a locally Lipschitz continuous mapping. From Clarke [71, 72], Ψ is differentiable almost everywhere in $\mathbb{R}^{n+m_0+m}$. Let B be any nonempty, open subset of R^{n+m_0+m}, and Ψ be Lipschitz of B. Then, for $z \in B$, define a set

$$Jac\ \Psi(z) = \{\lim_{j\to\infty} \nabla\Psi(z^j)|\ \{z^j\} \to z,\ \text{and } \Psi \text{ is differentiable at } z^j\}. \tag{3.36}$$

The convex hull of $Jac\ \Psi(z)$ is subdifferential of Ψ at z, and is denoted by $\partial\Psi(z)$. Thus,

$$\partial\Psi(z) = Conv\{g_{\Psi(z)} \in \mathbb{R}^{n+m} \mid g_{\Psi(z)} = H(z)^\top F(z),\ H(z) \in Jac\ F(z)\}.$$

3.4.4 Algorithm: Inexact Newton method for jointly convex GNEPs

In the following algorithm, we provide a stepwise procedure to apply the inexact Newton method to solve the GNEP. The essence of notations and parameters used in the algorithm is identical to that in Subsection 3.4.2.

Algorithm 4 Computing z_k such that $\|\Psi(z_k)\| < \epsilon$ to solve (3.35)

Step 0. (Initialization step). Choose the constants $\rho > 0$ and $\kappa > 2$.

Take an initial value $z^0 \in \mathbb{R}^n \times \mathbb{R}^{m_0} \times \mathbb{R}^m$.

Given precision scalar $\epsilon > 0$.

Set the iteration counter $k = 0$.

Step 1. (Terminating condition). If $\|G(z^k)\| < \epsilon$, then give the output z^k as an ϵ-precision solution and stop.

Step 2. (Main steps).

Substep 2.1: (Descent direction choice). Choose an $H^k \in \text{Jac}\,\Psi(z^k)$, where $\Psi(z) = \frac{1}{2}\|G(z)\|^2$.

Find a vector $d^k \in \mathbb{R}^{n+m_0+m}$ that satisfies the following conditions

$$H^k d = -G(z^k) + r^k \text{ and } \|r^k\| \leq \eta_k \|G(z^k)\| \tag{3.37}$$

and

$$\langle (H^k)^\top G(z^k), d^k \rangle \leq -\rho \|d^k\|^\kappa. \tag{3.38}$$

If d^k does not exist with the above conditions, set $d^k = -(H^k)^\top G(z^k)$.

Step 2.2: (Step length choice). Choose i_k, the smallest nonnegative integer i that satisfies the following pair of inequalities

$$\left.\begin{aligned} &\Psi(z^k + 2^{-i}d^k) \leq \Psi(z^k) + c2^{-i}\langle (H^k)^\top G(z^k), d^k \rangle \text{ and} \\ &\Psi(z^k + 2^{-i}d^k) \geq \Psi(z^k) + (1-c)2^{-i}\langle (H^k)^\top G(z^k), d^k \rangle. \end{aligned}\right\} \tag{3.39}$$

Step 2.3: (Increase the iteration counter k and update z^k). Set $z^{k+1} = z^k + 2^{-i_k} d^k$.

Set $k = k + 1$.

Choose an $\eta_{k+1} \geq 0$ such that $\eta_{k+1} \leq \eta_k$ and go to Step 1.

The following theorem proves the global quadratic convergence of Algorithm 4.

Theorem 3.3 *Let $\{z^k\}$ be the sequence generated by Algorithm 4. Assume that the forcing sequence $\{\eta_k\}$ with $\eta_k \geq \frac{\|r^k\|}{\|G(z^k)\|}$ is a sequence of positive numbers such that $\eta_k \leq \overline{\eta} < 1$ for every k. Then,*

(i) for every accumulation point $\overline{z}$ of $\{z^k\}$, $0 \in \partial\Psi(\overline{z})$, and

(ii) for any strongly BD-regular solution *$\overline{z}$ of the system $G(z) = 0$, if $z^k \to \overline{z}$, then the rate of convergence of this sequence $\{z^k\}$ is Q-superlinear, if $\eta_k \to 0$. Furthermore, if $\eta_k = O(\|G(z^k)\|)$, then the rate of convergence of this sequence $\{z^k\}$ is Q-quadratic.*

Proof: The proof is analogous to that of Theorem 3.2. □

3.5 Numerical results

In this section, we solve a few examples by applying Algorithms 3 and 4. The numerical computations are performed in Matlab (version 2018b). The performances of Algorithms 3 and 4 are compared with the semismooth Newton method II in [1]. We give the comparison only with the semismooth Newton method [1] due to the known fact that semismooth Newton method performs better than other existing methods for GNEPs (see [1]). In the following results, the nonzero $\xi\ e-a$ stands for $\xi \times 10^{-a}$.

Problem 3.1 *Consider the following game with two players:*

$$\left.\begin{array}{l} \min\limits_{x}\,(x-1)^2 \\[2ex] \text{subject to } x+y \leq 1 \end{array}\right\} \qquad \text{and} \qquad \left\{\begin{array}{l} \min\limits_{y}\,\left(y-\frac{1}{2}\right)^2 \\[2ex] \text{subject to } x+y \leq 1. \end{array}\right.$$

This problem was considered by Facchinei et al. [1], and has infinitely many solutions. Its solutions are given by $(\alpha, 1-\alpha)$, $\alpha \in [0.5, 1]$. It is observed that for this problem, Algorithm 3 converges to the point $(\alpha, 1-\alpha)$ for an $\alpha \in [0.5, 1]$ in 2 to 4 iterations starting from any feasible point, which shows much faster convergence than that of the semismooth Newton method II [1] (see Table 3.1).

In Table 3.1, we draw a comparision of the performances of Algorithm 3 and the semismooth Newton method II [1].

This problem has many generalized Nash equilibria, for example, $(0.2, 0.3)$, $(0, 0)$, $(0.1, 0.2)$, *etc. The different initial value (starting point) gives different GNE points. In this problem, if we choose the initial point as a feasible point, then both the numerical schemes (Algorithm 3 and the semismooth Newton method II [1]) converge in almost same number of iterations. However, if we take an infeasible point as the initial point, Algorithm 3 converges rapidly in* 2 *to* 4 *number of iterations, while semismooth Newton method II [1] takes a higher number of iterations.*

We have also given the graphical view in Figures 3.1 and 3.2 on how both the numerical schemes converge starting from the same initial point $(2, 4)$. *We see that inexact Newton method converges in* 3 *iterations, while the semismooth Newton method II takes* 28 *iterations to converge to the same GNE point* $(1, 0)$.

Table 3.1: Comparison of the performances of Algorithm 3 and the semismooth Newton method II [1] on Problem 3.1

Starting point $(x^0, y^0, \lambda^0, \mu^0)$	Tol. (ϵ)	Solution $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\mu})$	No. of iterations in inexact-Newton method 3	Computation time (sec.)	No. of iterations in Newton method II [1]	Computation time (sec.)
$(0.2, 0.3, 0.5, 0.6)$	$e-8$	$(0.775, 0.225, 0.45, 0.55)$	2	4.593	2	3.346
$(0, 0, 0, 0)$	$e-8$	$(0.75, 0.25, 0.5, 0.5)$	2	4.581	2	3.427
$(0.1, 0.2, 0.6, 0.9)$	$e-8$	$(0.8125, 0.1875, 0.375, 0.625)$	2	4.490	2	4.425
$(0.6, 0.9, 0.3, 0.9)$	$e-8$	$(0.9, 0.1, 0.2, 0.8)$	2	4.594	2	3.909
$(0.6, 0.3, 0.5, 0.6)$	$e-8$	$(0.775, 0.225, 0.45, 0.55)$	2	5.024	2	3.406
$(10, 20, 12, 3)$	$e-8$	$(0.5, 0.5, 1, 1.0e-10)$	3	5.615	30	16.182
$(90, 80, 60, 40)$	$e-8$	$(0.75, 0.25, 0.5, 0.5)$	3	5.824	31	16.396
$(90, 80, 60, 2)$	$e-8$	$(0.5, 0.5, 1, 5.344e-10)$	3	6.127	33	17.383
$(2, 4, 3, 6)$	$e-8$	$(1, -1.328e-11, 2.5e-11, 1)$	3	5.600	28	17.115
$(9, 8, 6, 4)$	$e-8$	$(0.5, 0.5, 1, 6.25e-11)$	3	5.615	27	14.595
$(10, 20, 30, 40)$	$e-8$	$(1, -9.57e-11, 1.844e-10, 1)$	3	5.705	30	15.816
$(20, 60, 140, 320)$	$e-8$	$(1, -2.124e-12, 3.162e-12, 1)$	3	5.860	35	18.517
$(200, 350, 620, 655)$	$e-8$	$(1, -2.125e-10, -3.653e-19, 1)$	4	6.535	32	16.907

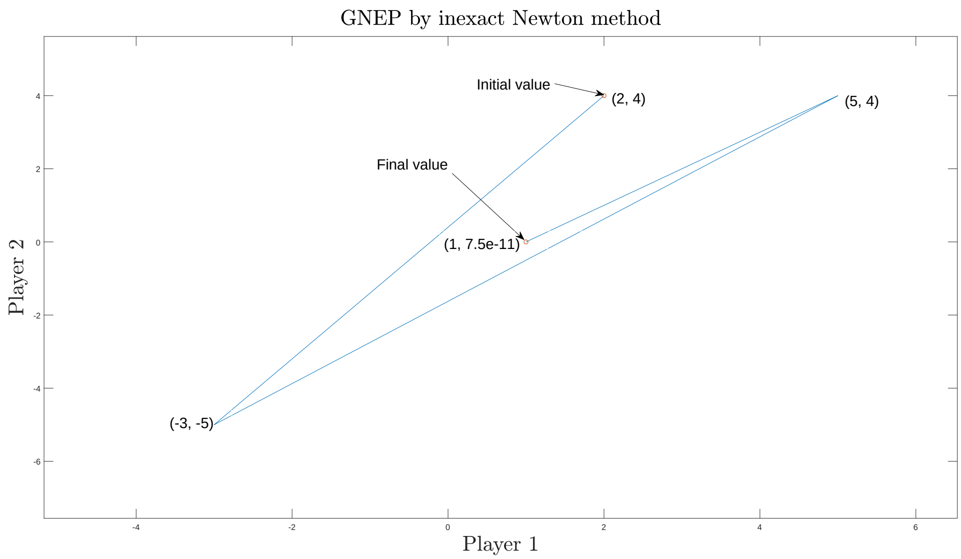

Figure 3.1: Graphical view of the movement of the iterative points generated by the proposed inexact Newton method for Problem 3.1 with the initial point $(2, 4)$: it takes 3 iterations to converge

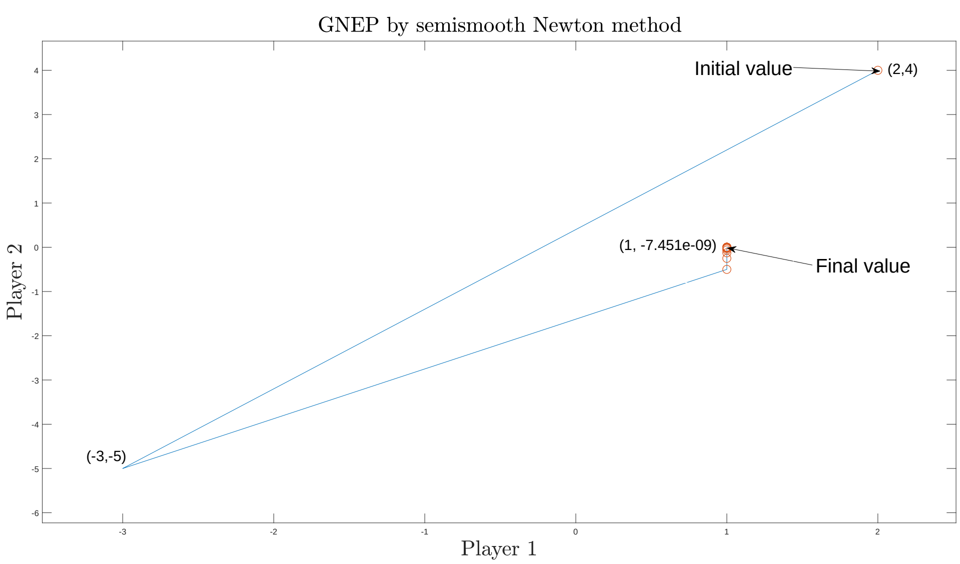

Figure 3.2: Graphical view of the movement of the iterative points generated by the semismooth Newton method II [1] for Problem 3.1 with the initial point $(2, 4)$: it takes 28 iterations to converge

Problem 3.2 *Consider the following game with two players and one shared constraint:*

$$\left.\begin{array}{l}\min\limits_{x_1} \frac{1}{2}x_1^2 - x_1x_2 \\ \\ \textit{subject to } x_1 + x_2 \geq 1,\ x_1 \geq 0\end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{l}\min\limits_{x_2} x_2^2 + x_1x_2 \\ \\ \textit{subject to } x_1 + x_2 \geq 1,\ x_2 \geq 0.\end{array}\right.$$

This problem has been introduced by Rosen [64]. In this problem, there are two constraints $h_1(x_1) = -x_1$ *and* $h_2(x_2) = -x_2$ *that depend only on the variables of a single player, and there is one shared constraint* $s(x_1, x_2) = 1 - x_1 - x_2$. *A KKT point for this problem is* $\overline{x}_1 = 1, \overline{x}_2 = 0, \overline{\lambda}_1 = 0, \overline{\lambda}_2 = 0, \overline{\mu} = 1$.

Starting with any feasible point $(x_1^0, x_2^0, \lambda_1^0, \lambda_2^0, \mu^0)$, *Algorithm 4 converges to* $(1, 0, 0, 0, 1)$. *For an initial point that is sufficiently close to* $(1, 0, 0, 0, 1)$, *both the numerical schemes — semismooth [1] and Algorithm 4 — converge rapidly taking almost same number of iterations,* 2 *or* 3. *However, if we take an initial point that is much away from the solution point* $(1, 0, 0, 0, 1)$, *then Algorithm 4 converges in approximately* 10 *to* 20 *iterations, while semismooth-Newton method [1] takes much higher number of iterations to converge. This algorithm converges for any initial point and converges to* $\overline{x}_1 = 1$ *and* $\overline{x}_2 = 1.713\text{e} - 9$. *In Table 3.2, we draw a comparison of the performances of Algorithm 4 and semismooth Newton method II.*

A graphical view on how both the numerical schemes converge starting from the same initial point $(20, 40)$ *is shown in Figures 3.3 and 3.4. We see that inexact Newton method converges in* 10 *iterations, while the semismooth Newton method II takes* 51 *iterations to converge to the GNE point* $(1, 0)$.

Problem 3.3 *Consider the following GNEP that has two players and one shared constraint* $s(x) = x_1 + x_2 - 15$*:*

$$\left.\begin{array}{l}\min\limits_{x_1} x_1^2 + \frac{8}{3}x_1x_2 - 34x_1 \\ \\ \textit{subject to } x_1 + x_2 \geq 15,\ 0 \leq x_1 \leq 10\end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{l}\min\limits_{x_2} x_2^2 + \frac{5}{4}x_1x_2 - \frac{97}{4}x_2 \\ \\ \textit{subject to } x_1 + x_2 \leq 15,\ 0 \leq x_2 \leq 10.\end{array}\right.$$

This game was introduced by Harker [63]. For this problem, Algorithm 4 converges to $\overline{x}_1 = 5, \overline{x}_2 = 9$ *starting from any feasible point. Numerical results for different starting points have been shown in the following Table 3.3. From Table 3.3, we see that*

Table 3.2: Comparison of the performances of Algorithm 4 and the semismooth Newton method II [1] on Problem 3.2

Starting point $(x_1^0, x_2^0, \lambda_1^0, \lambda_2^0, \mu^0)$	Tol. (ϵ)	Solution $(\overline{x_1}, \overline{x_2}, \overline{\lambda_1}, \overline{\lambda_2}, \overline{\mu})$	No. of iterations in Algorithm 3	Computation time(sec.)	No. of iterations in semismooth Newton method II [1]	Computation time(sec.)
$(4, 2, 1, 2, 1)$	$e-8$	$(1.0, 1.0e-20, 5.0e-21, 1.0e-20, 1.0)$	2	7.198	2	6.118
$(0, 0, 0, 0, 0)$	$e-8$	$(1.0, 0, 0, 0, 1.0)$	1	5.065	2	4.572
$(125, 235, 102, 30, 355)$	$e-8$	$(1.0, 1.175e-18, 5.1e-19, 1.5e-19, 1.0)$	2	7.158	2	5.649
$(20, 36, 5, 2, 3)$	$e-8$	$(1.0, 1.8e-19, 2.5e-20, 1.0e-20, 1.0)$	2	7.340	2	5.604
$(100, 200, 2, 3, 5)$	$e-8$	$(1, 5.926e-23, 5.926e-25, 8.889e-25, 1)$	2	9.147	2	5.740
$(20, 40, 24, 54, 21)$	$e-8$	$(1, 3.316e-14, 0, 6.395e-14, 1)$	10	23.855	51	69.470
$(100, 300, 352, 652, 129)$	$e-8$	$(1, 5.329e-15, 0, 0, 1)$	13	30.376	59	78.746
$(120, 30, 224, 153, 100)$	$e-8$	$(1, 0, 0, 0, 1)$	10	23.551	57	79.689
$(100, 20, 110, 230, 108)$	$e-8$	$(1, 0, 0, 0, 1)$	13	31.333	57	83.299

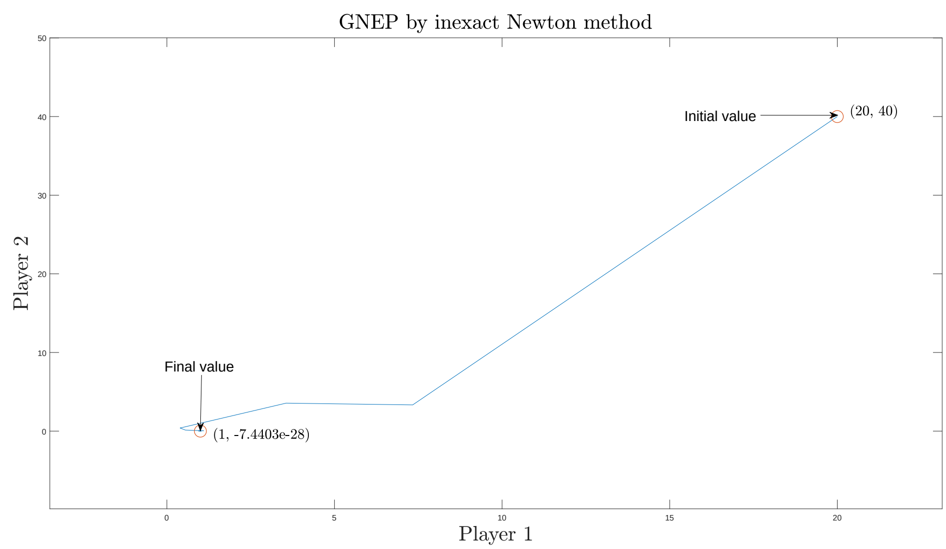

Figure 3.3: Graphical view of the movement of the iterative points generated by Algorithm 4 for Problem 3.2 with the initial point $(20, 40)$: it takes 10 iterations to converge

for quite a few initial points, the semismooth Newton method II does not converge, but Algorithm 4 converges in just 5 to 6 iterations.

The graphical comparison of the numerical schema—Algorithm 4 and semismooth-Newton method [1] is shown in Figures 3.5 and 3.6.

Problem 3.4 *(*A model of internet switching with selfish users*). Kesselman et al. [65] proposed this model in 2005. It analyzes the problem of internet switching, where the*

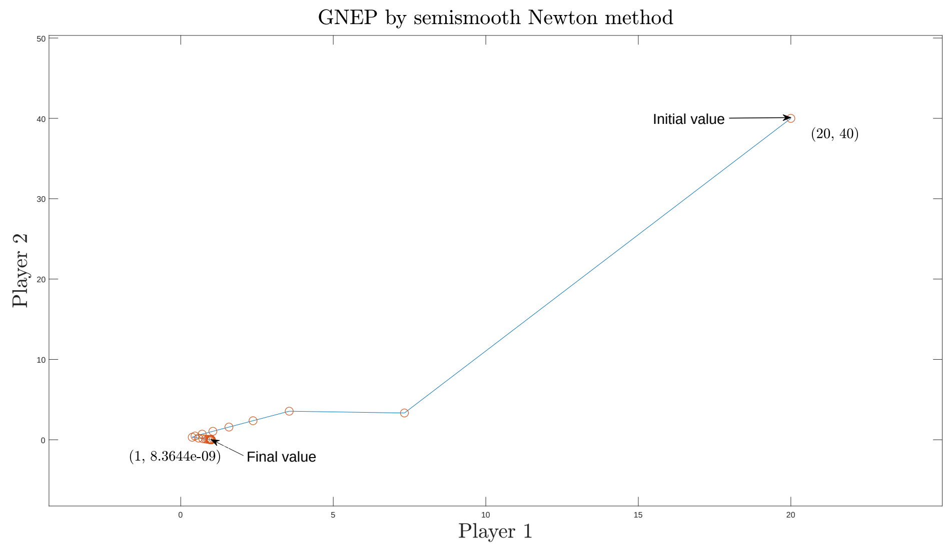

Figure 3.4: Graphical view of the movement of the iterative points generated by the semismooth Newton method II [1] for Problem 3.2 with the initial point $(20, 40)$: it takes 51 iterations to converge

Table 3.3: Comparison of the performances of Algorithm 4 and the semismooth Newton method II [1] on Problem 3.3

Starting point $(x_1^0, x_2^0, \lambda_{11}^0, \lambda_{12}^0, \lambda_{21}^0, \lambda_{22}^0, \mu^0)$	Tol. (ϵ)	Solution $(\overline{x_1}, \overline{x_2}, \overline{\lambda_{11}}, \overline{\lambda_{12}}, \overline{\lambda_{21}}, \overline{\lambda_{22}}, \overline{\mu})$	No. of iterations in Algorithm 3	Computation time(sec.)	No. of iterations in semismooth Newton method II [1]	Computation time(sec.)
$(0,0,0,0,0,0,0)$	$e-8$	$(5,9,0,0,0,0,0)$	1	10.940	1	7.103
$(1,2,3,5,2,1,9)$	$e-8$	$(5,9,3.125e-17,3.125e-12, 3.125e-22,1.562e-22,3.906e-12)$	3	25.891	3	11.089
$(2,5,4,2,6,5,2)$	$e-8$	$(5,9,6.698e-16,6.698e-21, 3.349e-16,1.674e-20,1.004e-11)$	3	25.816	3	11.047
$(0.25,0.36,0.65, 0.1,0.25,0.36,0.45)$	$e-8$	$(5,9,2.315e-11,5.787e-17, 1.447e-16,2.083e-16,2.604e-16)$	2	16.590	2	9.004
$(6,5,0,0,0,0,0)$	$e-8$	$(5,9,0,0,0,0,0)$	1	7.778	1	7.128
$(2,5,4,6,9,7,5)$	$e-8$	$(5,9,1.953e-18,3.906e-13, 3.906e-18,5.127e-18,5.859e-13)$	4	30.386	Not converging	∞
$(12,2,3,4,1,2,2)$	$e-8$	$(5,9,4.102e-20,1.172e-19, 1.367e-20,6.042e-11,2.954e-20)$	4	30.579	Not converging	∞
$(3,6,7,1,4,3,1)$	$e-8$	$(5,9,5.0e-10,8.75e-10, 5.0e-19,3.75e-19,4.375e-10)$	4	26.253	Not converging	∞
$(8,6,9,2,8,3,1)$	$e-8$	$(5,9,1.953e-18,1.952e-13, 1.953e-18,3.052e-18,5.371e-13)$	5	30.018	Not converging	∞
$(9,4,8,7,3,2,8)$	$e-8$	$(5,9,1.172e-12,2.246e-12, 2.93e-18,1.953e-18,2.209e-12)$	5	30.429	Not converging	∞

traffic is generated by selfish users. In this model, the routers have First-In-First-Out buffers of bounded capacity with the drop-tail policy. The utility of each user depends on its transmission rate and the congestion level. In this model, there are N users and

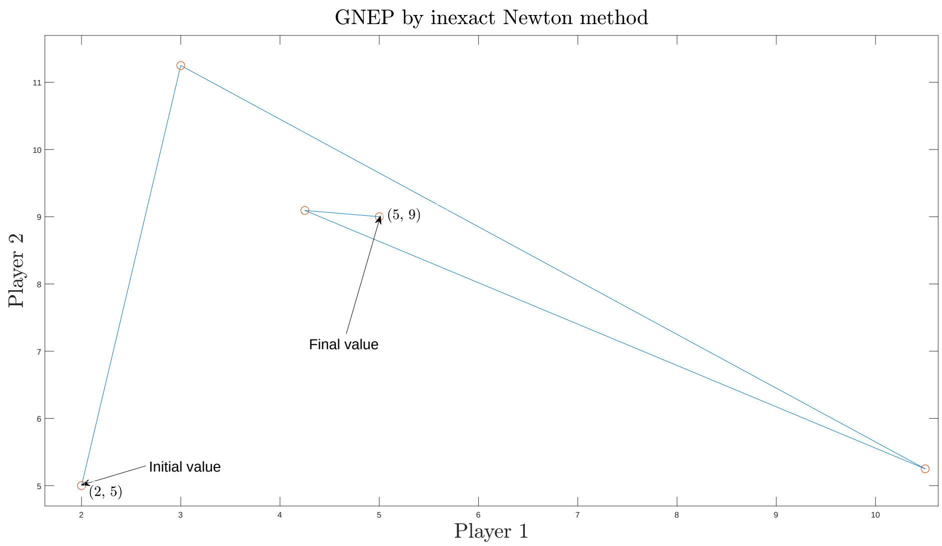

Figure 3.5: Graphical view of the movement of the iterative points generated by Algorithm 4 for Problem 3.3 with the initial point $(2, 5)$: it takes 4 iterations to converge

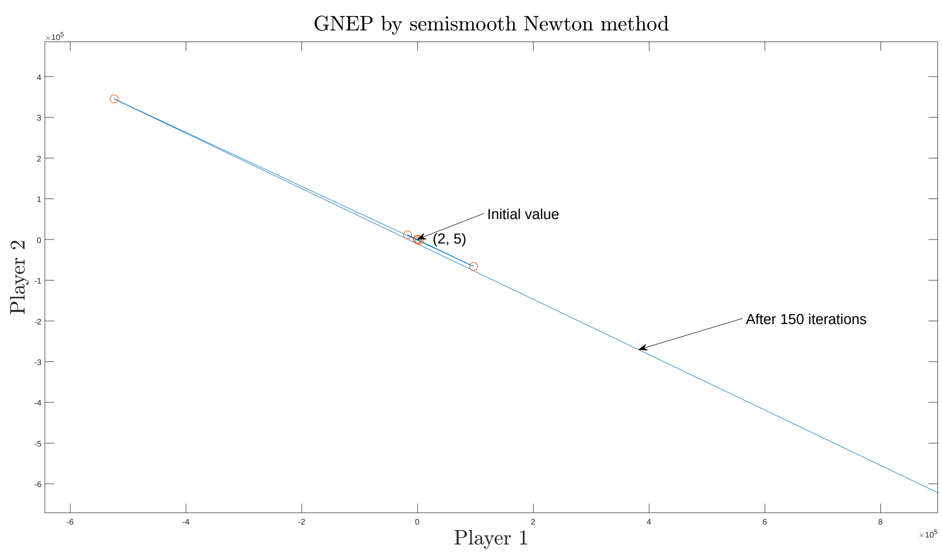

Figure 3.6: Graphical view of the movement of the iterative points generated by the semismooth Newton method II [1] for Problem 3.3 with the initial point $(2, 5)$: it does not converge

the buffer capacity B. Each user v controls the amount of his packets in the buffer, denoted by $x_v \in [0, \infty)$. If the minimal amount of data for Player v is l_v, the problem *associated to* v^{th} *player is*

$$\min_{x_v} \frac{x_v}{B} - \frac{x_v}{\sum_{v=1}^{N} x_v}$$
$$\textit{subject to } \sum_{v=1}^{N} x_v \leq B,\ x_v \geq l_v.$$

For this game with four players ($v = 4$)*, Table 3.4 shows the performance of Algorithm 4 where the minimial amount of data* ($l_v > 0$) *for the players are* $l_v = 0.01$ *for all* $v = 1, 2, 3, 4$ *and* $B = 1$*. Algorithm 4 converges to the GNE point* $\overline{x}_1 = 0.1875, \overline{x}_2 = 0.1875, \overline{x}_3 = 0.1875, \overline{x}_4 = 0.1875$ *for any initial point in* 4 *and* 5 *iterations. Also, it is observed that both numerical techniques–Algorithm 4 and Semismooth newton method II [1]–take almost same number of iterations.*

Table 3.5 shows the performance for 10 *players and* $l_v = 0.01$ *for every* $v = 1, 2, \ldots, 10$ *and* $B = 1$*, Algorithm 4 converges to an equilibrium point starting from any feasible point in* 8 *iterations.*

Table 3.4: Performance of Algorithm 4 on Problem 3.4 with 4 players

Starting points $(x_1^0, x_2^0, x_3^0, x_4^0)$	Tol. (ϵ)	Solutions $(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4)$	Number of iterations
$(0.09091, 0.04892, 0.2006, 0.08469)$	$e-8$	$(0.1875, 0.1875, 0.1875, 0.1875)$	5
$(0.2367, 0.1278, 0.1274, 0.09105)$	$e-8$	$(0.1875, 0.1875, 0.1875, 0.1875)$	4
$(0.1413, 0.08112, 0.1887, 0.05535)$	$e-8$	$(0.1875, 0.1875, 0.1875, 0.1875)$	5
$(0.02947, 0.2331, 0.1962, 0.1268)$	$e-8$	$(0.1875, 0.1875, 0.1875, 0.1875)$	4

3.6 Conclusion

In this chapter, an inexact Newton method to solve generalized Nash equilibrium problems is proposed for both the cases of player convex GNEP and jointly convex GNEP. In the proposed approach, we have reformulated GNEP into a nonsmooth system of

Table 3.5: Performance of Algorithm 4 on Problem 3.4 with 10 players

Starting points $(x_1^0, x_2^0, x_3^0, x_4^0, x_5^0, x_6^0, x_7^0, x_8^0, x_9^0, x_{10}^0)$	Tol. (ϵ)	Solutions $(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5, \overline{x}_6, \overline{x}_7, \overline{x}_8, \overline{x}_9, \overline{x}_{10})$	Number of iterations
$(0.0134, 0.0116, 0.0179, 0.0131, 0.0153, 0.0117, 0.0160, 0.0126, 0.0165, 0.0169)$	$e-8$	$(0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09)$	8
$(0.0108, 0.0109, 0.0101, 0.0109, 0.0106, 0.0101, 0.0103, 0.0106, 0.0110, 0.0110)$	$e-8$	$(0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09)$	8
$(0.0100, 0.0109, 0.0109, 0.0107, 0.0108, 0.0107, 0.0104, 0.0107, 0.0101, 0.0107)$	$e-8$	$(0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09)$	8
$(0.0108, 0.0108, 0.0102, 0.0105, 0.0104, 0.0106, 0.0107, 0.0108, 0.0103, 0.0107)$	$e-8$	$(0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09, 0.09)$	8

equations and then solved it by the inexact Newton method. Under some mild conditions (see Theorems 3.2 and 3.3), the numerical Algorithms 3 and 4 globally converges Q-quadratically, which is a faster rate of convergence for such equilibrium problems. The numerical Algorithms 3 and 4 have been tested on various problems found in specialized literature on GNEPs (see [41, 63–65]). Previously, GNEP was solved by other conventional methods, such as smoothing Newton method [77], a feasible direction interior point method [78], etc., but it has been reported that the proposed numerical scheme converges faster than semismooth Newton method II and hence than all the existing method (see [1]).

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