Chapter 1

Introduction

Optimization, in simplest terms, is the act of making the best of anything. Broadly speaking, optimization seeks to change an existing process to increase the occurrence of favorable outcomes and decrease the occurrence of undesirable results. Concerning mathematics, optimization adopts the most significant component of a particular criterion from a set of viable alternatives. Mathematical optimization aims to find a combination of input variables that maximizes or minimizes the output return of a multivariable function. These days mathematical optimization has been transformed into an innovatory tool for powerful modeling and decision-making occurrences in all quantitative disciplines, from computer science and engineering to operations research and economics.

Mathematically, optimization models comprise three significant components: decision variables, objective function, and constraints.

- *Decision variables* designate a value that may vary within the scope of a given optimization problem.

- In a mathematical optimization problem, the *objective function* expresses the problem's main criteria, whose value is either minimized or maximized over the set of feasible alternatives.

- *Constraints* are the logical conditions or allowable values or scopes for the variables in an optimization problem that the solution of a given problem must satisfy.

In an optimization problem, the types of mathematical relationships among the decision variables, the objective function, and the constraints determine how intricate it is to operate and the algorithms that can be used for optimization. There are numerous applications of optimization theory and methods in the fields of applied math, computation math, and operations research, including science, engineering, business management, military, and space technology. It involves

- the construction of model problems,
- the study of optimality conditions of the problems,
- the determination of the algorithmic method of the solution,
- the establishment of convergence theory of the algorithms, and
- numerical experiments with typical and real-life problems.

One of the most common and primary problems in scientific research and engineering practice is optimization problems. In our thesis, we focus on a specific optimization problem: the generalized Nash equilibrium problem. A generalized Nash Equilibrium Problem (GNEP) is a noncooperative Nash equilibrium problem in which the strategy set of each player may depend on the strategies of the rival player. It was first formally introduced by Debreu [2] as a social equilibrium in 1952, and later as an abstract economy [3]. GNEPs have been an interesting area of research during the last two decades and it has several real-world applications in the areas of economics, computer science and engineering, for example, the abstract economy model given by Arrow and Debreu [3], a power allocation problem in telecommunications [4], a competition among countries that arises from the Kyoto protocol to reduce the air pollution [5], etc. A few other application areas of GNEPs include wireless communication [6, 7], cloud computing [8], electricity generation [9], etc. As an application of GNEPs, Robinson [10, 11] discussed the problem of measuring effectiveness in optimization-based combat models and gave several mathematical formulations.

In the next section, we provide a literature review on GNEPs.

1.1 Literature review of GNEPs

GNEP is a non-cooperative game in which the strategy set of each player may depend on the strategies of the rival player. It was first formally introduced by Debreu [2] as a social equilibrium in 1952, and later as an abstract economy [3] in 1954. In the early '50s, Nash [12, 13] introduced a notion of equilibrium, called Nash equilibrium, for non-cooperative N-player games where the payoff function of each player depends on the others' strategies. Arrow and Debreu [3] extended this notion to the generalized Nash equilibrium for games where both the payoff function and the set of feasible strategies depend on others' strategies. GNEPs have been a major area of research during the last two decades, which have several real-world applications in the areas of economics, computer science, and engineering, e.g., the abstract economy model [3], a power allocation problem in telecommunications [4], a competition among countries that arises from the Kyoto protocol to reduce the air pollution [5], social science [14], energy problems [15–17], wireless communication [6, 7], cloud computing [8], electricity generation [9], etc. Robinson [10, 11] discussed the problem of measuring the effectiveness in optimization-based combat models and gave several mathematical formulations. All these applications have motivated the evolution of the generalized Nash equilibrium concept and its use in complex games that now require a deep understanding of theoretical and computational mathematics.

Several numerical approaches have been proposed in the literature to solve GNEPs: decomposition methods [18–20], (quasi)-variational inequality type methods [21–24], penalty methods [25], Nikaido-Isoda function type methods [26, 27], and Newton-type approaches [28–31]. For a rigorous review of the commonly used numerical approaches, we also refer to the articles by Facchinei et al. [32, 33], Fisher et al. [34], Cojocaru et al. [35], Migot et al. [36] and Nabetani et al. [37]. Interested readers can see the survey articles [34,38] and the references therein for a complete overview of the existing techniques. In most methods, researchers have analyzed the case of player convex GNEPs or jointly convex GNEPs.

According to Facchinei and Kanzow [38] and Fischer et al. [34], the two most popular methods for solving GNEPs are the Jacobi method and the Gauss-Seidel method. In the former, optimization problems of the players are solved simultaneously to compute the next iterate, while in the latter, the problems are solved one after another. One advantage of the Jacobi method is the possibility of exploiting parallel computation. Pang et al. [4] provided a study of a GNEP formulation of an interference channels problem and proved the convergence of a Jacobi type and a Gauss-Seidel type approach for this particular problem. Facchinei et al. [38] studied two special cases of GNEPs: a nonconvex version where nonconvex problems have to be solved globally and a 2-player game. Sagratella [39, 40], consider a Jacobi-type method for computing solutions of a Nash model with mixed-integer variables. Facchinei and Pang [32] discussed conditions under which an algorithm based on the regularized Jacobi method converges to a (classical) Nash equilibrium. Pang and Tao [18] studied a general algorithm (Block Coordinate Descent) for nonconvex nondifferentiable optimization and showed that their framework is applicable to compute the stationary point of a generalized potential game with linear shared constraints.

In the next section, the mathematical formulation of GNEP is discussed.

1.2 Mathematical formulation of GNEP

In this section, we provide a formal definition of GNEP and the associated notations and assumptions. The described notations and terminologies are used throughout the thesis.

In general, a GNEP has N players, namely, Player 1, Player 2, ..., Player N. In the description of a GNEP, we associate a variable $x^v \in \mathbb{R}^{n_v}$ corresponding to Player v, $v = 1, 2, \ldots, N$. Accordingly, we form a vector $x \in \mathbb{R}^n$ by

$$x = \left(x^1, x^2, \ldots, x^N\right)^\top.$$

Denoting $n = n_1 + n_2 + \cdots + n_N$, we see that $x \in \mathbb{R}^n$. To differentiate v^{th} player's variable in x, we write $(x^v, \boldsymbol{x}^{-v})$, where $\boldsymbol{x}^{-v}$ is a vector formed by all players' variables except that of Player v.

The strategy set of a player depends on the rival players' strategies. We denote the strategy set of Player v by

$$X_v(\boldsymbol{x}^{-v}) \subseteq \mathbb{R}^{n_v}.$$

In a game, the aim of Player v, for a given other players' strategy $\boldsymbol{x}^{-v}$, is to choose a strategy x^v such that x^v solves the following optimization problem:

$$\begin{aligned} &\min_{x^v} \; \theta_v(x^v, \boldsymbol{x}^{-v}) \\ &\text{subject to } x^v \in X_v(\boldsymbol{x}^{-v}), \end{aligned} \tag{1.1}$$

where $-\theta_v$ is the payoff function of the v^{th} player. We denote the solution set of (1.1) by $S_v(\boldsymbol{x}^{-v})$ for the given $\boldsymbol{x}^{-v}$. The GNEP is the problem of finding a vector $\overline{x} = \left(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N\right)^\top$ such that

$$\overline{x}^v \in S_v(\overline{\boldsymbol{x}}^{-v}), \quad \text{for every } v \in \{1, 2, \ldots, N\}. \tag{1.2}$$

The point $\overline{x}$ is called a generalized Nash equilibrium point to the game.

In practical applications, the feasible set $X_v(\boldsymbol{x}^{-v})$ of v^{th} player is defined by a finite number of constraints. Let $g^v : \mathbb{R}^n \to \mathbb{R}^{m_v}$ be the constraint mapping for Player v.

Then, the feasible set for the v^{th} player is given by

$$X_v(\boldsymbol{x}^{-v}) = \left\{x^v \in \mathbb{R}^{n_v} : g^v(x^v, \boldsymbol{x}^{-v}) \leq 0\right\}, \tag{1.3}$$

where the inequality $g^v(x^v, \boldsymbol{x}^{-v}) \leq 0$ is in the sense of componentwise. It is also possible to add equality constraints in (1.3), but we omit them for the notational simplicity. Note that for GNEP (1.1)–(1.2), the constraint functions are $g^1, g^2, \ldots, g^N$. Thus, the total number of constraints in the GNEP (1.1)–(1.2) is $m = m_1 + m_2 + \cdots + m_N$.

From now on, unless otherwise mentioned, we will always assume that the following assumptions for the cost function θ_v, $v = 1, 2, \ldots, N$, and

1.2.1 Assumptions

We call a function $\psi : \mathbb{R}^n \to \mathbb{R}^m$ as a C^p-function if ψ is p times continuously differentiable. A C^p-function is said to be an LC^p-function if its p-th derivative is locally Lipschitz. Throughout the chapter, for each $v = 1, 2, \ldots, N$, we assume the following:

1. the pay-off function $\theta_v : \mathbb{R}^n \to \mathbb{R}$ is an LC^1-function, and
2. the constraint mapping $g^v : \mathbb{R}^n \to \mathbb{R}^{m_v}$ is an LC^2-function.
3. The strategy set X_v is nonempty, closed, and convex.

1.2.2 Karush-Kuhn-Tucker conditions for GNEP

Let $\overline{x} = \left(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N\right)^\top$ be a solution to GNEP (1.1)–(1.2). For the v^{th} player, if a suitable constraint qualification (for example, Mangasarian-Fromovitz, Slater, etc.) holds, then there exists a vector $\overline{\lambda}^v \in \mathbb{R}^{m_v}$ such that $(\overline{x}^v, \overline{\lambda}^v)$ is a solution to the following KKT system:

$$\left.\begin{aligned} \nabla_{x^v} L_v(x^v, \overline{\boldsymbol{x}}^{-v}, \lambda^v) = 0 \\ \lambda^v \perp -g^v(x^v, \overline{\boldsymbol{x}}^{-v}) \\ \lambda^v \geq 0,\ g^v(x^v, \overline{\boldsymbol{x}}^{-v}) \leq 0, \end{aligned}\right\} \tag{1.4}$$

where $L_v(x^v, \boldsymbol{x}^{-v}, \lambda^v) = \theta_v(x^v, \boldsymbol{x}^{-v}) + g^v(x^v, \boldsymbol{x}^{-v})^\top \lambda^v$ is the Lagrangian associated with the v^{th} player's optimization problem (1.1).

Assume that a suitable constraint qualification holds for all the players. Then, for every player $v = 1, 2, \ldots, N$, KKT system (1.4) has a solution $(\overline{x}^v, \overline{\lambda}^v)$. Concatenating these N KKT systems, if $\overline{x}$ is a solution of GNEP (1.1)–(1.2), then there exists a multiplier $\overline{\lambda} \in \mathbb{R}^m$ such that $(\overline{x}, \overline{\lambda})$ satisfies the system

$$\left.\begin{aligned} L(x, \lambda) &= 0 \\ \lambda &\perp -g(x) \\ \lambda \geq 0,\ g(x) &\leq 0, \end{aligned}\right\} \tag{1.5}$$

where

$$\lambda = \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \vdots \\ \lambda^N \end{pmatrix}, \quad g(x) = \begin{pmatrix} g^1(x^1, \boldsymbol{x}^{-1}) \\ g^2(x^2, \boldsymbol{x}^{-2}) \\ \vdots \\ g^N(x^N, \boldsymbol{x}^{-N}) \end{pmatrix}, \quad L(x, \lambda) = \begin{pmatrix} \nabla_{x^1} L_1(x^1, \boldsymbol{x}^{-1}, \lambda^1) \\ \nabla_{x^2} L_1(x^2, \boldsymbol{x}^{-2}, \lambda^2) \\ \vdots \\ \nabla_{x^N} L_N(x^N, \boldsymbol{x}^{-N}, \lambda^N) \end{pmatrix}$$

and $\lambda^v,\ g^v(x^v, \boldsymbol{x}^{-v}) \in \mathbb{R}^{m_v}$. Thus, in the context of a suitable constraint qualification, system (1.5) can be considered as a first order necessary optimality condition for GNEP (1.1)–(1.2). In addition, in the context of further convexity assumptions, the x-component of the solution to system (1.5) solves GNEP (1.1)–(1.2).

Definition 1.1 [1] *Consider a function $f_v : \mathbb{R}^n \to \mathbb{R}$ associated to the v^{th} player that depends on every players' variable. The function f_v is said to be a player convex function if the function $f_v(x^v, \boldsymbol{x}^{-v})$ is convex in x^v for every fixed $\boldsymbol{x}^{-v}$. If f_v is convex with respect to $x = (x^v, \boldsymbol{x}^{-v})$, then f_v is called a jointly convex function.*

Let us consider GNEP (1.1)–(1.2) whose feasible set is defined by (1.3). GNEP (1.1) is called player convex if the objective function θ_v and the constraint functions $g_i^v, i = 1, 2, \ldots, m_v$ are player convex, for every player $v = 1, 2, \ldots, N$, i.e., for a given $\overline{\boldsymbol{x}}^{-v}$, the minimization problem (1.1) of the v^{th} player is a convex optimization problem.

In the next section, we will consider both cases of GNEPs: Player convex GNEP and Jointly convex GNEP.

1.2.3 GNEP reformulations: Player convex GNEP

In player convex GNEP, we need the following theorem and definitions.

Theorem 1.1 [41] *If the GNEP is player convex, then for each solution* $(\overline{x}, \overline{\lambda})$ *to system* (1.5)*, the vector* $\bar{x}$ *is a generalized Nash equilibrium point.*

Definition 1.2 [42] *A function* $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ *is called a complementarity function if*

$$\phi(x, y) = 0 \quad \Leftrightarrow \quad (x, y) \geq 0,\ xy = 0. \tag{1.6}$$

With the help of a complementarity function $\phi : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, defined by

$$\phi(x, y) = \begin{pmatrix} \phi(x_1, y_1) \\ \phi(x_1, y_1) \\ \vdots \\ \phi(x_m, y_m) \end{pmatrix},$$

system (1.5) can be reformulated as

$$\begin{pmatrix} L(x, \lambda) \\ \phi(-g(x), \lambda) \end{pmatrix} = 0. \tag{1.7}$$

reformulated system (1.7) becomes

$$F(x, \lambda) = 0, \tag{1.8}$$

where

$$F(x, \lambda) = \begin{pmatrix} L(x, \lambda) \\ \phi(-g(x), \lambda) \end{pmatrix}. \tag{1.9}$$

1.2.4 GNEP reformulations: Jointly convex GNEP

In this section, we assume that the objective function $\theta_v(x^v, \boldsymbol{x}^{-v})$ of (1.1) is convex in x^v, and the set $X_v(\boldsymbol{x}^{-v})$ is closed and convex for every v, $v = 1, 2, \ldots, N$. Accumulating the strategy sets of all the players, we get the strategy set for the GNEP as

$$X \coloneqq \prod_{v=1}^{N} X_v(\boldsymbol{x}^{-v}).$$

Definition 1.3 [38] *A GNEP (1.1) with $\theta_v(x^v, \boldsymbol{x}^{-v})$ of (1.1) being convex in x^v is said to be a jointly convex GNEP if X is closed and convex, and*

$$X_v(\boldsymbol{x}^{-v}) = \{x^v \in \mathbb{R}^{n_v} : (x^v, \boldsymbol{x}^{-v}) \in X\} \tag{1.10}$$

for every $v = 1, 2, \ldots, N$.

From the definition (1.3) of the set $X_v(\boldsymbol{x}^{-v})$, it is easy to check that (1.10) is equivalent to the requirement that $g^1 = g^2 = \cdots = g^N \coloneqq g$. Here, $g(x)$ is componentwise convex with respect to all variables x. In this case, we have

$$X = \{x \in \mathbb{R}^n : g(x) \leq 0\}.$$

To characterize a generalized Nash equilibrium of the GNEP in Definition 1.3, we note that the strategy sets are defined by

$$X_v(\boldsymbol{x}^{-v}) = \{x \in \mathbb{R}^n : g(x^v, \boldsymbol{x}^{-v}) \leq 0\}, \tag{1.11}$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and convex in x. Then, the v^{th} player's problem is given by

$$\begin{cases} \min\limits_{x^v} \theta_v(x^v, \boldsymbol{x}^{-v}) \\ \text{subject to } g(x^v, \boldsymbol{x}^{-v}) \leq 0. \end{cases} \tag{1.12}$$

Assume that the KKT conditions are satisfied at every solution of (1.12), for every player $v = 1, 2, \ldots, N$. Let x^* be a solution of the game (1.12). Then, for each player v, a vector λ^v of multipliers exists such that

$$\left.\begin{aligned} \nabla_{x^v}\theta_v(x^*) + \nabla_{x^v} g(x^*)\lambda^v = 0 \\ 0 \leq \lambda^v \perp -g^v(x^*) \geq 0. \end{aligned}\right\}$$

In general, the multipliers λ^{v_1} of the player v_1 need not be equal to the multipliers λ^{v_2} of the player v_2, i.e., $\lambda^{v_1} \neq \lambda^{v_2}$ for $v_1 \neq v_2$. It is easy to see that at a generalized Nash equilibria of the GNEP in Definition 1.3, the multipliers are the same for all players, i.e., $\lambda^1 = \lambda^2 = \cdots = \lambda^N$ (see [43]).

Theorem 1.2 [44] *Consider a GNEP in Definition 1.3 with continuously differentiable functions θ_v and g_v for every $v = 1, 2, \ldots, N$.*

(i) Let $\overline{x}$ be a generalized Nash equilibrium at which all player's subproblems satisfy a constraint qualification. Then, there exists $\overline{\lambda}$ which together with $\overline{x}$ solves the system (1.5).

(ii) Assume that $(\overline{x}, \overline{\lambda})$ solves the system (1.5) and that $\theta_v(x^v, \boldsymbol{x}^{-v})$ of (1.1) is convex for all $v = 1, 2, \ldots, N$. Then, $\overline{x}$ is a generalized Nash equilibrium point.

For a jointly convex GNEP, we let the feasible set X have the following explicit expression

$$X = \{x \in \mathbb{R}^n : s(x) \leq 0,\ h^v(x^v) \leq 0,\ v = 1, 2, \ldots, N\}, \tag{1.13}$$

where $s : \mathbb{R}^n \to \mathbb{R}^{m_0}$ defines those constraints which are shared by all players and can depend on all variables. The total number of such constraints in the GNEP is m_0. These constraints are known as shared constraints. The function s is same for all players and assumed to be a componentwise convex function. Here, $h^v(x^v)$ is the constraint that depends only on the variables of Player v, and we assume that h^v is also componentwise convex function. The functions h^v and s are assumed to be continuously differentiable. Let the strategy set for the v^{th} player is

$$X_v(\boldsymbol{x}^{-v}) = \{x^v \in \mathbb{R}^{n_v} : s(x^v, \boldsymbol{x}^{-v}) \leq 0,\ h^v(x^v) \leq 0\}. \tag{1.14}$$

Then, the KKT conditions for the v^{th} player's optimization problem (1.12) are

$$\left.\begin{aligned} \nabla_{x^v}\theta_v(x^v, \overline{\boldsymbol{x}}^{-v}) + \nabla_{x^v} s(x^v, \overline{\boldsymbol{x}}^{-v})\lambda^v + \nabla_{x^v} h(x^v)\mu^v = 0 \\ 0 \leq \lambda^v \perp -s(x^v, \overline{\boldsymbol{x}}^{-v}) \geq 0 \\ 0 \leq \mu^v \perp -h^v(x^v) \geq 0 \end{aligned}\right\} \tag{1.15}$$

for some multiplier $\lambda \in \mathbb{R}^{m_0}$ and $\mu^v \in \mathbb{R}^{m_v}$.

Theorem 1.3 [38] *Consider a jointly convex GNEP with θ_v, g, and h^v are continuously differentiable. Then, $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a solution to the GNEP in Definition 1.3 if and only if $(\overline{x}, \overline{\lambda}, \overline{\mu})$ satisfies the KKT conditions (1.15) with $\lambda^1 = \lambda^2 = \cdots = \lambda^N := \lambda$.*

Thus, to find a generalized Nash equilibrium point of a jointly convex GNEP, we attempt to find a solution of (1.15) with $\lambda := \lambda^1 = \lambda^2 = \cdots = \lambda^N$.

With the help of a complementarity function ϕ as defined in (1.6), system (1.5) for the GNEP in Definition 1.3 can be reformulated into the following system

$$G(x,\lambda,\mu) = \begin{pmatrix} L(x,\lambda,\mu) \\ \phi(-s(x),\lambda) \\ \phi(-h^1(x^1),\mu^1) \\ \vdots \\ \phi(-h^v(x^v),\mu^v) \\ \vdots \\ \phi(-h^N(x^N),\mu^N) \end{pmatrix} = 0, \tag{1.16}$$

where $\mu = \begin{pmatrix} \mu^1 \\ \mu^2 \\ \vdots \\ \mu^N \end{pmatrix}$, $L(x,\lambda,\mu) = \begin{pmatrix} \nabla_{x^1} L_1(x^1, \boldsymbol{x}^{-1}, \lambda, \mu) \\ \nabla_{x^2} L_1(x^2, \boldsymbol{x}^{-2}, \lambda, \dots, \mu) \\ \vdots \\ \nabla_{x^N} L_N(x^N, \boldsymbol{x}^{-N}, \lambda, \mu) \end{pmatrix}$ and

$$L_v(x^v, x^{-v}, \lambda, \mu) = \theta_v(x^v, \overline{\boldsymbol{x}}^{-v}) + \lambda s(x^v, \overline{\boldsymbol{x}}^{-v}) + h(x^v)\mu^v, \quad v = 1, 2, \dots, N.$$

1.3 Motivation and objective of the thesis

The generalized Nash equilibrium problems are typically challenging to solve by Newtonian methods because the problems generally have locally nonunique solutions.

Facchinei et al. [1] analyzed GNEPs with shared constraints and proposed Newton-type methods– semi-smooth Newton methods and Levenberg-Marquardt method to solve them. The semi-smooth Newton method in [1] converges Q-quadratically, but they have a drawback: they do not converge globally. Solving a system of linear (or nonlinear) equations by the semi-smooth Newton method at each stage can be expensive if the number of unknowns is large and may not be justified when the initial guess is far from a solution. This motivates us to develop an improved BFGS method that consumes lesser computation costs (number of iterations and CPU time). Therefore, we aim to solve GNEPs using an improved BFGS method such that it converges globally. To minimize the computation costs, we use Armijo-type line search techniques, which are cost-effective compared to the Wolfe-type line search techniques. Therefore, we solve GNEPs by BFGS method using the two-line search techniques: Armijo-Goldstein and MWWP [45], and provide their numerical performances.

But the proposed improved BFGS method converges globally and has a superlinear rate of convergence. Therefore, we propose to solve the GNEP by a Newtonian method: the inexact Newton method. The inexact Newton method has a nice convergence property. Under some mild conditions, the inexact Newton method converges globally and Q-quadratically too. We provide the numerical performance of the inexact Newton method for solving GNEPs.

The main challenge in solving GNEPs is that the solution sets are mostly local and nonunique. Some reliable techniques have attractive global convergence properties as well, for example, the augmented Lagrangian-type method [46] and the interior-point-type scheme [47], but they are not locally fast convergent. To develop a method for GNEPs, which is both locally and globally convergent, Tong et al. [48] have proposed a monotone trust region method for constrained optimization problems, which is globally and quadratically convergent under the local error-bound assumption. Further, Galli et al. [49] modified this method using nonmonotone strategy and obtained a nice local convergence as well as global convergence under mild error-bound conditions. We develop this method using a new nonmonotone technique and adaptive trust region radius. Also, with some mild error bound constraints [50], we use a smooth GNEP reformulation, and using the proposed method, we solve a dataset of 35 different GNEPs.

Further, we try to solve an interval-valued GNEP, and for that, we need an optimality condition in interval optimization problems. Therefore, we propose an extended KKT condition to characterize efficient solutions to constrained interval optimization problems. Also, we extend Gordan's theorems of the alternative for the existence of a solution to a system of interval linear inequalities. Using Gordan's theorem, we extend Fritz John condition.

In this thesis, we develop three optimization methods to solve GNEPs considering both cases of GNEPs: the player convex case and the joint convex case. In the next section, we have given a literature review on GNEPs.

1.4 Organization of the thesis

The thesis is composed of six chapters, including an introductory chapter and a chapter containing a conclusion and future scopes. In the introductory chapter, we have introduced the GNEP and provided an adequate literature review on the GNEPs. The outline of the thesis is as follows.

In Chapter 2, we propose to solve GNEPs by a quasi-Newton method. Therefore, we consider the smooth version of GNEP reformulation with the help of the Fischer-Burmeister function and solve the GNEP using a globally convergent improved BFGS method. In Chapter 2, we solve GNEP by two algorithms: improved BFGS method with Armijo-Goldstein line search technique and improved BFGS method with MWWP line search technique. We provide the convergence analysis for both algorithms. In the numerical part of Chapter 2, we compare the numerical performances of both algorithms.

In Chapter 3, we propose to solve GNEPs by an inexact Newton method. In this chapter, we consider the two GNEPs: player convex GNEP and jointly convex GNEP. Also, in both GNEP reformulations, we use a semismooth complementarity function, and therefore the reformulated system is nonsmooth. It is shown that the proposed numerical scheme has the global convergence property for both types of GNEPs. It is observed that the strongly BD-regularity assumption for the reformulated system of GNEP plays a crucial role in global convergence. In fact, the strongly BD-regularity assumption and a suitable choice of a forcing sequence expedite the convergence behavior of the inexact Newton method for GNEPs to Q-quadratic convergence. The performance of the proposed numerical scheme is shown for a collection of problems, including the internet switching problem, where the traffic is generated by selfish users. A comparison of the proposed method with the existing semismooth Newton method II for GNEP indicates that the proposed scheme is more efficient.

In Chapter 4, we propose an improved nonmonotone adaptive trust region (INATR) method to solve constrained nonlinear system of equations and provide its application to solve GNEPs. Also, we provide its numerical performances. The INATR method maintains the local convergence properties of its nonmonotone counterpart, and also it is proven that the proposed INATR method has global convergence properties. The numerical results indicate that the INATR method performs better compared to the nonmonotone trust region method.

Chapter 5 presents an extended KKT condition to characterize efficient solutions to constrained interval optimization problems. The theory n this chapter has been developed on the fact that at an optimal solution, the cone of feasible directions and the set of descent directions have an empty intersection. Using this developed theory, a set of first-order optimality conditions has been derived to solve unconstrained optimization problems. Further, Gordan's theorems for the existence of a solution to a system of interval linear inequalities have been extended. Moreover, with the help of Gordan's theorem, we proposed Fritz John and KKT necessary optimality conditions for constrained interval optimization problems. It is also observed that these optimality conditions appear with inclusion relations instead of equations. Lastly, we apply the derived KKT condition to the binary classification problem with interval-valued data using support vector machines.

Finally, Chapter 6 summarizes the main conclusions and forecasts potential directions for future research.

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