Optimization Methods to Solve Generalized Nash Equilibrium Problems

Thesis submitted in partial fulfillment

for the Award of Degree

Doctor of Philosophy

by

Abhishek Singh

DEPARTMENT OF MATHEMATICAL SCIENCES

INDIAN INSTITUTE OF TECHNOLOGY

(BANARAS HINDU UNIVERSITY)

VARANASI - 221005

Roll No. 16121010 *Year 2022*

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It is certified that the work contained in the thesis titled ***"Optimization Methods to Solve Generalized Nash Equilibrium Problems"*** by ***Abhishek Singh*** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all requirements of Comprehensive Examination, Candidacy, and SOTA for the award of Ph.D. Degree.

Dr. Debdas Ghosh

(Supervisor)
Assistant Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

Dr. Anuradha Banerjee

(Co-Supervisor)
Assistant Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

DECLARATION BY THE CANDIDATE

I, ***Abhishek Singh***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Dr. Debdas Ghosh*** and co-supervison of ***Dr. Anuradha Banerjee*** from ***July, 2016*** to ***December 2022*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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Place: Varanasi

Abhishek Singh

(Abhishek Singh)

CERTIFICATE BY THE SUPERVISOR

It is certified that the above statement made by the candidate is correct to the best of our knowledge.

Dr. Debdas Ghosh

(Supervisor)
Assistant Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

Dr. Anuradha Banerjee

(Co-Supervisor)
Assistant Professor
Department of Mathematical Sciences
Indian Institute of Technology
(Banaras Hindu University)
Varanasi-221005

30.12.2022

Signature of the Head of the Department

Department of Mathematical Sciences
Indian Institute of Technology (BHU), Varanasi-221005

विभागाध्यक्ष /HEAD
गणितीय विज्ञान विभाग
Department of Mathematical Sciences
भारतीय प्रौद्योगिकी संस्थान
Indian Institute of Technology
(काशी हिन्दू विश्वविद्यालय)
(Banaras Hindu University)
वाराणसी /Varanasi-221005

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Dedicated to my parents

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Place: Varanasi

Abhishek Singh
(Abhishek Singh)

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List of Symbols

Symbol	Description
$\mathbb{R}^n$	Euclidean space of dimension n
$\mathcal{X}_{\sqsubseteq}$	Feasible set for the v^{th}
$I_{n\times n}$	Identity matrix of dimension n

Abbreviations

Abbreviation	**Description**
GNEP	Generalized Nash Equilibrium Problem
BFGS	Broyden–Fletcher–Goldfarb–Shanno
VI	Variational inequality
WWP	Weak Wolfe Powell
MWWP	Modified Weak Wolfe Powell
IPM	Interior Point Method
TR	Trust Region
MTR	Monotone Trust Region
NTR	Nonmonotone Trust Region
INATR	Improved Nonmonotone Adaptive Trust Region
KKT	Karush-Kuhn-Tucker
SVM	Support Vector Machine

PREFACE

Optimization is not only important in its own right, but also integral to many applied sciences such as operations research, management sciences, economics and finance, and all branches of math-oriented engineering. It provides a unique insight into any situation. Optimization introduces a computational situation where the goal is to obtain the best of all possible circumstances. Also, optimization techniques assist us in finding the best under prespecified circumstances. Essentially, the optimization is used to detect, characterize and compute the maxima or minima of a function for a set of acceptable points and certain predefined conditions.

The mode of optimization is not just confined to the mathematical arena. Optimization methods can be applied in many spheres of study to find solutions that maximize or minimize some study parameters, such as in producing a good or service, minimizing the cost of production, and maximizing profits. Such instances often have special structures: convex, nonconvex, linear, nonlinear, quadratic, semidefinite, dynamic, integer, stochastic programming, etc. Optimization is the source of vast theoretical foundations and advanced algorithms. Mathematically, identification of the solution is the essence of optimization, i.e., The minimization or maximization of a function or a set of functions in conjunction with a set of constraints, regardless of the number of decision variables.

This thesis describes a study of various optimization methods to solve GNEPs. Also in Chapter 5, we have proposed an extended KKT condition, so that in the future we can solve an interval-valued GNEPs This thesis is organized as follows.

Chapter 1 begins with the introduction to GNEP. Then, a literature review of GNEPs will be provided. After that, we derive KKT conditions for GNEP. Further, a discussion on GNEP reformations will be provided.

In Chapter 2, a study of a globally convergent improved BFGS method to solve GNEP is discussed. The author has solved GNEP by using two versions of the improved BFGS method by considering two inexact line search techniques: the Armijo-Goldstein line search technique and the MWWP line search technique. The algorithmic implementation and convergence analysis of the method with an estimate of the number of iterations to reach an ϵ-precise solution is provided. A performance comparison between the proposed methods is provided on some numerical problems.

Chapter 3 analyzes two types of GNEPs: player convex GNEP and jointly convex GNEP by considering the nonsmooth GNEP reformulation. The author solves both types of GNEPs by an inexact Newton method. The global convergence of the proposed method is also shown under some mild conditions. The numerical performances of the inexact Newtone method have been provided.

In Chapter 4, the author proposes an improved nonmonotone adaptive trust region (INATR) method to solve a constrained nonlinear system of equations, and uses this to solve GNEPs. The proposed optimization method converges globally. The numerical performance of INATR method with nonmonotone trust region (NTR) method and monotone trust region (MTR) method has been provided.

In Chapter 5, the author proposes extended first Gordan's theorem, extended second Gordan's theorem, and extended Fritz John condition for interval optimization problems. After that, the author proposes extended Karush-Kuhn-Tucker conditions for interval optimization problems and discusses its application to Support Vector Machines (SVM).

Finally, in Chapter 6, we conclude the thesis with some suggestions for future work.