Chapter 6

Conclusion and future work

These chapters provide a compilation of the main conclusions that can be derived from this thesis. It also presents some future directions for the research work presented in this thesis.

6.1 General conclusions

In this thesis, the following contributions are made.

- A three-variable reference function, representation of space fuzzy point by a reference function, fuzzy number along a direction, the addition operation of two space fuzzy points, a general expression of same and inverse points by a reference function, scalar multiplication of space fuzzy point, and a linear combination of two space fuzzy points are introduced. The fuzzy distance between two space fuzzy points and the space fuzzy line segments are also evaluated. The proposed results are compared with existing ones.
- Rigorous formulation of space fuzzy line passing through two continuous fuzzy points and a symmetric fuzzy line have been studied. The formulation of skew fuzzy lines and the shortest distance between skew fuzzy lines are also provided. In addition, we define three different forms of fuzzy planes— a three-point form, an intercept form, and a fuzzy plane passing through an S-type

space fuzzy point and perpendicular to a given crisp direction. Constructions of all the different forms of fuzzy planes supported with step-wise algorithms are given. Our results are discussed and compared with the existing methods.

- The construction of all the different forms of fuzzy spheres and their intersection by a crisp plane have been studied. The formulation of a fuzzy cone and its intersection by a crisp plane is also explained. This thesis includes discussing and comparing the proposed fuzzy sphere and fuzzy cone with existing ideas.

- The fuzzy Hough transform technique to detect fuzzy lines and fuzzy circles has been delineated. A brief study on the generalized version of the fuzzy Hough transform is also described. This thesis shows the effective implementation of the proposed method to detect fuzzy lines and circles in authentic images.

6.2 Contributions of the thesis

This thesis appertains to the study of analytical fuzzy space geometry and its applications in fuzzy image processing. The following highlights the chapter-wise contributions to this dissertation.

Chapter 2 formulates a basic idea of space fuzzy point with the help of a three-variable reference function, namely, S-type space fuzzy point. We also have proposed the idea of the same and inverse points with respect to continuous S-type space fuzzy points. With the help of these concepts, analysis of the fuzzy distance between two S-type space fuzzy points, convex combinations of S-type space fuzzy points, the coincidence of two S-type space fuzzy points, and space fuzzy line segment have been performed. Importantly, since the proposed ideas depend on a unified representation of space fuzzy points by three-variable reference functions, just by extending the number of variables, all the three-dimensional fuzzy geometrical concepts can find the corresponding concepts in n-dimensional Euclidean space.

With the help of S-type representation of fuzzy points, the general expressions of the same and inverse points have been given that are used throughout the paper. Future studies can find the general explicit expressions of the same and inverse points for two fuzzy points in $\mathbb{R}^n$. Towards this, one needs an S-type representation of a fuzzy point in $\mathbb{R}^n$, which can be found by replacing $\mathbb{S}^2$ by $\mathbb{S}^{n-1}$ in the definition of reference function given in Section 2.3.1. One can also try to develop the idea of the same and inverse points for discontinuous fuzzy points.

Chapter 3 discusses space fuzzy lines and three different forms of a fuzzy plane in $\mathbb{R}^3$. Notably, new ideas of the skew fuzzy lines and the shortest distance between skew fuzzy lines are presented. Moreover, the three different forms of fuzzy planes, namely, three-point form, intercept form, and a fuzzy plane passing through an S-type space fuzzy point and perpendicular to a given crisp direction, have been proposed. We have also developed the algorithms for finding the membership values of all the proposed formulations of the space fuzzy lines and the fuzzy planes and added suitable numerical examples. The geometric properties of all these proposed forms of fuzzy planes and their interrelations are also investigated. The intersection of two space fuzzy lines may not be a space fuzzy point as the α-cuts of space fuzzy line may not be convex (see Figure 3.2). However, the intersection of two symmetric fuzzy lines is a fuzzy point (see Observation 3.2.2). Also, the interrelations between the three-point form and intercept form are found to be equivalent. Proposition 3.4.1 shows that the membership value of a point $P \in \widetilde{\Pi}_{Pn}(0)$ is equal to the membership grade of the closest point on $\widetilde{P}(0) \cap \Pi$ from the core point of $\widetilde{P}$, where Π is the plane passing through P and perpendicular to the given crisp direction.

We have reported that the support of the proposed fuzzy plane must be bounded by two crisp planes that lie on either side of the core plane. Accordingly, we have formulated a fuzzy plane keeping in mind that the support may not suddenly widen as considered in [7], and hence the intersection of fuzzy planes can be viewed as a space fuzzy line. Furthermore, an Observation 3.4.1 is added to show that the intersection of two fuzzy planes represents a space fuzzy line. In a sequel, we have defined the notion of the angle between two fuzzy planes. Theorem 3.4.5 shows that the angle between two fuzzy planes is a fuzzy number. Also, we have defined the distance between an S-type space fuzzy point and a fuzzy plane, which is a fuzzy number (see Theorem 3.4.6). Some proposed ideas are the extension of the concepts investigated by Ghosh and Chakraborty in a two-dimensional plane [1, 2]. The proposed formulations and algorithms of space fuzzy lines and all the forms of fuzzy planes facilitate the evaluation of the membership values compared to that of [7]. It is noted that the Algorithms 3.3.1 and 3.3.2 are used for symmetric skew fuzzy lines to evaluate the fuzzy shortest distances and to find the membership grade of a point in the fuzzy shortest distance, respectively. A brief analysis of the shortest distance between two non-symmetric skew fuzzy lines has been given in this paper. However, a detailed study of the shortest distance between two non-symmetric skew fuzzy lines will be discussed in the future.

Chapter 4 provides three different methods to construct the fuzzy sphere. Method 1 deals with the formulation of the fuzzy sphere when the center and radius of a sphere are given imprecisely. This formulation of the fuzzy sphere is based on the study that there is a fuzzy number at a predetermined distance from a given fuzzy number. Method 2 depends on the diameter of a fuzzy sphere. The diameter of the fuzzy sphere is the fuzzy line segment joining two continuous fuzzy points. We have given two methodologies to define the diameter form of a fuzzy sphere. At first, the construction of the fuzzy sphere depends on the translation of fuzzy points along the perpendicular directions passing through the core points of the fuzzy points. Second, we have extended the conventional definition of the classical diameter form of a sphere. Method 3 defines the fuzzy sphere passing through the four S-type space fuzzy points whose core points are not co-planar. Importantly, we have proved that there is a unique sphere passing through the four S-type space fuzzy points. In addition, we have investigated that the intersection of a fuzzy sphere with a crisp plane is a fuzzy circle. The idea of finding a center and radius of the fuzzy circle is based on the study of translation and rotation of space fuzzy points. An interesting concept of a great fuzzy circle and its rotation are also included in this study. An extensive idea of a fuzzy cone is presented thereafter. Relevantly, we have delineated the intersection of a fuzzy cone with a crisp plane. Theorem 4.3.1 gives the necessary and sufficient conditions to form a convex fuzzy cone. One can easily deploy the proposed Algorithms 4.2.1 and 4.2.2 to get the membership value of a point in the proposed Definitions 4.2.6 and 4.2.7 of the fuzzy spheres, respectively. Sequentially, the properties of the fuzzy sphere and the fuzzy cone are also discussed. One of the properties is the degree of intersection of two fuzzy spheres, which is just an extension of the degree of intersection of two fuzzy circles (see [3]) from two variables to three variables. Such that the degree of intersection of two fuzzy spheres $\widetilde{S}_1$ and $\widetilde{S}_2$, (ζ) say, can be defined as

$$\zeta = \begin{cases} 0 & \text{if } \widetilde{S}_1(1) \bigcap \widetilde{S}_2(1) = \emptyset \\ 1 & \text{if } \widetilde{S}_1 = \widetilde{S}_2 \\ 1 - \sup\limits_{s \in \mathbb{R}^3} \left| \mu\left(s \middle| \widetilde{S}_1\right) - \mu\left(s \middle| \widetilde{S}_2\right) \right| & \text{if } \widetilde{S}_1(1) \bigcap \widetilde{S}_2(1) \neq \emptyset \text{ but } \widetilde{S}_1 \neq \widetilde{S}_2. \end{cases}$$

Chapter 5 presents the study of the application of fuzzy plane geometrical elements like fuzzy lines and fuzzy circles. We have introduced a technique, say fuzzy Hough transform (FHT), for detecting fuzzy lines and fuzzy circles in the image space. This technique aims to find imprecise objects within a certain class of imprecise shapes by a voting procedure. The fuzzy Hough transform maps the α-cuts of fuzzy shapes (fuzzy lines or circles) in (x, y)-space to a fuzzy number in parameter space. Also, there is a one-to-one correspondence between the α-cuts of the fuzzy line to the α-cuts of a fuzzy number in parameter space for $\alpha \in [0, 1]$. A set of fuzzy points lying on the fuzzy circles form the corresponding fuzzy circles, which must intersect in a fuzzy number in the parameter space. The accumulator cells corresponding to those parameters, which are the points of the support of the fuzzy number in the parameter space, have the maximum number of voting. These accumulator cells give the α-cuts of the fuzzy shapes (fuzzy lines or circles) in image space. In a similar manner, the fuzzy Hough transform can be applied to detect generalized fuzzy geometrical curves (fuzzy triangles and fuzzy conics) by extending the dimension of the accumulator array. A brief study on the generalized version of the fuzzy Hough transform is also described. Sequentially, an idea of similarity measure between two fuzzy shapes is described.

One of the basic cons of the FHT is that it is not applicable for the detection of non-symmetric fuzzy lines since a point in the accumulator cell with a maximum number of elements represents an infinite line in the (x, y)-space. Hence FHT is valid only for detecting symmetric fuzzy lines. For the implementation of the proposed FHT and similarity measure, we have taken symmetric fuzzy lines (see Definition 5.2.1).

6.3 Future work

In light of the work presented in this thesis, several scopes for further development are evident. Following are some potential areas for future research.

- The development of tools and techniques in optimization reveals that the interlink between geometry and calculus plays an important role in optimization theory. In the future, we will use the proposed concepts of fuzzy geometry to analyze fuzzy optimization problems [119]. One can also use a fuzzy geometrical approach to develop fuzzy calculus and its applications. The efficiency of the theory of the same and inverse points for fuzzy numbers is shown in [87]. In the near future, we will show the efficiency of the formulated the same and inverse points for space fuzzy points in fuzzy optimization.

- Our next research on fuzzy space geometry will be continued with a detailed analysis of fuzzy paraboloids, fuzzy hyperboloids, fuzzy ellipsoids, and their properties. All the ideas can be extended to an n-dimensional space, $n \geq 4$. Future work can focus on such an extension.

- In the future, we may also focus on a detailed analysis of the properties of the fuzzy conic sections (fuzzy parabola, fuzzy ellipse, and fuzzy hyperbola). The proposed fuzzy space geometrical concepts can be used to analyze fuzzy optimization problems. We will work on this in the future.

- As discussed in the Subsection 1.4.3, there are several dimensions where the proposed fuzzy geometry may be applied, such as fuzzy medical imaging, fuzzy geometric object detection, fuzzy extrapolation, interpolation, etc. In medical imaging, it is seen that the aura boundaries of the cells, organs, etc., are hazy in nature. So along with digital geometry, fuzzy geometry may add a new dimension to the study of medical image analysis. In this line, we will develop the fuzzy Hough transform technique to detect the space fuzzy lines, fuzzy spheres, and fuzzy cones. Also, future research will be based on the detection of generalized fuzzy curves (fuzzy triangles, fuzzy conics, etc.). In addition, we will apply this technique of fuzzy Hough transform to medical image analysis.

- Further research can focus on the differentiability and integrability of fuzzy-valued functions in the fuzzy geometrical sense and their applications. There are numerous applications of fuzzy space geometry in fuzzy linear programming, medical imaging, fuzzy geometric object detection, fuzzy extrapolation, interpolation, etc (see Subsection 1.4.3). Further work on this line may be pursued in the application areas of fuzzy space geometry, which will explicitly explore the excellence of the suggested ideas on fuzzy space geometry when compared to existing literature.

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