Chapter 4

Analytical fuzzy space geometry III

4.1 Introduction

A thorough study of fuzzy space geometry with the introduction of related basic concepts has been provided by Ghosh et al. in [117]. In continuation of the study of fuzzy space geometry, we have initiated the construction of fuzzy spheres and fuzzy cones in this paper. Detailed literature on fuzzy geometry has been delineated in [117].

4.1.1 Motivation and novelty

As described in the Subsection 1.4.3, fuzzy geometry has been successfully applied to many areas, such as fuzzy optimization, fuzzy medical imaging, fuzzy geometrical object detection, fuzzy extrapolation or interpolation, etc.

In [117, 3, 1, 2], it is observed that the same and inverse points are indispensable tool to develop fuzzy geometry. With the help of these tools, in [117], we have developed fuzzy distance between two space fuzzy points and space fuzzy line segments. It is perceived in [117] that there is a need to develop fuzzy space geometrical elements, and these fuzzy elements can be successfully applied in many realistic fields. Therefore, in continuation of our study in fuzzy space geometry, in this paper, we have investigated the fuzzy sphere and the fuzzy cone.

The paper has following *novelties*.

This paper has dealt with the construction of fuzzy spheres and fuzzy cones. Mainly, three different forms of a fuzzy sphere and a fuzzy cone are presented in this study. Proposed analysis in this study are as follows:

(i) We give three different methodologies to formulate fuzzy spheres depending on the information available for the fuzzy sphere, such as a space fuzzy point and a fuzzy distance or a diameter of the fuzzy sphere or four space fuzzy points.

(ii) With the help of Theorem 4.2.1, we construct a fuzzy sphere as a collection of space fuzzy points that are at a predetermined fuzzy distance from a given fuzzy point.

(iii) We define the notions of translation and rotation of a space fuzzy point. With the help of these notions, we have constructed the diameter form of a fuzzy sphere and a fuzzy cone.

(iv) We give two methods to construct the diameter form of a fuzzy sphere. One is based on the translation of space fuzzy points. The other one is the extension of the classical definition of the diameter form to the fuzzy environment.

(v) We discuss a fuzzy sphere passing through four space fuzzy points whose core points are not co-planar. A detailed study on the intersection of the fuzzy sphere with a crisp plane has been conducted.

(vi) This study incorporates the concept of a great fuzzy circle and its rotation. We show that the rotation of a great fuzzy circle about its diameter is a fuzzy sphere.

(vii) The notions of a fuzzy cone, convex fuzzy cone, and its intersection with a crisp plane are initiated here.

(viii) Also, an idea of degenerated and non-degenerated fuzzy conic sections is explored. The types of fuzzy conics depend on how a crisp plane intersects a fuzzy cone.

(ix) We discuss the construction of the membership functions of the fuzzy conics and their classification as a fuzzy parabola, fuzzy ellipse, and fuzzy hyperbola.

The following section discusses three different forms of fuzzy spheres and their properties. The formulation of these forms of fuzzy spheres depends on the same and inverse points.

4.2 Fuzzy sphere

This section explores the fuzzy spheres' mathematical formulations when center and radius are known imprecisely. In classical geometry, a sphere is a collection of equidistant (radius) points from a fixed point (center). Analogously, we have investigated that a fuzzy sphere is a collection of fuzzy points equidistant (fuzzy radius) from a fixed fuzzy point (fuzzy center).

To define a fuzzy sphere, first, we focus on whether a fuzzy point exists at a pre-decided fuzzy distance from a given fuzzy point, as in classical geometry.

The following theorem exhibits the condition based on which one can get a fuzzy point at a pre-decided fuzzy distance from a given fuzzy point.

Theorem 4.2.1. Let $\widetilde{P}_1(a_1, b_1, c_1)$ be a fuzzy point and $\widetilde{d} = (d - \beta/d/d + \gamma)_{LR}$ be an LR-type fuzzy number. Then, a fuzzy point $\widetilde{P}_2(a_2, b_2, c_2)$ exists such that $\widetilde{D}(\widetilde{P}_1, \widetilde{P}_2) = \widetilde{d}$ or $\widetilde{D}(\widetilde{P}_2, \widetilde{P}_1) = \widetilde{d}$ if and only if

$$\beta L^{-1}(\alpha), \gamma R^{-1}(\alpha) \geq \phi_1^{-1}(\alpha) \text{ for all } \alpha \in [0, 1].$$

Proof. Similar to Theorem 2.1 in [3]. □

Now, we give a numerical example that illustrates Theorem 4.2.1. The condition of Theorem 4.2.1 gives a fuzzy point at a pre-decided fuzzy distance from a given fuzzy point.

Example 4.2.1. Let $\widetilde{P}_1(1, 0, -1)$ be an S-type space fuzzy point with the membership function

$$\mu\left((x, y, z)\middle|\widetilde{P}_1(1, 0, -1)\right) = \begin{cases} 1 - \sqrt{(x-1)^2 + y^2 + (z+1)^2} & \text{if } (x-1)^2 + y^2 + (z+1)^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{d} = (5.1140/7.8102/10.5101)_{LR}$ be an LR-type fuzzy number. As per the notations of Theorem 4.2.1, $L(x) = R(x) = \max\{0, 1 - x\}$, $\beta = 2.6962$, $\gamma = 2.6999$. Also,

$$\phi_1(\lambda) = 1 - \sqrt{(\lambda \sin\varphi \cos\theta)^2 + (\lambda \sin\varphi \sin\theta)^2 + (\lambda \cos\varphi)^2} = \alpha \tag{4.1}$$

is the membership function of the fuzzy number along the line

$$L : \tfrac{x-1}{\sin\varphi\cos\theta} = \tfrac{y}{\sin\varphi\sin\theta} = \tfrac{z+1}{\cos\varphi}$$

on the support of $\widetilde{P}_1$. By (4.1), $\phi_1(\lambda) = 1-\lambda = \alpha$, i.e., $\phi_1^{-1}(\alpha) = \lambda = 1-\alpha$. Note that $2.6962(1-\alpha), 2.6999(1-\alpha) \geq (1-\alpha)$ for all $\alpha \in [0, 1]$. This satisfies the restriction of the Theorem 4.2.1 for the existence of the fuzzy point at a predetermined fuzzy distance from a given fuzzy point. According to Theorem 4.2.1, we get

$$\phi_2^{-1}(\alpha) = 2.6962(1-\alpha) - (1-\alpha),$$

i.e.,

$$\phi_2^{-1}(\alpha) = 1.6962(1-\alpha)$$

and the core point of $\widetilde{P}_2$ is

$$(a_2, b_2, c_2) = (1 + 7.8102 \sin\varphi\cos\theta, 7.8102 \sin\varphi\sin\theta, -1 + 7.8102\cos\varphi)$$

along the line L. Now,

$$\widetilde{P}_2(\alpha) = (a_2 + 1.6962(1-\alpha)\sin\varphi\cos\theta, b_2 + 1.6962(1-\alpha)\sin\varphi\sin\theta, c_2 + 1.6962(1-\alpha)\cos\varphi),$$

for some $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$ and $\alpha \in [0, 1]$.
The representation of the membership function of $\widetilde{P}_2(a_2, b_2, c_2)$ is

$$\mu\left((x,y,z)\middle|\widetilde{P}_2(a_2,b_2,c_2)\right) = \begin{cases} 1 - \frac{1}{1.6962}\sqrt{(x-a_2)^2 + (y-b_2)^2 + (z-c_2)^2} & \text{if } (x-a_2)^2 + (y-b_2)^2 + (z-c_2)^2 \leq 2.8770 \text{ and} \\ & \frac{x-a_2}{\sin\varphi\cos\theta} = \frac{y-b_2}{\sin\varphi\sin\theta} = \frac{z-c_2}{\cos\varphi} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to note that $\widetilde{P}_2$ is monotonically non-increasing along the ray

$$\frac{x-a_2}{\sin\varphi\cos\theta} = \frac{y-b_2}{\sin\varphi\sin\theta} = \frac{z-c_2}{\cos\varphi}.$$

In Method 1, we formulate a fuzzy sphere based on the classical definition of a sphere: the loci of a moving point in the space, whose distance from fixed point is constant. The fixed point and constant distance are called the center and radius of the sphere, respectively.

We propose a fuzzy sphere as a set of imprecise locations at a pre-decided imprecise distance from a fixed imprecise location. Here, the imprecise location and the imprecise distance are expressed by a fuzzy point and a fuzzy number, respectively.

Definition 4.2.1. (*Fuzzy sphere* ($\widetilde{S}_1$)). Let $\widetilde{P}_1(a_1, b_1, c_1)$ be a fuzzy point and $\widetilde{R}$ be a fuzzy number. Consider a line passing through the (a_1, b_1, c_1), i.e.,

$$L : \tfrac{x-a_1}{\sin\varphi\cos\theta} = \tfrac{y-b_1}{\sin\varphi\sin\theta} = \tfrac{z-c_1}{\cos\varphi} = \lambda.$$

Let $\widetilde{\phi_1^{\theta\varphi}}$ be a fuzzy number with the membership function

$$\phi_1^{\theta\varphi}(\lambda) = f_1\left(\lambda\sin\varphi\cos\theta, \lambda\sin\varphi\sin\theta, \lambda\cos\varphi\right)$$

along the line L on the support of $\widetilde{P}_1(a_1, b_1, c_1)$.

A fuzzy sphere, say $\widetilde{S}_1$, with center $\widetilde{P}_1(a_1, b_1, c_1)$ and radius $\widetilde{R}$ is formulated as a cluster of fuzzy numbers that are at a fuzzy distance $\widetilde{R}$ from the fuzzy number $\widetilde{\phi_1^{\theta\varphi}}$ along L, for $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$. More precisely, the fuzzy sphere $\widetilde{S}_1$ with center $\widetilde{P}_1(a_1, b_1, c_1)$ and radius $\widetilde{R}$ is evaluated as

$$\widetilde{S}_1 = \bigvee_{\substack{\theta\in[0,2\pi]\\ \varphi\in[0,\pi]}} \{\widetilde{\phi_2^{\theta\varphi}} : \widetilde{D}\left(\widetilde{\phi_1^{\theta\varphi}}, \widetilde{\phi_2^{\theta\varphi}}\right) = \widetilde{R}, \text{ where } \widetilde{\phi_1^{\theta\varphi}} \text{ and } \widetilde{\phi_2^{\theta\varphi}}$$
$$\text{are fuzzy numbers along the line } L\}. \tag{4.2}$$

Note 10. In the next theorem, by $\widetilde{0}$, we mean a fuzzy subset of $\mathbb{R}$ with the following properties:

(i) $\mu(x|\widetilde{0}) = 1$, for $x = 0$.

(ii) $\mu(x|\widetilde{0}) = 0$, for $x < 0$.

(iii) $\mu(x|\widetilde{0})$ is decreasing for $x \in (0, r)$ for some $r > 0$ and $\mu(x|\widetilde{0}) = 0$, for $x \geq r$.

A fuzzy number $\widetilde{R}$ is called positive ($\widetilde{R} > 0$) if its membership function $\mu(x|\widetilde{R}) = 0$, for $x \leq 0$. A fuzzy number $\widetilde{R}$ is called negative ($\widetilde{R} < 0$) if its membership function $\mu(x|\widetilde{R}) = 0$, for $x \geq 0$.

The following theorem explains the representation of the equation (4.2) according as $\widetilde{R} > 0$, $\widetilde{R} = \widetilde{0}$, or $\widetilde{R} < 0$, respectively.

Theorem 4.2.2. The equation (4.2) represents a fuzzy sphere, a fuzzy point, or no fuzzy point according as $\widetilde{R} > 0$, $\widetilde{R} = \widetilde{0}$, or $\widetilde{R} < 0$, respectively.

Proof. By the Definition 4.2.1, it is clear that $\widetilde{P}_1(a_1, b_1, c_1)$ is a fuzzy point and $\widetilde{R}$ is a fuzzy number in the equation (4.2).
For $\widetilde{R} > 0$, the equation (4.2) represents a fuzzy sphere since $\widetilde{D}\left(\widetilde{\phi_1^{\theta\varphi}}, \widetilde{\phi_2^{\theta\varphi}}\right) = \widetilde{R} > 0$ which ensures that the core of the fuzzy sphere is a crisp sphere not a crisp point. In this case, the center and radius of the fuzzy sphere are the fuzzy point $\widetilde{P}_1(a_1, b_1, c_1)$ and the fuzzy number $\widetilde{R} > 0$, respectively, in (4.2).
For $\widetilde{R} = \widetilde{0}$, the equation (4.2) reduces to

$$\bigvee_{\substack{\theta\in[0,2\pi]\\ \varphi\in[0,\pi]}} \{\widetilde{\phi_2^{\theta\varphi}} : \widetilde{D}\left(\widetilde{\phi_1^{\theta\varphi}}, \widetilde{\phi_2^{\theta\varphi}}\right) = \widetilde{0}\},$$

which ensures that the core of the fuzzy sphere is a crisp point. The fuzzy sphere $\widetilde{S}_1$ reduces to a fuzzy point $\widetilde{P}$ which can be evaluated by

$$\widetilde{P} = \bigvee_{\substack{\theta\in[0,2\pi]\\ \varphi\in[0,\pi]}} \{\widetilde{\phi_2^{\theta\varphi}} : \widetilde{D}\left(\widetilde{\phi_1^{\theta\varphi}}, \widetilde{\phi_2^{\theta\varphi}}\right) = \widetilde{0}\}.$$

Now, Definition 4.1 (in [117]) clearly shows that the fuzzy distance $\widetilde{D}\left(\widetilde{\phi_1^{\theta\varphi}}, \widetilde{\phi_2^{\theta\varphi}}\right) = \widetilde{R} < 0$ is not a possible case since the distance between two points can not be negative. This completes the proof. □

Example 4.2.2. (*Fuzzy sphere* $(\widetilde{S}_1)$). Let us consider a fixed fuzzy point $\widetilde{P}_1(1,0,-1)$ and a fixed fuzzy number $\widetilde{d}$ as in Example 4.2.1. Let $\widetilde{\phi_1^{\theta\varphi}}$ be a fuzzy number along the line

$$L : \tfrac{x-1}{\sin\varphi\cos\theta} = \tfrac{y}{\sin\varphi\sin\theta} = \tfrac{z+1}{\cos\varphi}$$

on the support of $\widetilde{P}_1$. Here, the core point (a_2, b_2, c_2) of $\widetilde{\phi_2^{\theta\varphi}}$ is

$$(1 + 7.8102\sin\varphi\cos\theta, 7.8102\sin\varphi\sin\theta, -1 + 7.8102\cos\varphi).$$

The union of all possible such fuzzy numbers

$$\widetilde{\phi_2^{\theta\varphi}}(\alpha) = (a_2 + 1.6962(1-\alpha)\sin\varphi\cos\theta, b_2 + 1.6962(1-\alpha)\sin\varphi\sin\theta, c_2 + 1.6962(1-\alpha)\cos\varphi)\,,$$

for all $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$ and $\alpha \in [0, 1]$ along the line L, forms the fuzzy sphere $\widetilde{S}_1$. One can note that $\widetilde{D}\left(\widetilde{\phi_1^{\theta\varphi}}, \widetilde{\phi_2^{\theta\varphi}}\right) = \widetilde{d}$.

Choose a point $(-4, 0, 5) \in \widetilde{S}_1(0)$ whose membership value has to be evaluated. Consider the line joining the points $(1, 0, -1)$ and $(-4, 0, 5)$, i.e.,

$$L : \tfrac{x-1}{-5} = \tfrac{z+1}{6}, y = 0,$$

for $\theta = \pi$ and $\varphi = 39.8056$. The core point (a_2, b_2, c_2) of $\widetilde{\phi_2^{\theta\varphi}}$ is $(-3.9999, 0, 4.9999)$ along the line L, for $\theta = \pi$, $\varphi = 39.8056$. Now, we evaluate $\widetilde{\phi_2^{\theta\varphi}}(\alpha)$ for $\theta = \pi$, $\varphi = 39.8056$ and $\alpha \in [0, 1]$. The α-cuts of $\widetilde{\phi_2^{\theta\varphi}}$ is

$$\widetilde{\phi_2^{\theta\varphi}}(\alpha) = (-3.9999 - 1.0858(1-\alpha), 0, 4.9999 + 1.3030(1-\alpha)),$$

for $\theta = \pi$, $\varphi = 39.8056$ and $\alpha \in [0, 1]$. The membership value for the point $(-4, 0, 5) \in \widetilde{S}_1(0)$ is 0.9999.

Let us represent the α-cuts of the fuzzy sphere $\widetilde{S}_1$.

Theorem 4.2.3. Let $\widetilde{S}_1$ be a fuzzy sphere with center $\widetilde{P}_1(a_1, b_1, c_1)$ and radius $\widetilde{R}$. Let $\phi_1^{-1}(\alpha) = \widetilde{P}_1(\alpha) \cap \{(x, y, z) : \frac{x-a_1}{\sin\varphi\cos\theta} = \frac{y-b_1}{\sin\varphi\sin\theta} = \frac{z-c_1}{\cos\varphi} = \lambda_1\}$ and $\phi_2^{-1}(\alpha)$ be a line segment in $\frac{x-a_1}{\sin\varphi\cos\theta} = \frac{y-b_1}{\sin\varphi\sin\theta} = \frac{z-c_1}{\cos\varphi} = \lambda_2$, where

$$\phi_1(\lambda_1) = f_1\left(\lambda_1 \sin\varphi\cos\theta, \lambda_1 \sin\varphi\sin\theta, \lambda_1\cos\varphi\right) = \alpha.$$

Then,

$$\widetilde{S}_1(\alpha) = \bigcup_{\theta\in[0,2\pi]}\bigcup_{\varphi\in[0,\pi]} \{\phi_2^{-1}(\alpha) : \widetilde{R}(\alpha) = [\min_{\substack{\lambda_1\in\phi_1^{-1}(\alpha)\\ \lambda_2\in\phi_2^{-1}(\alpha)}} d(\lambda_1, \lambda_2), \max_{\substack{\lambda_1\in\phi_1^{-1}(\alpha)\\ \lambda_2\in\phi_2^{-1}(\alpha)}} d(\lambda_1, \lambda_2)]\}.$$

Proof. The proof is similar to Theorem 3.1 in [3]. □

The following definitions are mainly dealt with the notions of translations and rotations of a fuzzy point which will be the tools for investigating the fuzzy sphere $\widetilde{S}_2$ in the second form and the fuzzy cone $\widetilde{\mathscr{C}}$.

Definition 4.2.2. (*Translation of a fuzzy point along a direction* (ℓ, m, n)). Let $\widetilde{P}(a, b, c)$ be a fuzzy point whose membership grade at a point (x, y, z) be evaluated by

$$\mu\left((x, y, z)\middle|\widetilde{P}\right) = f(x - a, y - b, z - c).$$

Let

$$\tfrac{x-a}{\ell} = \tfrac{y-b}{m} = \tfrac{z-c}{n} = \lambda$$

be the line passing through (a, b, c). Translation of $\widetilde{P}(a, b, c)$, say $\widetilde{P}_T$, along a direction (ℓ, m, n) is defined by the membership function as

$$\mu\left((x, y, z)\middle|\widetilde{P}^T\right) = f(x - (a + \lambda\ell), y - (b + \lambda m), z - (c + \lambda n)).$$

Explicitly, the translation of $(x, y, z) \in \widetilde{P}(0)$ along a direction (ℓ, m, n) to a new position (x', y', z') is obtained by applying the translation matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & \lambda\ell \\ 0 & 1 & 0 & \lambda m \\ 0 & 0 & 1 & \lambda n \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

such that $T(x, y, z) = (x', y', z')$.

During translation, $\widetilde{P}(0)$ gets shifted to another position with keeping the shape and size of the fuzzy point intact. If $\widetilde{P}_T$ is obtained by the translation of $\widetilde{P}$ along a direction, then $\widetilde{P}$ and $\widetilde{P}_T$ are said to be translation copies of each other.

Example 4.2.3. (*Translation of a fuzzy point along a direction* (ℓ, m, n)).
Let $\widetilde{P}(5, 0, -2)$ be a fuzzy point whose membership grade at a point (x, y, z) is evaluated by

$$\mu\left((x, y, z)\middle|\widetilde{P}(5, 0, -2)\right) = \begin{cases} 1 - \sqrt{(x-5)^2 + y^2 + (z+2)^2} & \text{if } (x-5)^2 + y^2 + (z+2)^2 \leq 25 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{P}_T$ be the translated fuzzy point along the direction $(0, 0, 1)$. Consider the line $x = 5, y = 0$ passing through the core point $(5, 0, -2)$ along the direction $(0, 0, 1)$. Note that the foot of the perpendicular on the line $x = 5, y = 0$ from the point $(1, 0, -1)$ will be the core point of the translated $\widetilde{P}_T(0)$ on which the point $(1, 0, -1)$ lies. On calculation, we find that $(5, 0, -1)$ is the core point of the translated $\widetilde{P}_T(0)$ on which the point $(1, 0, -1)$ lies.

Now, the membership grade $\mu\left((x, y, z)\middle|\widetilde{P}_T\right)$ can be evaluated by

$$\mu\left((x, y, z)\middle|\widetilde{P}_T\right) = \begin{cases} 1 - \sqrt{(x-5)^2 + y^2 + (z+1)^2} & \text{if } (x-5)^2 + y^2 + (z+1)^2 \leq 25 \\ 0 & \text{otherwise.} \end{cases}$$

Let us define the rotation of a fuzzy point about co-ordinate axes.

Definition 4.2.3. (*Rotation of a fuzzy point about co-ordinate axes*). Let $\widetilde{P}(a, b, c)$ be a fuzzy point whose membership function is

$$\mu\left((x, y, z)\middle|\widetilde{P}\right) = f(x - a, y - b, z - c).$$

Let

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}, R_y = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}$$

and

$$R_z = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be the rotation matrices about a co-ordinate axis x, or y, or z by an angle ψ, respectively. The membership function of rotation of $\widetilde{P}$ about a co-ordinate axis x, or y, or z by an angle ψ can be defined by

$$\begin{aligned} &\mu\left((x, y\cos\psi - z\sin\psi, y\sin\psi + z\cos\psi)\middle|\left(\widetilde{P}\right)_{R_x}^{\psi}\right) \\ &= f(x - a, y - b, z - c), \end{aligned}$$

or

$$\begin{aligned} &\mu\left((x\cos\psi + z\sin\psi, y, -x\sin\psi + z\cos\psi)\middle|\left(\widetilde{P}\right)_{R_y}^{\psi}\right) \\ &= f(x - a, y - b, z - c), \end{aligned}$$

or

$$\mu\left((x\cos\psi - y\sin\psi, x\sin\psi + y\cos\psi, z)\middle|\left(\widetilde{P}\right)^{\psi}_{R_z}\right)$$
$$= f(x-a, y-b, z-c),$$

respectively. Here, $\left(\widetilde{P}\right)^{\psi}_{R_x}$, $\left(\widetilde{P}\right)^{\psi}_{R_y}$ and $\left(\widetilde{P}\right)^{\psi}_{R_z}$ denote the rotation of $\widetilde{P}(a,b,c)$ about a co-ordinate axis x, or y, or z, by an angle ψ.

Definition 4.2.4. (*Rotation of a fuzzy point about any arbitrary line passing through the core point*). Let $\widetilde{P}(a,b,c)$ be a fuzzy point whose membership function is

$$\mu\left((x,y,z)\middle|\widetilde{P}\right) = f(x-a, y-b, z-c).$$

Apply translation and a combination of rotations by a required angle on $\widetilde{P}$ such that the arbitrary axis passing through the core point (a,b,c) coincides with any co-ordinate axis, say z-axis. The steps to do same, we refer [117] (see p. 10). After applying the transformations, rotation of $\widetilde{P}$ about any arbitrary axis passing through the core point (a,b,c) by an angle ψ can be obtained as $\left(\widetilde{P}\right)^{\psi}_{R_z}$.

Example 4.2.4. (*Rotation of a fuzzy point about any arbitrary axis passing through the core point*). Let us consider an S-type space fuzzy point $\widetilde{P}(0,0,0)$ with the membership function

$$\mu\left((x,y,z)\middle|\widetilde{P}(0,0,0)\right) = \begin{cases} 1 - \frac{1}{2}\sqrt{x^2+y^2+z^2} & \text{if } x^2+y^2+z^2 \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

Consider a line

$$L : \frac{x}{\frac{1}{\sqrt{11}}} = \frac{y}{\frac{3}{\sqrt{11}}} = \frac{z}{\frac{-1}{\sqrt{11}}}$$

passing through the $\widetilde{P}(1)$.

Let $T_{(0,0,0)} = I_4$, the identity matrix of order 4,

$$R_x^{71.56^\circ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & 0 \\ 0 & \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ R_y^{17.54^\circ} = \begin{pmatrix} \sqrt{\frac{10}{11}} & 0 & -\frac{1}{\sqrt{11}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{11}} & 0 & \sqrt{\frac{10}{11}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$R_z^{90^\circ} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(see p. 10 in [117]). Then,

$$R_z^{90^\circ} R_y^{17.54^\circ} R_x^{71.56^\circ} T_{(0,0,0)} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & 0 \\ -\sqrt{\frac{10}{11}} & \frac{3}{\sqrt{110}} & -\frac{1}{\sqrt{110}} & 0 \\ \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{11}} & -\frac{1}{\sqrt{11}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For instance, choose a point $(0.6325, -0.5721, 1.5076)$ for which the membership value of $\widetilde{P}_{R_z^{90^\circ}}$ has to be calculated. The membership value is given by

$$
\begin{aligned}
& \mu\left((0.6325, -0.5721, 1.5076)\middle|\widetilde{P}_{R_z^{90^\circ}}\right) \\
= & \mu\left((1, 1, -1)\middle|\widetilde{P}\right) \\
= & \begin{cases} 1 - \frac{1}{2}\sqrt{x^2 + y^2 + z^2} & \text{if } x^2 + y^2 + z^2 \leq 4 \\ 0 & \text{otherwise} \end{cases} \\
= & 0.1339.
\end{aligned}
$$

In Method 2, we formulate a fuzzy sphere when a fuzzy line segment as the diameter is given. We give two methodologies to define the diameter form of the fuzzy sphere. In the first methodology, the construction of the fuzzy sphere depends on the translation of fuzzy points along the perpendicular directions passing through the core points of the fuzzy points. In the second methodology, we extend the classical definition of the diameter form of a sphere in a fuzzy environment.

A sphere can be obtained in the classical geometry when a line segment joining two points as a diameter is given. Let $L_{P_1P_2}$ be the line segment joining two given points P_1 and P_2. Consider a point P on the sphere. Then, the line segment joining PP_1 and PP_2 must be perpendicular. A question may arise what if the given points are imprecise? Is the condition of perpendicularity of the fuzzy line segment joining PP_1 and PP_2 necessary when the given points P_1 and P_2 are imprecise? The following definition is the answer to these questions. The analogous idea, as in classical geometry, can be applied to describe a fuzzy sphere, say $\widetilde{S}_2$, when the given points are imprecise. The idea of the perpendicularity of the line segment joining PP_1 and PP_2 prompted us to define the fuzzy sphere in the following manner.

Definition 4.2.5. (*Fuzzy sphere $\widetilde{S}_2$ (diameter form)*). Let $\widetilde{P}_1$ and $\widetilde{P}_2$ be two S-type space fuzzy points, and $\tilde{\tilde{L}}_{P_1P_2}$ be the fuzzy line segment joining $\widetilde{P}_1$ and $\widetilde{P}_2$. The fuzzy line segment $\tilde{\tilde{L}}_{P_1P_2}$ is a diameter of the fuzzy sphere. Let $l_\theta\varphi$ and $l_{\theta'\varphi'}$ be two perpendicular directions passing through the core points of $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively. Consider a fuzzy line $\widetilde{L}_\theta\varphi$ generated by the translation copies of $\widetilde{P}_1$ along the direction $l_\theta\varphi$. Consider another fuzzy line $\widetilde{L}_{\theta'\varphi'}$ generated by the translation copies of $\widetilde{P}_2$ along the direction $l_{\theta'\varphi'}$.

The diameter form of the fuzzy sphere is the collection of fuzzy points $\widetilde{P}$, which are the intersection of $\widetilde{L}_{\theta\varphi}$ and $\widetilde{L}_{\theta'\varphi'}$ such that

$$\widetilde{S}_2 = \bigvee\{\widetilde{P} : \text{the fuzzy points which are the intersection of perpendicular fuzzy lines } \widetilde{L}_{\theta\varphi} \text{ and } \widetilde{L}_{\theta'\varphi'}\}.$$

More explicitly, the diameter form of the fuzzy sphere can be formulated by the membership function

$$\mu\left((x,y,z)\middle|\widetilde{S}_2\right) = \min\{\mu\left((x,y,z)\middle|\widetilde{L}_{\theta\varphi}\right), \mu\left((x,y,z)\middle|\widetilde{L}_{\theta'\varphi'}\right)\}.$$

A geometrical view of the diameter form of the fuzzy sphere $\widetilde{S}_2$ is shown in Figure 4.1, where the fuzzy line segment $\tilde{\tilde{L}}_{P_1P_2}$ joining $\widetilde{P}_1$ and $\widetilde{P}_2$ is the diameter of $\widetilde{S}_2$. The lines $l_\theta\varphi$ and $l_{\theta'\varphi'}$ are two perpendicular directions passing through the core points of $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively. The point $\widetilde{P}$ is a representative fuzzy point on $\widetilde{S}_2(0)$, and the points (a,b,c) and (a_i,b_i,c_i) are the core points of $\widetilde{P}$ and $\widetilde{P}_i$, respectively, for $i = 1,2$.

To evaluate $\mu\left((x,y,z)\middle|\widetilde{S}_2\right)$, we detect a fuzzy point in $\widetilde{S}_2$ on which the point (x,y,z) lies. We observe that there are infinite number of fuzzy points $\widetilde{P}_i(a_i,b_i,c_i)$'s at

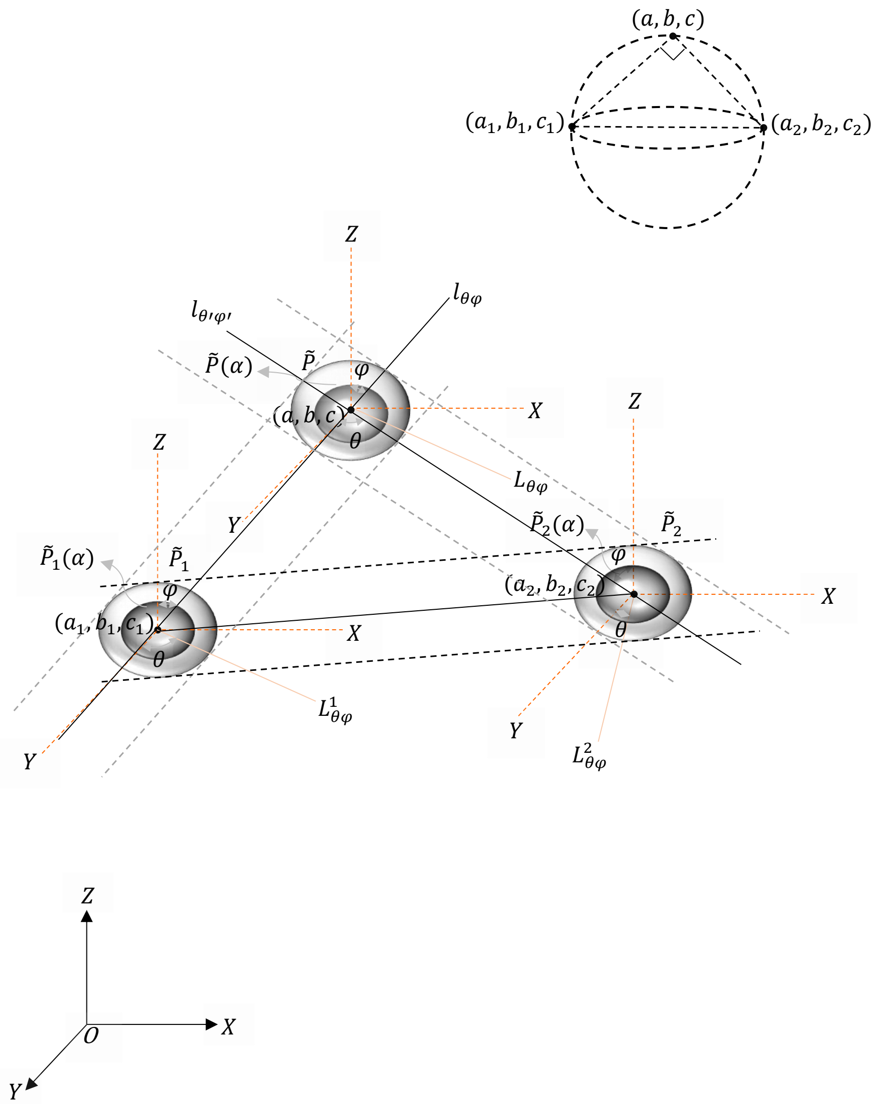

FIGURE 4.1: Diameter form of fuzzy sphere $\widetilde{S}_2$

which (x, y, z) lies and $\mu\left((x, y, z)\middle|\widetilde{P}_i\right) > 0$, $i \in \mathbb{R}$ (by Definition 4.2.5). The points (a_i, b_i, c_i)'s lie on the core sphere $\widetilde{S}_2(1)$. Suppose the membership function of $\widetilde{P}_i$'s is

$$\mu\left((x, y, z)\middle|\widetilde{P}_i\right) = 1 - \frac{d((x,y,z),(a_i,b_i,c_i))}{r}, \tag{4.3}$$

where 'd' is the Euclidean distance and $r \in \mathbb{R}$ is any given positive number. By (4.3), higher membership value is associated with the minimum distance $d((x, y, z), (a_i, b_i, c_i))$.

Hence, we search for a fuzzy point on which (x, y, z) lies and the core of the fuzzy point has minimum distance from the point (x, y, z). Noticeably, by Definition 4.2.5, $\widetilde{P}_i$ lies on $\widetilde{L}_{\theta\varphi}$ and $\widetilde{L}_{\theta'\varphi'}$ at (a_i, b_i, c_i) such that $d((x, y, z), (a_i, b_i, c_i))$ is minimum. The support of $\widetilde{P}_i(a_i, b_i, c_i)$ must be the intersection of $\widetilde{L}_{\theta\varphi}$ and $\widetilde{L}_{\theta'\varphi'}$. The membership value of the point $(x, y, z) \in \widetilde{S}_2(0)$ is given by $\mu\left((x, y, z)\middle|\widetilde{P}_i\right)$.

Example 4.2.5. (*Fuzzy sphere* $\widetilde{S}_2$ *(diameter form)*). Let us consider two fuzzy points $\widetilde{P}_1(-2, 0, 0)$ and $\widetilde{P}_2(3, 1, -1)$ with the membership functions

$$\mu\left((x, y, z)\middle|\widetilde{P}_1(-2, 0, 0)\right) = \begin{cases} 1 - \frac{1}{4}\sqrt{(x+2)^2 + y^2 + z^2} & \text{if } (x+2)^2 + y^2 + z^2 \leq 16 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} &\mu\left((x, y, z)\middle|\widetilde{P}_2(3, 1, -1)\right) \\ = &\begin{cases} 1 - \frac{1}{3}\sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2} & \text{if } (x-3)^2 + (y-1)^2 + (z+1)^2 \leq 9 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The diameter $\tilde{\tilde{L}}_{P_1P_2}$ of $\widetilde{S}_2$ can be obtained by joining the $\widetilde{P}_1(-2, 0, 0)$ and $\widetilde{P}_2(3, 1, -1)$, and evaluated by Definition 2.5.1. The core line $\tilde{\tilde{L}}_{P_1P_2}(1)$ is

$$\tfrac{x+2}{5} = y = -z = \lambda.$$

The equation

$$x^2 + y^2 + z^2 - x - y + z - 6 = 0$$

describes the core sphere $\widetilde{S}_2(1)$ whose one diameter is the core line $\tilde{\tilde{L}}_{P_1P_2}(1)$. Note that the center of $\widetilde{S}_2(1)$ is $(0.5, 0.5, -0.5)$.

Suppose the membership value of a point $(2, -3, 0) \in \widetilde{S}_2(0)$ has to be evaluated. Consider a line joining the points $(2, -3, 0)$ and $(0.5, 0.5, -0.5)$, i.e.,

$$L : \tfrac{x-2}{1.5} = \tfrac{y+3}{-3.5} = \tfrac{z}{0.5} = \lambda.$$

According to Definition 4.2.5, our main task is to find the core point (x', y', z') of the fuzzy point in $\widetilde{S}_2$ in which the point $(2, -3, 0)$ belongs. It is consider that the lines joining '(x', y', z') and $(-2, 0, 0)$' and '(x', y', z') and $(3, 1, -1)$' are perpendicular since $(x', y', z') \in \widetilde{S}_2(1)$. To find (x', y', z'), evaluate the intersection points of the $\widetilde{S}_2(1)$ with the line L. After simple calculations, the intersection of L with

$$x^2 + y^2 + z^2 - x - y + z - 6 = 0$$

yields the points $(0.04, -1.77, 0.68)$ and $(-1.55, 1.96, 0.15)$, for $\lambda = -1.6764, -0.3235$, respectively. Choose the point $(0.04, -1.77, 0.68)$ or $(-1.55, 1.96, 0.15)$ which one is the nearest from the point $(2, -3, 0)$. Here, it is easy to check that the point $(0.04, -1.77, 0.68)$ is nearest from the point $(2, -3, 0) \in \widetilde{S}_2(0)$. Suppose the point $(0.04, -1.77, 0.68)$ is the core point of the fuzzy points, say $\widetilde{P}_{T_i}$, which includes the point $(2, -3, 0)$ on its supports, for $i = 1, 2$.

Now, the membership functions of the fuzzy points $\widetilde{P}_{T_i}(0.04, -1.77, 0.68)$ are

$$\mu\left((x, y, z)\middle|\widetilde{P}_{T_1}(0.04, -1.77, 0.68)\right)$$
$$= \begin{cases} 1 - \frac{1}{4}\sqrt{(x-0.04)^2 + (y+1.77)^2 + (z-0.68)^2} & \text{if } (x-0.04)^2 + (y+1.77)^2 + z^2 \leq 16 \\ 0 & \text{otherwise,} \end{cases}$$

if $\widetilde{P}_1$ is translated along the direction

$$l_{\theta\varphi} : \tfrac{x-0.04}{-2.04} = \tfrac{y+1.77}{1.77} = \tfrac{z-0.68}{-0.68}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_{T_2}(0.04,-1.77,0.68)\right)$$
$$=\begin{cases}1-\frac{1}{3}\sqrt{(x-0.04)^2+(y+1.77)^2+(z-0.68)^2} & \text{if } (x-0.04)^2+(y+1.77)^2+(z-0.68)^2\leq 9\\ 0 & \text{otherwise,}\end{cases}$$

if $\widetilde{P}_2$ is translated along the direction

$$l_{\theta'\varphi'}:\frac{x-0.04}{2.96}=\frac{y+1.77}{2.77}=\frac{z-0.68}{-1.68},$$

respectively. By Definition 4.2.5, the membership value of the point $(2,-3,0)$ is

$$\min\left\{\mu\left((2,-3,0)\middle|\widetilde{P}_{T_1}(0.04,-1.77,0.68)\right),\mu\left((2,-3,0)\middle|\widetilde{P}_{T_2}(0.04,-1.77,0.68)\right)\right\}$$
$$=\ \min\{0.2287, 0.4216\}$$
$$= 0.2287.$$

The forthcoming Theorem 4.2.4 refers that there is no lack of inner conformity to the perpendicularity of line segments joining a point on the sphere to the extreme points of the spheres' diameter when it is extended to the fuzzy geometry.

Theorem 4.2.4. Let $\widetilde{\widetilde{L}}_{P_1P_2}$ be a diameter of a fuzzy sphere and $\widetilde{P}$ be a fuzzy point on the fuzzy sphere such that the core points of $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}$ are not collinear. Then, the space fuzzy line segments $\widetilde{\widetilde{L}}_{P_1P}$ and $\widetilde{\widetilde{L}}_{P_2P}$ joining $\widetilde{P}_1$, $\widetilde{P}$ and $\widetilde{P}_2$, $\widetilde{P}$, respectively, are perpendicular.

Proof. Definition 4.2.5 indicates that the fuzzy point $\widetilde{P}$ on the $\widetilde{S}_2$ is the intersection of $\widetilde{L}_{\theta\varphi}$ and $\widetilde{L}_{\theta'\varphi'}$. The fuzzy line $\widetilde{L}_{\theta}\varphi$ is generated by the translation copies of $\widetilde{P}_1$ along the direction $l_{\theta}\varphi$. Also, the fuzzy line $\widetilde{L}_{\theta'\varphi'}$ is generated by the translation copies of $\widetilde{P}_2$ along the direction $l_{\theta'\varphi'}$. Since $l_{\theta}\varphi$ and $l_{\theta'\varphi'}$ are two perpendicular directions passing through the core points of $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively. It is easy to say that the fuzzy line segments $\tilde{\tilde{L}}_{P_1P}$ and $\tilde{\tilde{L}}_{P_2P}$ joining $\widetilde{P}_1$, $\widetilde{P}$ and $\widetilde{P}_2$, $\widetilde{P}$, respectively, are perpendicular. This completes the proof. □

Note 11. Theorem 4.2.4 ensures that the diameter form of a fuzzy sphere is a true extension of the diameter form of a crisp sphere in the classical geometry to the fuzzy geometry. This is equivalent to the fact that any fuzzy diameter of a fuzzy sphere subtends a right angle at any space fuzzy point on the fuzzy sphere, except the two endpoints of the fuzzy diameter.

The following definition is the another way to construct a fuzzy sphere $\widetilde{S}_3$ in the diameter form.

Definition 4.2.6. (*Fuzzy sphere $\widetilde{S}_3$ (diameter form)*). Let $\widetilde{P}_1$ and $\widetilde{P}_2$ be two S-type space fuzzy points, and $\tilde{\tilde{L}}_{P_1P_2}$ be the fuzzy line segment joining $\widetilde{P}_1$ and $\widetilde{P}_2$. The fuzzy line segment $\tilde{\tilde{L}}_{P_1P_2}$ is the diameter of the fuzzy sphere. The membership value of a point (x, y, z) in the fuzzy sphere $\widetilde{S}_3$ is defined as

$$\mu\left((x,y,z)\middle|\widetilde{S}_3\right) = \sup\{\alpha : (x,y,z) \text{ belongs to the sphere whose diameter is a crisp line segment joining the same points of } \widetilde{P}_1 \text{ and } \widetilde{P}_2 \text{ with the membership value } \alpha\}.$$

More explicitly,

$$\begin{aligned}&\mu\left((x,y,z)\middle|\widetilde{S}_3\right)\\ =&\sup\{\alpha : \left(x-(x_1)^\alpha_{\theta\varphi}\right)\left(x-(x_2)^\alpha_{\theta\varphi}\right)+\left(y-(y_1)^\alpha_{\theta\varphi}\right)\left(y-(y_2)^\alpha_{\theta\varphi}\right)+\left(z-(z_1)^\alpha_{\theta\varphi}\right)\left(z-(z_2)^\alpha_{\theta\varphi}\right)=0,\\ &\quad \text{where } \left((x_i)^\alpha_{\theta\varphi},(y_i)^\alpha_{\theta\varphi},(z_i)^\alpha_{\theta\varphi}\right) \text{ are the co-ordinates of the same points of } \widetilde{P}_i, \text{ for } i=1,2\}.\end{aligned}$$

The following Algorithm 4.2.1 can be applied to evaluate $\mu\left((x,y,z)\middle|\widetilde{S}_3\right)$.

Algorithm 4.2.1: To evaluate $\mu\left((x,y,z)\middle|\widetilde{S}_3\right)$

Input: Given two continuous S-type space fuzzy points $\widetilde{P}_1(a_1,b_1,c_1)$ and $\widetilde{P}_2(a_2,b_2,c_2)$ whose membership functions are strictly decreasing along the rays emanated from their respective core points.

Given a point (x,y,z) whose membership value in $\widetilde{S}_3$ is to be calculated.

Output: The membership value $\mu\left((x,y,z)\middle|\widetilde{S}_3\right) = \alpha_{\text{sup}}$.

Initialize $\alpha_{\text{sup}} \leftarrow 0$

loop:

for $\alpha = 0$ to 1 with step size $\delta\alpha$ **do**

  for $\theta = 0$ to 2π with step size $\delta\theta$ **do**

    for $\varphi = 0$ to π with step size $\delta\varphi$ **do**

      Compute the same points $(u^1)^\alpha_{\theta\varphi} : \left((x_1)^\alpha_{\theta\varphi}, (y_1)^\alpha_{\theta\varphi}, (z_1)^\alpha_{\theta\varphi}\right)$ and $(u^2)^\alpha_{\theta\varphi} : \left((x_2)^\alpha_{\theta\varphi}, (y_2)^\alpha_{\theta\varphi}, (z_2)^\alpha_{\theta\varphi}\right)$ using (2.4) and (2.5), respectively

      Compute

$$S^\alpha_{\theta\varphi} = \left(x-(x_1)^\alpha_{\theta\varphi}\right)\left(x-(x_2)^\alpha_{\theta\varphi}\right) + \left(y-(y_1)^\alpha_{\theta\varphi}\right)\left(y-(y_2)^\alpha_{\theta\varphi}\right) + \left(z-(z_1)^\alpha_{\theta\varphi}\right)\left(z-(z_2)^\alpha_{\theta\varphi}\right)$$

      if $S^\alpha_{\theta\varphi} = 0$ **then**

        if $\alpha_{\text{sup}} < \alpha$ **then**

          $\alpha_{\text{sup}} \leftarrow \alpha$

        else

          goto loop

        end

      end

    end

  end

end

return $\mu\left((x,y,z)\middle|\widetilde{S}_3\right) = \alpha_{\text{sup}}$

Example 4.2.6. (*Evaluation of the membership values of some points in the fuzzy* *sphere* $\widetilde{S}_3(0)$). Consider the fuzzy points $\widetilde{P}_1(1,0,1)$ and $\widetilde{P}_2(-1,1,5)$ with the membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_1(1,0,1)\right) = \begin{cases} 1-\frac{1}{2}\sqrt{(x-1)^2+y^2+(z-1)^2} & \text{if } (x-1)^2+y^2+(z-1)^2 \le 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_2(-1,1,5)\right) = \begin{cases} 1-\sqrt{(x+1)^2+(y-1)^2+(z-5)^2} & \text{if } (x+1)^2+(y-1)^2+(z-5)^2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

The general expressions of the same points with the membership value $\alpha \in [0,1]$ on $\widetilde{P}_1(1,0,1)$ and $\widetilde{P}_2(-1,1,5)$ are

$$\left(u^1\right)^{\alpha}_{\theta\varphi} : (1+2(1-\alpha)\sin\varphi\cos\theta, 2(1-\alpha)\sin\varphi\sin\theta, 1+2(1-\alpha)\cos\varphi)$$

and

$$\left(u^2\right)^{\alpha}_{\theta\varphi} : (-1+(1-\alpha)\sin\varphi\cos\theta, 1+(1-\alpha)\sin\varphi\sin\theta, 5+(1-\alpha)\cos\varphi)\,,$$

respectively. Table 4.1 shows the membership values of some points in the fuzzy sphere $\widetilde{S}_3(0)$ by execution of Algorithm 4.2.1.

(x, y, z)	Membership Value	Step size
$(-1.2893, 0.8947, 5.8457)$	0.1000	$\delta\alpha = 0.1000$, $\delta\theta = 0.6981$ and $\delta\varphi = 0.3491$
$(-1.6000, 1.2370, 5.4000)$	0.3000	$\delta\alpha = 0.1000$, $\delta\theta = 0.6981$ and $\delta\varphi = 0.3491$
$(1.4500, 0, 1)$	0.7750	$\delta\alpha = 0.2250$, $\delta\theta = 1.5708$ and $\delta\varphi = 0.7854$
$(1.6364, 0, 1.6364)$	1	$\delta\alpha = 0.2250$, $\delta\theta = 1.5708$ and $\delta\varphi = 0.7854$

Table 4.1: Membership values of some points of $\widetilde{S}_3(0)$ using Algorithm 4.2.1 for Example 4.2.6

In Method 3, we investigate a fuzzy sphere that passes through four given S-type space fuzzy points. Note that the core points of the fuzzy points must not be co-planar, and any three of the core points must not be collinear. In classical geometry, for any tetrahedron, there exists a sphere on which all four vertices lie. In a similar manner, we describe that if the same points of four fuzzy points form a tetrahedron of non-zero volume, then there is a fuzzy sphere containing all the fuzzy points. Although, there may be a fuzzy sphere in which only core points of fuzzy points form a tetrahedron of non-zero volume but not the same points of the fuzzy points. It may be noted that in this case, we may not find a unique sphere, in fact, this is demonstrated in the following study. First of all, we focus on formulating a fuzzy sphere $\widetilde{S}_4$.

Definition 4.2.7. (*Fuzzy sphere* $(\widetilde{S}_4)$). A fuzzy sphere, say $\widetilde{S}_4$, passing through four S-type space fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$ whose core points must not be co-planar and any three of the core points must not be collinear, can be defined by the membership function as

$$\begin{aligned} & \mu\left((x,y,z)\middle|\widetilde{S}_4\right) \\ = \; & \sup\{\alpha : \text{where } (x,y,z) \text{ belongs to the sphere passing through the four same} \\ & \quad \text{points of } \widetilde{P}_1(0),\ \widetilde{P}_2(0),\ \widetilde{P}_3(0) \text{ and } \widetilde{P}_4(0) \text{ with the membership value } \alpha\}. \end{aligned}$$

The following example explicits the construction of the fuzzy sphere by the third approach in which the positions of four points are given imprecisely. Also, a mathematical expression is illustrated using the concept of the same and inverse points of the fuzzy points. According to Definition 4.2.7, a fuzzy sphere is the union of all the crisp spheres that pass through the four same points of $\widetilde{P}_1(a_1,b_1,c_1)$, $\widetilde{P}_2(a_2,b_2,c_2)$, $\widetilde{P}_3(a_3,b_3,c_3)$ and $\widetilde{P}_4(a_4,b_4,c_4)$. The same approach can be perceived from Figure 4.2 which represents the fuzzy sphere through four fuzzy points $\widetilde{P}_1(a_1,b_1,c_1)$, $\widetilde{P}_2(a_2,b_2,c_2)$, $\widetilde{P}_3(a_3,b_3,c_3)$ and $\widetilde{P}_4(a_4,b_4,c_4)$, where $(u^i)^{\alpha}_{\theta\varphi}$ $\left((v^i)^{\alpha}_{\theta\varphi}\right)$ denote the same points of $\widetilde{P}_i(a_i,b_i,c_i)$, for $i=1,2,3,4$. The direction ratios of $L^i_{\theta\varphi}$'s that pass through the core points (a_i,b_i,c_i) of $\widetilde{P}_i$ are identical, for $i=1,2,3,4$.

Example 4.2.7. Let $\widetilde{P}_1(a_1,b_1,c_1)$, $\widetilde{P}_2(a_2,b_2,c_2)$, $\widetilde{P}_3(a_3,b_3,c_3)$ and $\widetilde{P}_4(a_4,b_4,c_4)$ be four fuzzy points whose core points are not co-planar and any three of the core points are not collinear. Let $\widetilde{S}_3$ be a fuzzy sphere passing through $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$. The same points of $\widetilde{P}_i(a_i,b_i,c_i)$ with the membership value α are

$$\left(u^i\right)^{\alpha}_{\theta\varphi} : \left(a_i+\phi_i^{-1}(\alpha)\sin\varphi\cos\theta,\, b_i+\phi_i^{-1}(\alpha)\sin\varphi\sin\theta,\, c_i+\phi_i^{-1}(\alpha)\cos\varphi\right), \quad (4.4)$$

for $i=1,2,3,4$.

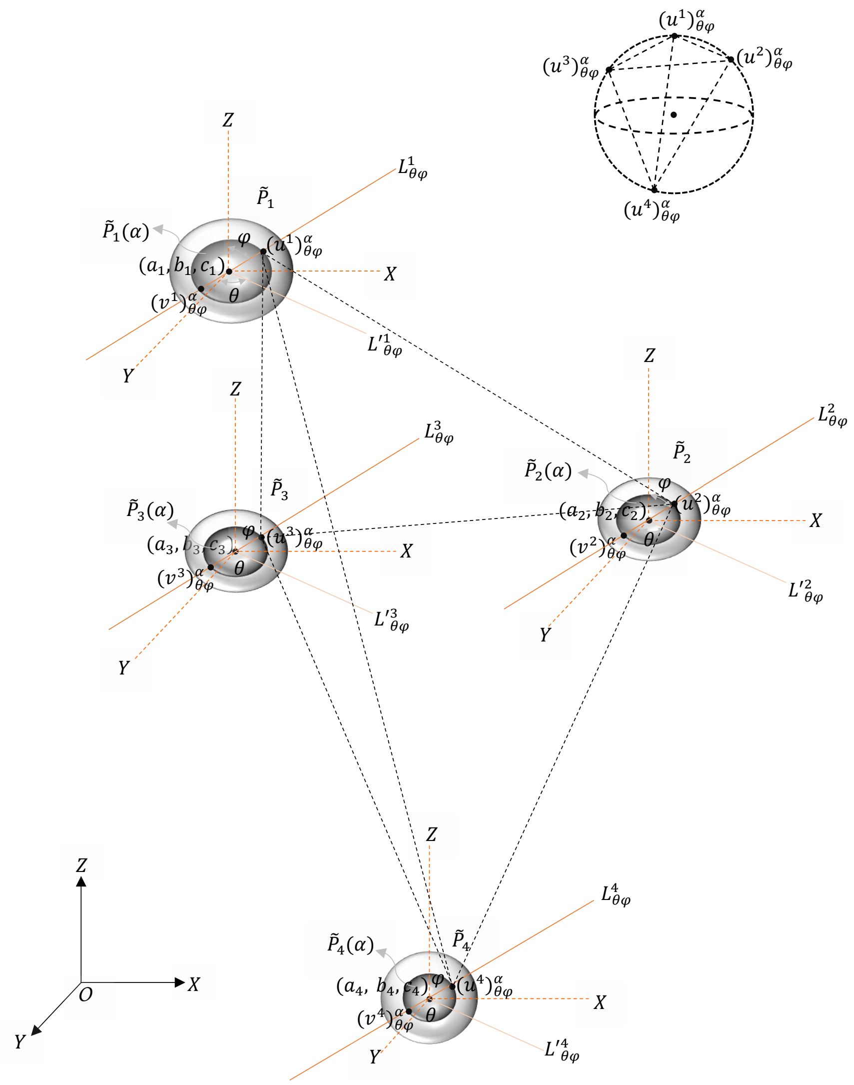

FIGURE 4.2: Fuzzy sphere passing through four fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$

The sphere, say $(S)^\alpha_{\theta\varphi}$, that passes through $(u^1)^\alpha_{\theta\varphi}$, $(u^2)^\alpha_{\theta\varphi}$, $(u^3)^\alpha_{\theta\varphi}$, and $(u^4)^\alpha_{\theta\varphi}$ can be determined by the equation

$$x^2 + y^2 + z^2 + 2x\,(u)^\alpha_{\theta\varphi} + 2y\,(v)^\alpha_{\theta\varphi} + 2z\,(w)^\alpha_{\theta\varphi} + (c)^\alpha_{\theta\varphi} = 0,$$

where

$$(u)^{\alpha}_{\theta\varphi} = \frac{1}{M}\begin{vmatrix} -\left(\left((x_1)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_1)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_1)^{\alpha}_{\theta\varphi}\right)^2\right) & 2\,(y_1)^{\alpha}_{\theta\varphi} & 2\,(z_1)^{\alpha}_{\theta\varphi} & 1 \\ -\left(\left((x_2)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_2)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_2)^{\alpha}_{\theta\varphi}\right)^2\right) & 2\,(y_2)^{\alpha}_{\theta\varphi} & 2\,(z_2)^{\alpha}_{\theta\varphi} & 1 \\ -\left(\left((x_3)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_3)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_3)^{\alpha}_{\theta\varphi}\right)^2\right) & 2\,(y_3)^{\alpha}_{\theta\varphi} & 2\,(z_3)^{\alpha}_{\theta\varphi} & 1 \\ -\left(\left((x_4)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_4)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_4)^{\alpha}_{\theta\varphi}\right)^2\right) & 2\,(y_4)^{\alpha}_{\theta\varphi} & 2\,(z_4)^{\alpha}_{\theta\varphi} & 1 \end{vmatrix},$$

$$(v)^{\alpha}_{\theta\varphi} = \frac{1}{M}\begin{vmatrix} 2\,(x_1)^{\alpha}_{\theta\varphi} & -\left(\left((x_1)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_1)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_1)^{\alpha}_{\theta\varphi}\right)^2\right) & 2\,(z_1)^{\alpha}_{\theta\varphi} & 1 \\ 2\,(x_2)^{\alpha}_{\theta\varphi} & -\left(\left((x_2)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_2)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_2)^{\alpha}_{\theta\varphi}\right)^2\right) & 2\,(z_2)^{\alpha}_{\theta\varphi} & 1 \\ 2\,(x_3)^{\alpha}_{\theta\varphi} & -\left(\left((x_3)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_3)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_3)^{\alpha}_{\theta\varphi}\right)^2\right) & 2\,(z_3)^{\alpha}_{\theta\varphi} & 1 \\ 2\,(x_4)^{\alpha}_{\theta\varphi} & -\left(\left((x_4)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_4)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_4)^{\alpha}_{\theta\varphi}\right)^2\right) & 2\,(z_4)^{\alpha}_{\theta\varphi} & 1 \end{vmatrix},$$

$$(w)^{\alpha}_{\theta\varphi} = \frac{1}{M}\begin{vmatrix} 2\,(x_1)^{\alpha}_{\theta\varphi} & 2\,(z_1)^{\alpha}_{\theta\varphi} & -\left(\left((x_1)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_1)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_1)^{\alpha}_{\theta\varphi}\right)^2\right) & 1 \\ 2\,(x_2)^{\alpha}_{\theta\varphi} & 2\,(z_2)^{\alpha}_{\theta\varphi} & -\left(\left((x_2)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_2)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_2)^{\alpha}_{\theta\varphi}\right)^2\right) & 1 \\ 2\,(x_3)^{\alpha}_{\theta\varphi} & 2\,(z_3)^{\alpha}_{\theta\varphi} & -\left(\left((x_3)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_3)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_3)^{\alpha}_{\theta\varphi}\right)^2\right) & 1 \\ 2\,(x_4)^{\alpha}_{\theta\varphi} & 2\,(z_4)^{\alpha}_{\theta\varphi} & -\left(\left((x_4)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_4)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_4)^{\alpha}_{\theta\varphi}\right)^2\right) & 1 \end{vmatrix},$$

$$(c)^{\alpha}_{\theta\varphi} = \frac{1}{M}\begin{vmatrix} 2\,(x_1)^{\alpha}_{\theta\varphi} & 2\,(y_1)^{\alpha}_{\theta\varphi} & 2\,(z_1)^{\alpha}_{\theta\varphi} & -\left(\left((x_1)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_1)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_1)^{\alpha}_{\theta\varphi}\right)^2\right) \\ 2\,(x_2)^{\alpha}_{\theta\varphi} & 2\,(y_2)^{\alpha}_{\theta\varphi} & 2\,(z_2)^{\alpha}_{\theta\varphi} & -\left(\left((x_2)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_2)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_2)^{\alpha}_{\theta\varphi}\right)^2\right) \\ 2\,(x_3)^{\alpha}_{\theta\varphi} & 2\,(y_3)^{\alpha}_{\theta\varphi} & 2\,(z_3)^{\alpha}_{\theta\varphi} & -\left(\left((x_3)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_3)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_3)^{\alpha}_{\theta\varphi}\right)^2\right) \\ 2\,(x_4)^{\alpha}_{\theta\varphi} & 2\,(y_4)^{\alpha}_{\theta\varphi} & 2\,(z_4)^{\alpha}_{\theta\varphi} & -\left(\left((x_4)^{\alpha}_{\theta\varphi}\right)^2 + \left((y_4)^{\alpha}_{\theta\varphi}\right)^2 + \left((z_4)^{\alpha}_{\theta\varphi}\right)^2\right) \end{vmatrix},$$

and

$$M = \begin{vmatrix} 2\,(x_1)^{\alpha}_{\theta\varphi} & 2\,(y_1)^{\alpha}_{\theta\varphi} & 2\,(z_1)^{\alpha}_{\theta\varphi} & 1 \\ 2\,(x_2)^{\alpha}_{\theta\varphi} & 2\,(y_2)^{\alpha}_{\theta\varphi} & 2\,(z_2)^{\alpha}_{\theta\varphi} & 1 \\ 2\,(x_3)^{\alpha}_{\theta\varphi} & 2\,(y_3)^{\alpha}_{\theta\varphi} & 2\,(z_3)^{\alpha}_{\theta\varphi} & 1 \\ 2\,(x_4)^{\alpha}_{\theta\varphi} & 2\,(y_4)^{\alpha}_{\theta\varphi} & 2\,(z_4)^{\alpha}_{\theta\varphi} & 1 \end{vmatrix}.$$

The fuzzy sphere $\widetilde{S}_4$ that passes through $\widetilde{P}_1(a_1, b_1, c_1)$, $\widetilde{P}_2(a_2, b_2, c_2)$, $\widetilde{P}_3(a_3, b_3, c_3)$ and $\widetilde{P}_4(a_4, b_4, c_4)$ is the union of all possible spheres $(S)^{\alpha}_{\theta\varphi}$ that passes through $(u^1)^{\alpha}_{\theta\varphi}$, $(u^2)^{\alpha}_{\theta\varphi}$, $(u^3)^{\alpha}_{\theta\varphi}$, and $(u^4)^{\alpha}_{\theta\varphi}$, i.e.,

$$\widetilde{S}_4 = \bigvee_{\alpha\in[0,1]} \bigcup_{\theta\in[0,2\pi]} \bigcup_{\varphi\in[0,\pi]} \left\{x^2 + y^2 + z^2 + 2x\,(u)^{\alpha}_{\theta\varphi} + 2y\,(v)^{\alpha}_{\theta\varphi} + 2z\,(w)^{\alpha}_{\theta\varphi} + (c)^{\alpha}_{\theta\varphi} = 0\right\}.$$

Now we state a result that facilitates to get the membership value of the sphere S in $\widetilde{S}_4(0)$ by the idea of the same points.

Theorem 4.2.5. Suppose that S is a sphere in $\widetilde{S}_4(0)$ and four same points $(x_1, y_1, z_1) \in \widetilde{P}_1(0)$, $(x_2, y_2, z_2) \in \widetilde{P}_2(0)$, $(x_3, y_3, z_3) \in \widetilde{P}_3(0)$, and $(x_4, y_4, z_4) \in \widetilde{P}_4(0)$ with

$$\mu\left((x_1, y_1, z_1)\middle|\widetilde{S}_4\right) = \mu\left((x_2, y_2, z_2)\middle|\widetilde{S}_4\right) = \mu\left((x_3, y_3, z_3)\middle|\widetilde{S}_4\right) = \mu\left((x_4, y_4, z_4)\middle|\widetilde{S}_4\right) = \alpha$$

exist such that S is the sphere passing through (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4). Then

$$\mu\left(S\middle|\widetilde{S}_4\right) = \alpha.$$

Proof. Similar to Theorem 3.2 in [3]. □

The following Algorithm 4.2.2 treats how to find the membership value of a point in the fuzzy sphere $\widetilde{S}_4$.

Algorithm 4.2.2: To evaluate the membership value of a point in the fuzzy sphere $\widetilde{S}_4$

Input: Given four continuous fuzzy points $\widetilde{P}_1(a_1, b_1, c_1)$, $\widetilde{P}_2(a_2, b_2, c_2)$, $\widetilde{P}_3(a_3, b_3, c_3)$ and $\widetilde{P}_4(a_4, b_4, c_4)$ whose membership functions are strictly decreasing along the rays emanated from their respective core points.

Given a point (x, y, z) whose membership value in $\widetilde{S}_4$ is to be calculated.

Output: The membership value $\mu\left((x, y, z)\middle|\widetilde{S}_4\right) = \alpha_{\text{sup}}$.

Initialize $\alpha_{\text{sup}} \leftarrow 0$

loop:

for $\alpha = 0$ to 1; with step size $\delta\alpha$ **do**

  for $\theta = 0$ to 2π; with step size $\delta\theta$ **do**

    for $\varphi = 0$ to π; with step size $\delta\varphi$ **do**

      Compute the same points

      $(u^i)^\alpha_{\theta\varphi} : \left((x_i)^\alpha_{\theta\varphi}, (y_i)^\alpha_{\theta\varphi}, (z_i)^\alpha_{\theta\varphi}\right)$ using (4.4), for $i = 1, 2, 3, 4$

      Compute

$$S^\alpha_{\theta\varphi} = \begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ \left((x_1)^\alpha_{\theta\varphi}\right)^2 + \left((y_1)^\alpha_{\theta\varphi}\right)^2 + \left((z_1)^\alpha_{\theta\varphi}\right)^2 & (x_1)^\alpha_{\theta\varphi} & (y_1)^\alpha_{\theta\varphi} & (z_1)^\alpha_{\theta\varphi} & 1 \\ \left((x_2)^\alpha_{\theta\varphi}\right)^2 + \left((y_2)^\alpha_{\theta\varphi}\right)^2 + \left((z_2)^\alpha_{\theta\varphi}\right)^2 & (x_2)^\alpha_{\theta\varphi} & (y_2)^\alpha_{\theta\varphi} & (z_2)^\alpha_{\theta\varphi} & 1 \\ \left((x_3)^\alpha_{\theta\varphi}\right)^2 + \left((y_3)^\alpha_{\theta\varphi}\right)^2 + \left((z_3)^\alpha_{\theta\varphi}\right)^2 & (x_3)^\alpha_{\theta\varphi} & (y_3)^\alpha_{\theta\varphi} & (z_3)^\alpha_{\theta\varphi} & 1 \\ \left((x_4)^\alpha_{\theta\varphi}\right)^2 + \left((y_4)^\alpha_{\theta\varphi}\right)^2 + \left((z_4)^\alpha_{\theta\varphi}\right)^2 & (x_4)^\alpha_{\theta\varphi} & (y_4)^\alpha_{\theta\varphi} & (z_4)^\alpha_{\theta\varphi} & 1 \end{vmatrix}$$

      if $S^\alpha_{\theta\varphi} = 0$ **then**

        if $\alpha_{\text{sup}} < \alpha$ **then**

          $\alpha_{\text{sup}} \leftarrow \alpha$

        else

          goto loop

        end

      end

    end

  end

end

return $\mu\left((x, y, z)\middle|\widetilde{S}_4\right) = \alpha_{\text{sup}}$

Example 4.2.8. (*Evaluation of the membership values of some points in the fuzzy sphere* $\widetilde{S}_4(0)$). Consider four fuzzy points $\widetilde{P}_1(0, 3, 2)$, $\widetilde{P}_2(1, -1, 1)$, $\widetilde{P}_3(2, 1, 0)$ and

$\widetilde{P}_4(5,1,3)$ with the membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_1(0,3,2)\right)$$
$$= \begin{cases} 1-\frac{1}{2}\sqrt{x^2+(y-3)^2+(z-2)^2} & \text{if } x^2+(y-3)^2+(z-2)^2 \leq 4 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu\left((x,y,z)\middle|\widetilde{P}_2(1,-1,1)\right)$$
$$= \begin{cases} 1-\frac{1}{3}\sqrt{(x-1)^2+(y+1)^2+(z-1)^2} & \text{if } (x-1)^2+(y+1)^2+(z-1)^2 \leq 9 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu\left((x,y,z)\middle|\widetilde{P}_3(2,1,0)\right)$$
$$= \begin{cases} 1-\frac{1}{2}\sqrt{(x-2)^2+(y-1)^2+z^2} & \text{if } (x-2)^2+(y-1)^2+z^2 \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_4(5,1,3)\right)$$
$$= \begin{cases} 1-\sqrt{(x-5)^2+(y-1)^2+(z-3)^2} & \text{if } (x-5)^2+(y-1)^2+(z-3)^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the same points of $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$ can be expressed in general term as

$$\left(u^1\right)^\alpha_{\theta\varphi} : (2(1-\alpha)\sin\varphi\cos\theta, 3+2(1-\alpha)\sin\varphi\sin\theta, 2+2(1-\alpha)\cos\varphi)\,,$$

$$\left(u^2\right)^\alpha_{\theta\varphi} : (1+3(1-\alpha)\sin\varphi\cos\theta, -1+3(1-\alpha)\sin\varphi\sin\theta, 1+3(1-\alpha)\cos\varphi)\,,$$

$$\left(u^3\right)^\alpha_{\theta\varphi} : (2+2(1-\alpha)\sin\varphi\cos\theta, 1+2(1-\alpha)\sin\varphi\sin\theta, 2(1-\alpha)\cos\varphi) \text{ and}$$

$$\left(u^4\right)^\alpha_{\theta\varphi} : (5+(1-\alpha)\sin\varphi\cos\theta, 1+(1-\alpha)\sin\varphi\sin\theta, 3+(1-\alpha)\cos\varphi)\,,$$

respectively.

Table 4.2 shows the membership values of some points in the fuzzy sphere $\widetilde{S}_4$ by execution of Algorithm 4.2.2.

(x, y, z)	Membership Value	Step size
$(2, 1, 1.8000)$	0.1000	$\delta\alpha = 0.2250$, $\delta\theta = 1.5708$ and $\delta\varphi = 0.7854$
$(2, 2.2728, 1.2728)$	0.5500	$\delta\alpha = 0.2250$, $\delta\theta = 1.5708$ and $\delta\varphi = 0.7854$
$(1.5757, 3, 2.2778)$	0.6000	$\delta\alpha = 0.1000$, $\delta\theta = 0.6981$ and $\delta\varphi = 0.3491$
$(3.8000, 1, 0)$	1	$\delta\alpha = 0.2250$, $\delta\theta = 1.5708$ and $\delta\varphi = 0.7854$

TABLE 4.2: Membership values of some points $(x, y, z) \in \widetilde{S}_4(0)$ produced by Algorithm 4.2.2 for Example 4.2.8

In what follows we explore about center and radius of the fuzzy sphere $\widetilde{S}_4$.

Definition 4.2.8. (*Center* $\widetilde{C}$ *of the fuzzy sphere* $\widetilde{S}_4$). Let $\widetilde{S}_4$ be the fuzzy sphere passing through four S-type space fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$ whose core points must not be co-planar and any three of the core points must not be collinear. The center $\widetilde{C}$ of the fuzzy sphere $\widetilde{S}_4$ can be defined by the membership function as

$$\mu\left(c \middle| \widetilde{C}\right) = \sup\{\alpha : \text{where } c \text{ is the center of the sphere passing through the four same points of } \widetilde{P}_1(0),\ \widetilde{P}_2(0),\ \widetilde{P}_3(0) \text{ and } \widetilde{P}_4(0) \text{ with the membership value } \alpha\}.$$

Let us consider the Example 4.2.7. According to Definition 4.2.8, the center $\widetilde{C}$ of $\widetilde{S}_4$ is expressed by

$$\widetilde{C} = \bigvee_{\substack{\varphi\in[0,\pi] \\ \theta\in[0,2\pi] \\ \alpha\in[0,1]}} \left\{\left(-\,(u)^{\alpha}_{\theta\varphi}\,, -\,(v)^{\alpha}_{\theta\varphi}\,, -\,(w)^{\alpha}_{\theta\varphi}\right)\right\}. \tag{4.5}$$

Example 4.2.9. (*Center $\widetilde{C}$ of the fuzzy sphere $\widetilde{S}_4$*). Let us consider the fuzzy sphere in the Example 4.2.8. Figure 4.3 depicts the boundary of 0.7-cut of the $\widetilde{C}$ evaluated by (4.5) with step size $\delta\theta = \frac{\pi}{70}$ and $\delta\varphi = \frac{\pi}{70}$.

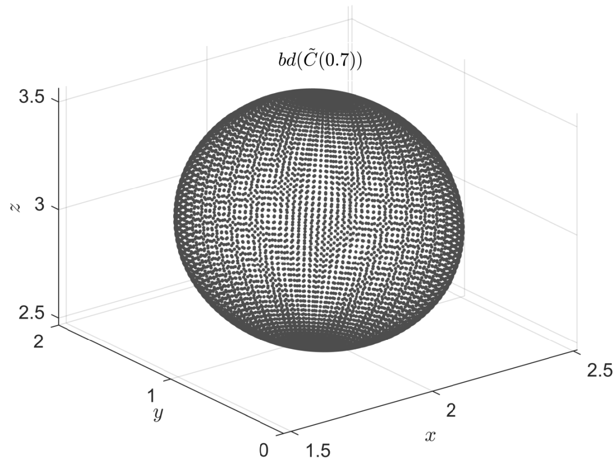

FIGURE 4.3: The boundary of 0.7-cut of the center of the fuzzy sphere

Theorem 4.2.6. Let $\widetilde{C}$ be the center of the fuzzy sphere $\widetilde{S}_4$ that passes through four fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$. If no pair of the same points of $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and

$\widetilde{P}_4$ are co-planar and any three of the same points are collinear. Then, $\widetilde{C}(\alpha)$ is a compact and connected set, and for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, $\widetilde{C}(\alpha_2) \subseteq \widetilde{C}(\alpha_1)$.

Proof. Analogous to the proof of Theorem 3.4 in [3], we can define a function

$$F : [\alpha, 1] \times [0, 2\pi] \times [0, \pi] \longrightarrow \mathbb{R}^3$$

by

$$F(\gamma, \theta, \varphi) = \left(-(u)^{\gamma}_{\theta\varphi}, -(v)^{\gamma}_{\theta\varphi}, -(w)^{\gamma}_{\theta\varphi} \right).$$

Since $(u^i)^{\alpha}_{\theta\varphi}$ are continuous and no pair of the same points in $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$ are co-planar and any three of the same points are collinear, for $i = 1, 2, 3, 4$. The expression of $(u)^{\alpha}_{\theta\varphi}, (v)^{\alpha}_{\theta\varphi}$ and $(w)^{\alpha}_{\theta\varphi}$ as represented in Example 4.2.7 are continuous. Since $(u)^{\alpha}_{\theta\varphi}, (v)^{\alpha}_{\theta\varphi}$ and $(w)^{\alpha}_{\theta\varphi}$ are continuous, the function $F(\gamma, \theta, \varphi)$ is continuous. We know that the continuous image of compact and connected set is compact and connected set. Hence, the set $F([\alpha, 1] \times [0, 2\pi] \times [0, \pi])$ must be a compact and connected set. Therefore, by (4.5), $\widetilde{C}(\alpha)$ is compact and connected. It directly follows from the (4.5) that for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, $\widetilde{C}(\alpha_2) \subseteq \widetilde{C}(\alpha_1)$.

□

Definition 4.2.9. (*Radius $\widetilde{R}$ of the fuzzy sphere $\widetilde{S}_4$*). Let $\widetilde{S}_4$ be the fuzzy sphere passing through four S-type space fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$ whose core points must not be co-planar and any three of the core points must not be collinear. The radius $\widetilde{R}$ of the fuzzy sphere $\widetilde{S}_4$ can be defined by the membership function as

$$\mu\left(r \middle| \widetilde{R}\right) = \sup\{\alpha : \text{where } r \text{ is the radius of the sphere passing through the four same points of } \widetilde{P}_1(0),\ \widetilde{P}_2(0),\ \widetilde{P}_3(0) \text{ and } \widetilde{P}_4(0) \text{ with the membership value } \alpha\}.$$

Let us consider Example 4.2.7. According to Definition 4.2.9, the radius $\widetilde{R}$ of $\widetilde{S}_4$ is expressed by

$$\widetilde{R} = \bigvee_{\substack{\varphi\in[0,\pi]\\ \theta\in[0,2\pi]\\ \alpha\in[0,1]}} \sqrt{(u)_{\theta\varphi}^{\alpha}{}^{2} + (v)_{\theta\varphi}^{\alpha}{}^{2} + (w)_{\theta\varphi}^{\alpha}{}^{2} - (c)_{\theta\varphi}^{\alpha}}. \tag{4.6}$$

Example 4.2.10. (*Radius $\widetilde{R}$ of the fuzzy sphere $\widetilde{S}_4$*). Consider the fuzzy sphere in the Example 4.2.8, then the α-cuts of the radius $\widetilde{R}$ of $\widetilde{S}_4$ in Example 4.2.8 are found in Table 4.3 as

α	$\widetilde{R}(\alpha)$
0	$[2.4141, 4.1111]$
0.1	$[2.4561, 3.9728]$
0.2	$[2.5012, 3.8406]$
0.3	$[2.5495, 3.7144]$
0.4	$[2.6011, 3.5942]$
0.5	$[2.6565, 3.4801]$
0.6	$[2.7158, 3.3721]$
0.7	$[2.7792, 3.2703]$
0.8	$[2.8476, 3.1745]$
0.9	$[2.9211, 3.0844]$
1	3

TABLE 4.3: α-cuts of $\widetilde{R}$ for Example 4.2.8

Theorem 4.2.7. Let $\widetilde{R}$ be the radius of the fuzzy sphere $\widetilde{S}_4$ that passes through four fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$. If no pair of the same points of $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$ are co-planar and any three of the same points are collinear. Then, radius $\widetilde{R}$ of the fuzzy sphere $\widetilde{S}_4$ is a fuzzy number.

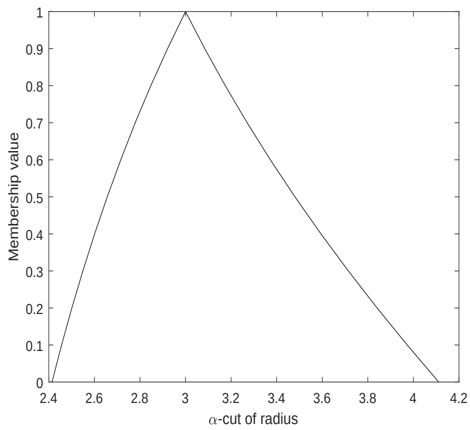

FIGURE 4.4: Membership function of the radius $\widetilde{R}$ in Example 4.2.8

Proof. Analogous to the proof of Theorem 4.2.6, we can define a function

$$R : [\alpha, 1] \times [0, 2\pi] \times [0, \pi] \longrightarrow \mathbb{R}$$

by

$$R(\gamma, \theta, \varphi) = \sqrt{{(u)^{\alpha}_{\theta\varphi}}^2 + {(v)^{\alpha}_{\theta\varphi}}^2 + {(w)^{\alpha}_{\theta\varphi}}^2 - (c)^{\alpha}_{\theta\varphi}},$$

which is continuous. The set $R([\alpha, 1] \times [0, 2\pi] \times [0, \pi])$ is compact and connected since $[\alpha, 1] \times [0, 2\pi] \times [0, \pi]$ is a compact and connected set. Therefore, the set $\widetilde{R}(\alpha)$ is a closed and bounded interval, for $\alpha \in [0, 1]$. It is obvious from (4.6) that for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, $\widetilde{R}(\alpha_2) \subseteq \widetilde{R}(\alpha_1)$. The membership function of $\widetilde{R}$ is upper semi-continuous since for all $\lambda \in \mathbb{R}$, the set $\left\{R : \mu\left(R \middle| \widetilde{R}\right) \geq \lambda\right\}$ is closed and bounded. Now, for the $\widetilde{R}(1)$, consider the radius of the sphere passing through $\widetilde{P}_1(1)$, $\widetilde{P}_2(1)$, $\widetilde{P}_3(1)$ and $\widetilde{P}_4(1)$. Hence, the theorem is proved. □

Note 12. In classical geometry, four points always determine a unique sphere if they are not co-planar. Here, we can exclude the restriction that 'if no three points are collinear' because in such a case, the four points will necessarily be co-planar. If they are co-planar, either there is no sphere through these four points or an infinity of them. Because if three points are collinear, they do not lie on any circle (and hence not on any sphere). Also, if they are not collinear, they determine a unique circle, which is contained by an infinity of spheres. Overall, a sphere passing through the four not co-planar points is a unique sphere.

Theorem 4.2.8. (Uniqueness theorem). Let $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$ be four fuzzy points. If no pair of the same points of $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$ are co-planar. Then, the fuzzy sphere $\widetilde{S}_4$ that passes through $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$ is unique.

Proof. The proof is trivial by Definition 4.2.7 and the Note 12. □

Next, we show that the intersection of fuzzy sphere $\widetilde{S}$ with a crisp plane $\mathit{\Pi}$ is a fuzzy circle.

Note 13. In further analysis, we have given a uniform notation for a fuzzy sphere as $\widetilde{S}$, formulated by either Method 1 or Method 2 or Method 3.

Definition 4.2.10. (*The intersection of fuzzy sphere $\widetilde{S}$ with a crisp plane $\mathit{\Pi}$*).

Let $\widetilde{S}$ be a fuzzy sphere and $\mathit{\Pi}$ be a crisp plane such that $\widetilde{S}(1)\bigcap \mathit{\Pi} \neq \emptyset$. The intersection of fuzzy sphere $\widetilde{S}$ with a crisp plane $\mathit{\Pi}$ is a fuzzy circle, say $\widetilde{C}_{\mathit{\Pi}}$, with the membership function as

$$\mu\left((x,y,z)\middle|\widetilde{C}_{\mathit{\Pi}}\right) = \begin{cases} \mu\left((x,y,z)\middle|\widetilde{S}\right) & \text{if } (x,y,z) \in \widetilde{S}\bigcap \mathit{\Pi} \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.2.11. (*The intersection of fuzzy sphere $\widetilde{S}$ with a crisp plane $\mathit{\Pi}$*). Let $\widetilde{P}(4,-2,-4)$ and $\widetilde{d} = (9-\beta/9/9+\gamma)_{LR}$ be a fuzzy point and a fuzzy number, where $\beta = 6$, $\gamma = 2$ and $L(x) = R(x) = \max\{0, 1-x\}$. The membership function of

$\widetilde{P}(4,-2,-4)$ is

$$\mu\left((x,y,z)\middle|\widetilde{P}(4,-2,-4)\right) = \begin{cases} 1-\frac{1}{2}\sqrt{(x-4)^2+(y+2)^2+(z+4)^2} & \text{if } (x-4)^2+(y+2)^2+(z+4)^2 \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

Let $x-2y+2z=3$ be a crisp plane. Let $\widetilde{S}$ be the fuzzy sphere formulated by the Method 1 whose center and radius are $\widetilde{P}(4,-2,-4)$ and $\widetilde{d}=(9-\beta/9/9+\gamma)_{LR}$, respectively. The α-cuts of $\widetilde{S}$ are

$$\widetilde{S}(\alpha) = \left(a_2+(1-\alpha)\sin\varphi\cos\theta, b_2+(1-\alpha)\sin\varphi\sin\theta, c_2+(1-\alpha)\cos\varphi\right),$$

for $\theta \in [0,2\pi]$, $\varphi \in [0,\pi]$ and $\alpha \in [0,1]$ along the ray

$$L : \tfrac{x-4}{\sin\varphi\cos\theta} = \tfrac{y+2}{\sin\varphi\sin\theta} = \tfrac{z+4}{\cos\varphi}.$$

Here, the points on the core sphere are

$$\{(a_2,b_2,c_2) : (a_2,b_2,c_2) = (4+9\sin\varphi\cos\theta, -2+9\sin\varphi\sin\theta, -4+9\cos\varphi)\},$$

for $\theta \in [0,2\pi]$, $\varphi \in [0,\pi]$. The membership function of $\widetilde{C}_{\Pi}$ is

$$\mu\left((x,y,z)\middle|\widetilde{C}_{\Pi}\right) = \begin{cases} \mu\left((x,y,z)\middle|\widetilde{S}\right) & \text{if } (x,y,z) \in \widetilde{S} \bigcap x-2y+2z=3 \\ 0 & \text{otherwise.} \end{cases}$$

The α-cuts $\widetilde{C}_{\Pi}(\alpha)$ is the set

$$\{(a_2 + (1-\alpha)\sin\varphi\cos\theta, b_2 + (1-\alpha)\sin\varphi\sin\theta, c_2 + (1-\alpha)\cos\varphi) :$$
$$(a_2 + (1-\alpha)\sin\varphi\cos\theta) - 2(b_2 + (1-\alpha)\sin\varphi\sin\theta) + 2(c_2 + (1-\alpha)\cos\varphi) = 3\},$$

for $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$.

Definition 4.2.11. (*Center of the fuzzy circle* $\widetilde{C}_{\Pi}$). Let $\widetilde{S}$ be a fuzzy sphere and Π be a crisp plane such that $\widetilde{S}(1) \bigcap \Pi \neq \emptyset$. Center of the fuzzy circle, say $\widetilde{c}$, can be obtained as a translation copy of the center $\widetilde{C}$ of the fuzzy sphere $\widetilde{S}$ along the direction drawn perpendicularly from $\widetilde{C}(1)$ to the plane Π. The membership function of the center $\widetilde{c}$ of the fuzzy circle $\widetilde{C}_{\Pi}$ is defined as

$$\mu\left((x, y, z)|\widetilde{c}\right) = \begin{cases} \mu\left((x, y, z)\middle|\widetilde{C}_T\right) & \text{if } (x, y, z) \in \widetilde{C}_T \bigcap \Pi \\ 0 & \text{otherwise,} \end{cases}$$

where $\widetilde{C}_T$ is the translation copy of the center $\widetilde{C}$ to the plane Π.

Example 4.2.12. (*Center of the fuzzy circle* $\widetilde{C}_{\Pi}$). Let us consider the fuzzy circle $\widetilde{C}_{\Pi}$ as in Example 4.2.11. Draw a perpendicular from $\widetilde{C}(1)$ to the plane $\Pi : x - 2y + 2z = 3$. We get that the point $(4.33, -2.66, -3.33)$ is the foot of perpendicular from $\widetilde{C}(1)$ to the plane Π. Hence, by Definition 4.2.11, the membership function of the center $\widetilde{c}$ of the fuzzy circle $\widetilde{C}_{\Pi}$ is

$$\mu\left((x, y, z)|\widetilde{c}\right) = \begin{cases} 1 - \frac{1}{2}\sqrt{(x-4.33)^2 + (y+2.66)^2 + (z+3.33)^2} & \text{if } (x-4.33)^2 + (y+2.66)^2 + (z+3.33)^2 \leq 4 \\ & \text{and } x - 2y + 2z = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Here, the core point $\widetilde{c}(1)$ is $(4.33, -2.66, -3.33)$.

Definition 4.2.12. (*Radius of the fuzzy circle* $\widetilde{C}_{\Pi}$). Let $\widetilde{S}$ be a fuzzy sphere and Π be a crisp plane such that $\widetilde{S}(1) \bigcap \Pi \neq \emptyset$. Let a fuzzy point, say $\widetilde{C}_R$, be a $90°$ rotation of the center $\widetilde{C}$ of the fuzzy sphere about the co-ordinate axis considering the direction drawn perpendicularly from $\widetilde{C}(1)$ to the plane Π as an axis of reference. Radius of the fuzzy circle, say $\widetilde{r}$, can be obtained as the fuzzy distance between the fuzzy points $\widetilde{C}_R$ and $\widetilde{c}$.

The membership function of the radius $\widetilde{r}$ of the fuzzy circle $\widetilde{C}_{\Pi}$ is defined as

$$\mu\left(r|\widetilde{r}\right) = \begin{cases} \mu\left(r\middle|\widetilde{D}(\widetilde{C}_R, \widetilde{c})\right) & \text{if } r \in \widetilde{D}(\widetilde{C}_R, \widetilde{c}) \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.2.13. (*Radius of the fuzzy circle* $\widetilde{C}_{\Pi}$). Let us consider the fuzzy circle $\widetilde{C}_{\Pi}$ as in Example 4.2.11. The α-cut of the center $\widetilde{P}(4, -2, -4)$ of the sphere $\widetilde{S}$ in Example 4.2.11 is

$$\left(4 + 2(1-\alpha)\sin\varphi\cos\theta, -2 + 2(1-\alpha)\sin\varphi\sin\theta, -4 + 2(1-\alpha)\cos\varphi\right),$$

for $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$ and $\alpha \in [0, 1]$ along the ray L. Consider a line

$$L' : \tfrac{x-4}{1} = \tfrac{y+2}{-2} = \tfrac{z+4}{2}$$

drawn from $(4, -2, -4)$ to the plane $x - 2y + 2z = 3$. Rotate all the points of $\widetilde{P}(4, -2, -4)(\alpha)$ perpendicular to the line L'. To do this, we need the following rotations by appropriate angles. Let

$$R_x^{45^\circ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } R_y^{90^\circ} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(see p. 10 in [117]). Then,

$$R_y^{90^\circ} R_x^{45^\circ} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The α-cuts of the fuzzy point $\widetilde{C}_R$ are $\{R_y^{90^\circ} R_x^{45^\circ}(4 + 2(1-\alpha)\sin\varphi\cos\theta, -2 + 2(1-\alpha)\sin\varphi\sin\theta, -4 + 2(1-\alpha)\cos\varphi)\}$. Here, $\widetilde{C}_R(1)$ is the point $(4.2426, 1.4142, 4)$.

According to Definition 4.2.12, the radius $\widetilde{r}$ of $\widetilde{C}_\Pi$ is the fuzzy distance between $\widetilde{c}$ and $\widetilde{C}_R$. We have evaluated the $\widetilde{r}(0.1) = [5.9683, 11.0916]$, $\widetilde{r}(0.3) = [6.4997, 10.4907]$, $\widetilde{r}(0.5) = [7.0360, 9.8903]$, $\widetilde{r}(0.8) = [7.8475, 8.9908]$, $\widetilde{r}(1) = 8.3922$.

Analogous to the definition of a great circle in classical geometry, we define the great fuzzy circle, say $\widetilde{G}_c$, as the intersection of a fuzzy sphere $\widetilde{S}$ by a crisp plane Π that passes through the core point of the center of the sphere $\widetilde{S}$. We know that the surface for $\theta =$ constant, in the spherical polar co-ordinate system, is a half-plane from any co-ordinate axis (say z-axis). Another part of the surface, where $\theta + \pi =$ constant, is also a half-plane. Both the half-planes, where $\theta =$ constant and $\theta + \pi =$ constant, form a complete plane. The intersection of $\widetilde{S}$ with that complete plane gives the $\widetilde{G}_c$. Note that the points of support of the center $\widetilde{C}(0)$ are represented by that spherical polar co-ordinate system in which origin is translated to the $\widetilde{C}(1)$.

Definition 4.2.13. (*Great fuzzy circle* $\widetilde{G}_c$). For each fixed $\theta, \theta+\pi \in [0, 2\pi]$, according to Definition 4.2.1, the membership function of the great fuzzy circle $\widetilde{G}_c$ can be expressed as

$$\widetilde{G}_c = \bigvee_{\varphi \in [0,\pi]} \{\widetilde{\phi_2^{\varphi}} : \widetilde{D}\left(\widetilde{\phi_1^{\varphi}}, \widetilde{\phi_2^{\varphi}}\right) = \widetilde{R}, \text{ where } \widetilde{\phi_1^{\varphi}} \text{ and } \widetilde{\phi_2^{\varphi}} \text{ are fuzzy numbers along the line } L\},$$

where $\widetilde{R}$ is the radius of $\widetilde{S}$, and

$$L : \tfrac{x-a}{\sin\varphi\cos\theta} = \tfrac{y-b}{\sin\varphi\sin\theta} = \tfrac{z-c}{\cos\varphi} = \lambda.$$

On the other hand, the membership function of the $\widetilde{G}_c$ can be represented as

$$\mu\left((x,y,z)\Big|\widetilde{G}_c\right) = \begin{cases} \mu\left((x,y,z)\Big|\widetilde{S}\right) & \text{if } (x,y,z) \in \widetilde{S} \bigcap \mathit{\Pi} \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathit{\Pi}$ is a crisp plane passing through the core point of the center of $\widetilde{S}$.

The center, say $\widetilde{c}_g$, and the radius, say $\widetilde{r}_g$, of the $\widetilde{G}_c$ is $\widetilde{C}(0) \cap \mathit{\Pi}$ and $\widetilde{R}$, where $\widetilde{C}$ and $\widetilde{R}$ are the center and radius of the fuzzy sphere $\widetilde{S}$, respectively.

Example 4.2.14. (*Great fuzzy circle* $\widetilde{G}_c$). Let us consider a fixed fuzzy point $\widetilde{P}_1$ and a fixed fuzzy number $\widetilde{d}$ as in the Example 4.2.2. For each fixed $\theta, \theta+\pi \in [0, 2\pi]$, the points on the α-cuts of the great fuzzy circle are

$$\widetilde{G}_c(\alpha) = (a_2 \pm 1.6962(1-\alpha)\sin\varphi\cos\theta, b_2 \pm 1.6962(1-\alpha)\sin\varphi\sin\theta, c_2 + 1.6962(1-\alpha)\cos\varphi),$$

for $\varphi \in [0, \pi]$ and $\alpha \in [0, 1]$ along the ray L. Here, the points on the core $\widetilde{G}_c(1)$ are

$$(1 \pm 7.8102 \sin\varphi \cos\theta, \pm 7.8102 \sin\varphi \sin\theta, -1 + 7.8102 \cos\varphi).$$

The center $\widetilde{c}_g = \widetilde{P}_1$ and the radius $\widetilde{r}_g = \widetilde{d}$, for each fixed $\theta, \theta + \pi \in [0, 2\pi]$.

As converse, the rotation of a great fuzzy circle $\widetilde{G}_c$ about an axis passing through $\widetilde{c}_g(1)$ that lies on the Π, is a fuzzy sphere.

Definition 4.2.14. (*Rotation of a great fuzzy circle* $\widetilde{G}_c$). Let $\widetilde{G}_c$ be the great fuzzy circle on $\widetilde{S}(0)$. A rotation of $\widetilde{G}_c$ about an axis that passes through the core point of the center of $\widetilde{G}_c$ and lies on the Π, can be perceived as a fuzzy sphere. For the appropriate rotation of $\widetilde{G}_c$, first coincide the $\widetilde{G}_c(0)$ to any co-ordinate plane (say xy-plane). Then, apply the rotations about any co-ordinate axis (say x-axis) by angle $\varphi \in [0, \pi]$ to obtain a fuzzy sphere. Let $(r \sin\theta, r \cos\theta, 0)$ be the points in $\widetilde{G}_c(0)$ (fuzzy circle in the xy-plane whose $\widetilde{c}_g(1)$ is origin), where $r \in \widetilde{r}_g(0)$. The rotation of $(r \sin\theta, r \cos\theta, 0) \in \widetilde{G}_c(0)$ with respect to x-axis by angle $\varphi \in [0, \pi]$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ 1 \end{pmatrix},$$

for $\varphi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. Hence, the rotation of $(r\cos\theta, r\sin\theta, 0)$ in $G_c(0)$, say $R_{c\theta\varphi}$, is

$$R_{c\theta\varphi} = (r\cos\theta, r\sin\theta\cos\varphi, r\sin\theta\sin\varphi).$$

The membership function of the fuzzy sphere $\widetilde{S}$ which is generated by the rotation of the fuzzy circle $\widetilde{G}_c$ can be defined by

$$\mu\left((x,y,z)\middle|\widetilde{S}\right) = \begin{cases} \mu\left((u,v,w)\middle|\widetilde{G}_c\right) & \text{if } (x,y,z) = R_{c\theta\varphi}(u,v,w) \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, by varying $\theta \in [0, 2\pi]$, in Definition 4.2.13, we get a fuzzy sphere.

Example 4.2.15. Let us consider a great fuzzy circle as in Example 4.2.14. For varying $\theta \in [0, 2\pi]$, the points of the α-cuts of the fuzzy sphere are

$$\widetilde{S}(\alpha) = (a_2 + 1.6962(1-\alpha)\sin\varphi\cos\theta, b_2 + 1.6962(1-\alpha)\sin\varphi\sin\theta, c_2 + 1.6962(1-\alpha)\cos\varphi)\,,$$

for $\varphi \in [0, \pi]$ and $\alpha \in [0, 1]$ along the ray L. Here, the points on the core $\widetilde{S}(1)$ are

$$(1 + 7.8102\sin\varphi\cos\theta, 7.8102\sin\varphi\sin\theta, -1 + 7.8102\cos\varphi).$$

The center and the radius are $\widetilde{P}_1$ and $\widetilde{d}$, respectively.

The following section deals with the formulation of a fuzzy cone and its intersection by a crisp plane.

4.3 Fuzzy cone

In classical geometry, a right circular cone is a surface generated by a straight line that passes through a fixed point and makes a constant angle with a fixed straight line. In analogy with this, we formulate a fuzzy cone as a collection of fuzzy lines passing through a fixed fuzzy point $\widetilde{P}$ and makes a constant angle with a fixed fuzzy line that passes through the fuzzy point $\widetilde{P}$.

To form a fuzzy cone, we first translate a given fuzzy point along a fixed axis. The fuzzy line generated by translating a given fuzzy point along a fixed axis is called a fixed fuzzy line. Then, draw a line making a constant angle from the fixed axis. Again, translate the same fuzzy point along that direction. The fixed fuzzy line is called the axis, and the given fuzzy point is called the vertex of the fuzzy cone. The fuzzy line that makes a constant angle with the fixed fuzzy line is called the fuzzy cone generator. We rotate the generator 360° around the axis to form the fuzzy cone.

Naturally, a question may arise as to why a constant angle and translation of fuzzy point has been taken to define a fuzzy cone? Further, is it possible to define a fuzzy cone in a general manner as other fuzzy space geometrical entities, such that the angle and the axis must be a fuzzy number and a fuzzy line segment, respectively? Answer to this question is geometrically depicted in Figure 4.5. For instance, we define a fuzzy cone, say $\widetilde{\mathscr{C}}$, as

$$\begin{aligned}
&\mu\left((x,y,z)\middle|\widetilde{\mathscr{C}}\right)\\
&= \sup\{\alpha :\ (x,y,z)\in\mathscr{C}_\psi^\alpha,\ \text{where } \mathscr{C}_\psi^\alpha \text{ is the cone generated by the line making angle}\\
&\qquad \psi \ \text{ with the line joining the same points of } \widetilde{P}_1(0) \text{ and } \widetilde{P}_2(0) \text{ with the}\\
&\qquad \text{membership value } \alpha\},
\end{aligned} \tag{4.7}$$

$\widetilde{\psi}$ is a fuzzy number and $\psi\in\widetilde{\psi}(0)$ with membership value α. Figure 4.5 gives the geometrical interpretation of (4.7), where $\widetilde{P}_1$ and $\widetilde{P}_2$ are two fuzzy points and $\mathscr{C}_\psi^\alpha$ is the crisp cone generated by the line, which makes angle ψ with the line joining the same points $(u^1)^\alpha_{\theta\varphi}\left((v^1)^\alpha_{\theta\varphi}\right)$ and $(u^2)^\alpha_{\theta\varphi}\left((v^2)^\alpha_{\theta\varphi}\right)$ of $\widetilde{P}_1(0)$ and $\widetilde{P}_2(0)$ with the membership value α, for some $\alpha\in[0,1]$. We denote the core cone by $\mathscr{C}_\psi^1$, which is a crisp cone generated by the line which makes angle $\psi' = \widetilde{\psi}(1)$ with the core

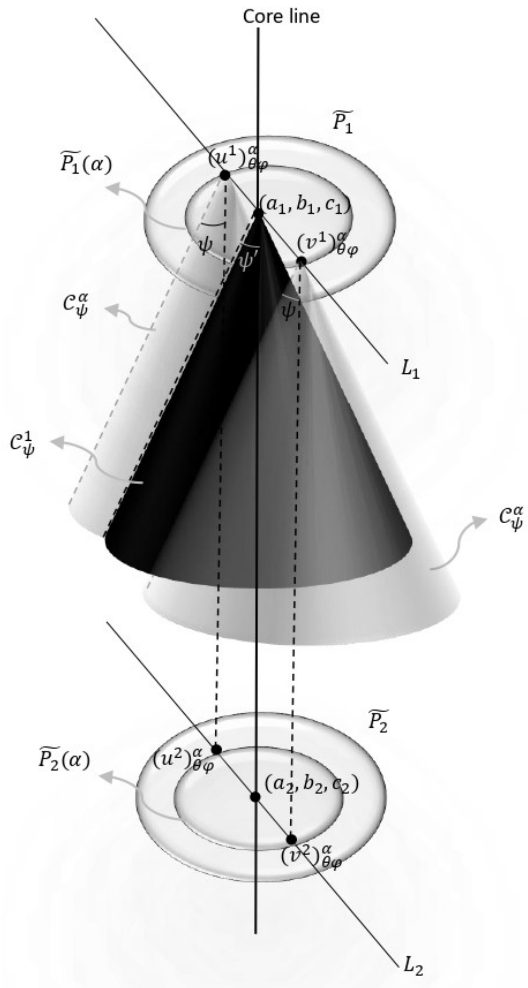

FIGURE 4.5: Fuzzy cone defined by Equation (4.7)

line of the fuzzy line segment joining $\widetilde{P}_1$ and $\widetilde{P}_2$. It can be perceived by the Figure 4.5 that $\widetilde{\mathscr{C}}(1) \nsubseteq \widetilde{\mathscr{C}}(\alpha)$, for $0 \leq \alpha < 1$. It contradicts an elementary property of the fuzzy sets by which the fuzzy sets and the crisp sets are connected. Hence the definition represented in (4.7) is not an appropriate way to define a fuzzy cone. To fulfill all the elementary properties of α-cuts of the fuzzy cone $\widetilde{\mathscr{C}}$ illustrated by (4.7), we define the fuzzy cone $\widetilde{\mathscr{C}}$ as follows.

Definition 4.3.1. (*Fuzzy cone* $\widetilde{\mathscr{C}}$). Let $\widetilde{P}(a, b, c)$ be an S-type space fuzzy point. Consider a fuzzy line $\widetilde{L}_\theta\varphi$ generated by a translation copies of $\widetilde{P}$ along the direction $l_\theta\varphi$. Consider another direction $l_{\theta'\varphi'}$ which makes angle ϕ with the $l_\theta\varphi$. We construct that the fuzzy cone is a collection of all the fuzzy lines $\widetilde{L}_{\theta'\varphi'}$ generated by the translation copies of $\widetilde{P}$ along the direction $l_{\theta'\varphi'}$. The membership function of the fuzzy cone $\widetilde{\mathscr{C}}$ is defined by

$$\mu\left((x,y,z)\middle|\widetilde{\mathscr{C}}\right) = \sup\{\alpha : \text{where } (x,y,z) \text{ belongs to the fuzzy line } \widetilde{L}_{\theta'\varphi'} \text{ with the membership value } \alpha\}.$$

As per notations of Definition 4.3.1, the geometrical representation of the fuzzy cone is depicted in Figure 4.6, where $\widetilde{P}_{T\theta'\varphi'}(a_1,b_1,c_1)$ is the translation copy of $\widetilde{P}$ along the direction $l_{\theta'\varphi'}$. The fuzzy point $\widetilde{P}_{T\theta\varphi}(a',b',c')$ is the translation copy of $\widetilde{P}$ along the direction $l_{\theta\varphi}$. The fuzzy line $\widetilde{L}'_{\theta'\varphi'}$ represents the another position of $\widetilde{L}_{\theta'\varphi'}$ in the fuzzy cone $\widetilde{\mathscr{C}}(0)$ which also makes angle ϕ with $l_\theta\varphi$.

Example 4.3.1. (*Fuzzy cone* $\widetilde{\mathscr{C}}$). Let $\widetilde{P}(0,0,0)$ be the vertex of the fuzzy cone $\widetilde{\mathscr{C}}$ with the membership function

$$\mu\left((x,y,z)\middle|\widetilde{P}(0,0,0)\right) = \begin{cases} 1-\frac{1}{4}\sqrt{x^2+y^2+z^2} & \text{if } x^2+y^2+z^2 \le 16 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{L}_\theta\varphi$ be a fuzzy line generated by a translation copies of $\widetilde{P}(0,0,0)$ along the z-axis. Let $l_{\theta'\varphi'}$ be another line which makes angle 45° with $l_\theta\varphi$. Let $\widetilde{L}_{\theta'\varphi'}$ be a fuzzy line generated by the translation copies of $\widetilde{P}(0,0,0)$ along the direction $l_{\theta'\varphi'}$. The surface of revolution of $\widetilde{L}_{\theta'\varphi'}(0)$ forms the fuzzy cone $\widetilde{\mathscr{C}}$. The equation of the core cone of the cone $\widetilde{\mathscr{C}}$ is $x^2+y^2=z^2$.

Choose a point $(1,4,0)$ whose membership value has to be evaluated. To evaluate the membership value, we have to find the point of minimum distance from $(1,4,0)$ to the core cone $x^2+y^2=z^2$. To find this, apply the Lagrange multiplier method.

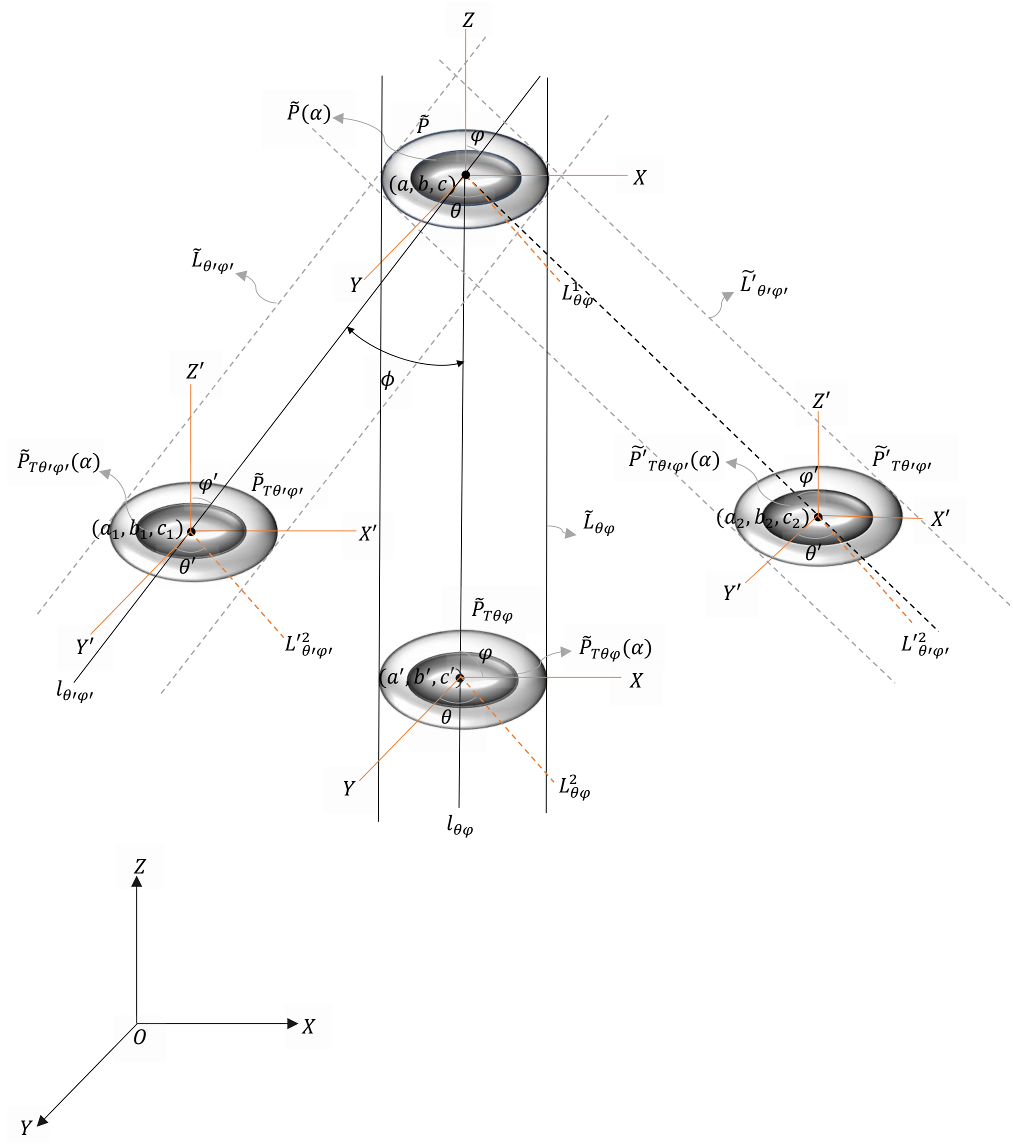

FIGURE 4.6: Fuzzy cone

This yields the point $(\frac{1}{2}, 2, \sqrt{\frac{17}{4}})$ on $x^2 + y^2 = z^2$ which is at the minimum distance from $(1, 4, 0)$. Note that the point $(\frac{1}{2}, 2, \sqrt{\frac{17}{4}})$ will be the core point of the translated fuzzy point $\widetilde{P}_T$ at which the point $(1, 4, 0)$ belongs. The fuzzy point $\widetilde{P}_T(\frac{1}{2}, 2, \sqrt{\frac{17}{4}})$ is the translated fuzzy point along the direction $(\frac{1}{2}, 2, \sqrt{\frac{17}{4}})$ with the membership function

$$\mu\left((x,y,z)\middle|\widetilde{P}_T\left(\tfrac{1}{2},2,\sqrt{\tfrac{17}{4}}\right)\right)$$
$$=\begin{cases}1-\frac{1}{4}\sqrt{(x-\frac{1}{2})^2+(y-2)^2+(z-\sqrt{\frac{17}{4}})^2} & \text{if } (x-\frac{1}{2})^2+(y-2)^2+(z-\sqrt{\frac{17}{4}})^2\leq 16\\ 0 & \text{otherwise.}\end{cases} \tag{4.8}$$

By (4.8),

$$\mu\left((1,4,0)\middle|\widetilde{P}_T\left(\tfrac{1}{2},2,\sqrt{\tfrac{17}{4}}\right)\right)=0.2712.$$

Next we define the notion of convex fuzzy cone.

Definition 4.3.2. A fuzzy cone $\widetilde{\mathscr{C}}$ (according to Definition 4.3.1) is a convex fuzzy cone if all the crisp cones $C\in\widetilde{\mathscr{C}}(\alpha)$ are convex, i.e., $\lambda_1x_1+\lambda_2x_2\in C$ for any $x_1,x_2\in C$ and $\lambda_1,\lambda_2\geq 0$.

Theorem 4.3.1. A fuzzy set $\widetilde{\mathscr{C}}$ is a convex fuzzy cone if and only if it is convex and $\widetilde{\mathscr{C}}(\lambda v)\geq\widetilde{\mathscr{C}}(v)$, $\forall v\in\mathbb{R}^3$ and $\lambda>0$.

Proof. Let $\widetilde{\mathscr{C}}$ be a convex fuzzy cone. We need to prove that it is convex and $\widetilde{\mathscr{C}}(\lambda v)\geq\widetilde{\mathscr{C}}(v)$, $\forall v\in\mathbb{R}^3$ and $\lambda>0$. Clearly, it is convex by Definition 4.3.2. Now, without loss of generality, we can write any arbitrary element $\lambda v\in\widetilde{\mathscr{C}}(0)$ as $\theta\lambda_1 v\in\widetilde{\mathscr{C}}(0)$, where $\lambda\geq 0$, $\lambda_1\geq 0$ and $\theta\in[0,1]$. Let us assign $\lambda_1 v$ as v'. Since $\widetilde{\mathscr{C}}$ is convex and $\theta\geq 0$,

$$\widetilde{\mathscr{C}}(\theta\lambda_1 v)\geq\theta\widetilde{\mathscr{C}}(\lambda_1 v)\geq\widetilde{\mathscr{C}}(\lambda_1 v).$$

This implies $\widetilde{\mathscr{C}}(\theta v')\geq\widetilde{\mathscr{C}}(v')$. Replace v' by v, the proof of this part is done.

Conversely, let $\widetilde{\mathscr{C}}$ be convex and $\widetilde{\mathscr{C}}(\lambda v)\geq\widetilde{\mathscr{C}}(v)$, $\forall v\in\mathbb{R}^3$ and $\lambda>0$. We have to prove $\widetilde{\mathscr{C}}$ is a convex fuzzy cone. Consider $v_1,v_2\in\widetilde{\mathscr{C}}(\alpha)$ such that $\widetilde{\mathscr{C}}(v_1),\widetilde{\mathscr{C}}(v_2)\geq\alpha$ for some $\alpha\in[0,1]$. By Definition 4.3.2, it is sufficient to prove that $\widetilde{\mathscr{C}}(\lambda_1v_1+\lambda_2v_2)\geq$

α, for $\lambda_1, \lambda_2 \geq 0$. It is easy to note that we can write $\lambda_1 = \theta_1 \lambda_1'$ and $\lambda_2 = \theta_2 \lambda_2'$, for $\theta_1, \theta_2 \in [0, 1]$. Since $\widetilde{\mathscr{C}}$ is convex,

$$\widetilde{\mathscr{C}}(\theta_1 \lambda_1' v_1 + \theta_2 \lambda_2' v_2) \geq \min\{\widetilde{\mathscr{C}}(\lambda_1' v_1), \widetilde{\mathscr{C}}(\lambda_2' v_2)\} \geq \alpha,$$

for some $\theta_1, \theta_2 \in [0, 1]$. Clearly, $\lambda_1 v_1 + \lambda_2 v_2 \in \widetilde{\mathscr{C}}(\alpha)$. This completes the proof. □

It is needed to give attention to fuzzy conics obtained by cutting a fuzzy cone by a crisp plane. The surface $\widetilde{\mathscr{C}}(0)$ is called the nappe. There are four types of non-degenerated fuzzy conics depending on how a crisp plane intersects the nappe. The types of non-degenerated fuzzy conics are the following:

(i) *Fuzzy parabola*: If the angle between the crisp plane and the fixed line is the same as the vertex angle.

(ii) *Fuzzy Circle*: If the angle between the crisp plane and the fixed line is right angle.

(iii) *Fuzzy ellipse*: If the angle between the crisp plane and the fixed line is greater than the vertex angle.

(iv) *Fuzzy hyperbola*: If the angle between the crisp plane and the fixed line is less than the vertex angle.

Note 14. The crisp plane must intersect the core cone of $\widetilde{\mathscr{C}}$ since the fuzzy conics are normal fuzzy sets. Also, in [3, 4], it is perceived that fuzzy circles and fuzzy parabolas are a collection of fuzzy points.

Some cases may arise for the degenerated fuzzy conics, i.e., the fuzzy conics that satisfy the requirement for a fuzzy parabola, a fuzzy circle, a fuzzy ellipse, and a fuzzy hyperbola, but do not form those fuzzy conics. There are three types of degenerated fuzzy conics as follows.

(i) *Fuzzy point*: If the crisp plane intersects the fuzzy cone at the vertex and at an angle greater than the vertex angle.

(ii) *Fuzzy line*: If the crisp plane intersects the fuzzy cone at the vertex and at an angle equal to the vertex angle.

(iii) *Two intersecting fuzzy lines*: If the crisp plane intersects the fuzzy cone at the vertex and at an angle less than the vertex angle.

Note 15. The crisp plane must pass through the core point of the vertex $\widetilde{P}$ since the fuzzy points and the fuzzy lines are normal fuzzy sets.

Now we discuss the construction of the membership functions of the fuzzy conics, say $\widetilde{FC}$. Consider a fuzzy point $\widetilde{P}(0,0,0)$ and a fixed fuzzy line $\widetilde{L}_{\theta}\varphi$ generated by a translation copies of $\widetilde{P}(0,0,0)$ along the z-axis. Let $l_{\theta'\varphi'}$ be a line that makes an angle ϕ with the z-axis and $\widetilde{L}_{\theta'\varphi'}$ be a fuzzy line generated by the translation copies of $\widetilde{P}(0,0,0)$ along the direction $l_{\theta'\varphi'}$. The surface of revolution of $\widetilde{L}_{\theta'\varphi'}(0)$ forms the fuzzy cone $\widetilde{\mathscr{C}}$. The equation of the core cone $\mathscr{C}$ of the fuzzy cone $\widetilde{\mathscr{C}}$ is $x^2 + y^2 = z^2 \tan\phi$.

Let $\mathit{\Pi}$ be a crisp plane making an angel β with the z-axis. Note that the plane $\mathit{\Pi}$ intersects the core cone $\mathscr{C}$. The intersection of cone $\mathscr{C}$ and plane $\mathit{\Pi}$ gives the core of the fuzzy conics $\widetilde{FC}$. It is easy to classify the conics as a parabola, an ellipse and a hyperbola according to $\beta = \phi$, $\beta > \phi$ and $\beta < \phi$, respectively, in the classical geometry.

Let FC be the core of the fuzzy conics $\widetilde{FC}$. For any point $(a,b,c) \in FC$, there must be a fuzzy line $\widetilde{L}_{\theta'\varphi'}(0)$ such that the intersection $\widetilde{L}_{\theta'\varphi'}(0) \bigcap \mathit{\Pi}$ will be a fuzzy point in the $\widetilde{FC}$ whose core is (a, b, c). Next, vary the point (a, b, c) over the core conic, this will form a collection of fuzzy points $\widetilde{P}(a, b, c) = \widetilde{L}_{\theta'\varphi'}(0) \bigcap \mathit{\Pi}$ which constitute the fuzzy conic $\widetilde{FC}$. The membership function of $\widetilde{FC}$ can be defined by

$$\mu\left((x, y, z)\middle|\widetilde{FC}\right) = \begin{cases} \mu\left((x, y, z)\middle|\widetilde{L}_{\theta'\varphi'}\right) & \text{if } (x, y, z) \in \widetilde{L}_{\theta'\varphi'} \bigcap \mathit{\Pi} \\ 0 & \text{otherwise.} \end{cases}$$

Note that each fuzzy point in $\widetilde{FC}$ will be translation copies of each other by the Definition of 4.3.1. The classification of core conic classifies the fuzzy conic as a fuzzy parabola, a fuzzy ellipse, and a fuzzy hyperbola. This is because each fuzzy point in $\widetilde{FC}$ is obtained by the rigid translation of each other along with the core conic.

4.4 Discussion and comparison

Some properties of the fuzzy spheres and the fuzzy cones have been dealt with in this study. These properties are as follows.

1. A fuzzy sphere is a collection of fuzzy points that are equidistant from a given fuzzy point. Under some restriction (Theorem 4.2.1), there always exists a fuzzy point at a preset fuzzy distance from a fixed fuzzy point.

2. The angle between the fuzzy line segments joining any fuzzy point on the fuzzy sphere with the extremities of the fuzzy diameter is a right angle (see Theorem 4.2.4).

3. The radius of the fuzzy sphere passing through four fuzzy points is a fuzzy number unless the same points of the fuzzy points are co-planar (see Figure 4.4).

4. All the different forms of fuzzy spheres (see Definitions 4.2.1, 4.2.5, 4.2.7) can be perceived as a union of the crisp spheres with varied membership values or a collection of space fuzzy points.

5. There always exists a crisp sphere S with $\mu\left(S\middle|\widetilde{S}\right) = 1$, i.e., $\widetilde{S}$ is a normal fuzzy set.

6. The α-cuts of the fuzzy spheres (see Definitions 4.2.1, 4.2.5, 4.2.7) and fuzzy cones (see Definition 4.3.1) are closed, connected and arc-wise connected but not necessarily convex.

7. The convex fuzzy cone is a fuzzy cone in which all the crisp cones are convex (see Definition 4.3.2).

8. A fuzzy cone is a normal fuzzy set whose core is a crisp cone.

9. The crisp plane sections of a fuzzy cone are fuzzy conics (fuzzy circle, fuzzy ellipse, fuzzy parabola, fuzzy hyperbola) (see Section 4.3).

In the introduction (see Subsection 1.4.1), it is noted that there are several papers based on the fuzzy spheres [50, 51, 52] and fuzzy cones [53, 54, 55, 56], in which these are not well defined and do not explicitly interpret the geometrical view. There are a few papers on the fuzzy space geometrical elements in $\mathbb{R}^3$. Only Qiu and Zhang [7] focused on the fuzzy space geometrical entities (space fuzzy lines and fuzzy planes) by extending the concepts of [5, 6]. They used sup-min compositions [15] for the well-known algebraic equations of lines, planes, etc., with the coefficients as fuzzy numbers [7]. Thus, as per the approach of [7], we give a comparison primarily based on Qiu and Zhang. A point-wise comparison is included below.

(i) ***Fuzzy sphere.***

As per the approach of [21], a fuzzy sphere is a fuzzy set in which the membership value of a point depends on its distance from a reference point in $\mathbb{R}^3$. The level sets of the fuzzy sphere are concentric spheres. Thus, the center of the fuzzy sphere would be a crisp point. Also, according to [20], the center of the fuzzy sphere is a crisp point. The definitions of the fuzzy spheres (according to [20, 21]) are not suitable since, in the general definition, a fuzzy point instead of a crisp point would have been more appropriate to represent a center of a fuzzy sphere. The core of the fuzzy sphere, according to [32, 33], is not a crisp sphere. However, the proposed definitions of the fuzzy spheres (see Definitions 4.2.1, 4.2.5, 4.2.7) are normal fuzzy sets, and the core of the fuzzy spheres are crisp spheres.

It may be noted that the ideas used in [7] are simply extensions of the formulated fuzzy geometrical entities in [6]. The sup-min compositions are also employed in [7] by increasing the number of variables from two to three. All the deficiencies of the fuzzy geometrical elements [6] are detailed in [3]. Furthermore, the proposed concepts are similar to the concepts studied in [3] when extending the number of variables from two to three. Also, the proposed Method 1 and Method 3 for fuzzy spheres behave similarly to the approaches delineated for the fuzzy circles in [3]. Hence, by [3], it is easy to note that the proposed methods are more rigorous than the formulations in [7].

Explicitly, as per the approach of [7], the fuzzy sphere $\widetilde{S}$ can be defined as

$$\widetilde{S} = \bigvee_{\alpha\in[0,1]} \{S : \text{ where } S \text{ is a sphere with center in } \widetilde{C}(\alpha) \text{ and radius in } \widetilde{R}(\alpha)\}.$$

The definition of a fuzzy sphere given by Qiu and Zhang [7] does not qualify the condition of the customary definition of a sphere, such as the collection of points that are at a fixed distance from a given fixed point. On the other hand, the proposed definition of the fuzzy sphere (Method 1) is the extension of the classical definition of the sphere in the fuzzy environment. For the apparent view, we can consider Example 4.2.2, where the fuzzy point $\widetilde{P}_1(1, 0, -1)$ with the membership function

$$\mu\left((x, y, z)\middle|\widetilde{P}_1\right) = 1 - \sqrt{(x-1)^2 + y^2 + (z+1)^2},$$

and the fuzzy number $\widetilde{d} = (5.1140/7.8102/10.5101)_{LR}$ are given. Note that the fuzzy distance $\widetilde{D}\left(\widetilde{\phi_1^{\theta\varphi}}, \widetilde{\phi_2^{\theta\varphi}}\right) = \widetilde{d}$, where $\widetilde{\phi_1^{\theta\varphi}}$ and $\widetilde{\phi_2^{\theta\varphi}}$ are fuzzy numbers along the line L in Example 4.2.2. However, as per the approach of Qiu and Zhang [7], $\widetilde{D}\left(\widetilde{\phi_1^{\theta\varphi}}, \widetilde{\phi_2^{\theta\varphi}}\right) = (1/7.8102/12.5101) \neq \widetilde{d}$. Here,

$$\widetilde{\phi_1^{\theta\varphi}} = ((1-a, -\lambda b, -1-c)/(1, 0, -1)/(1+a, b, -1+c))$$

and

$$\widetilde{\phi_1^{\theta\varphi}} = ((1, 0, -1)/(1 + 7.8102a, 7.8102b, -1 + 7.8102c)/(1 + 11.5101a, 11.5101b, -1 + 11.5101c))$$

are the fuzzy numbers along the line L, where $a = \sin\varphi\cos\theta$, $b = \sin\varphi\sin\theta$ and $c = \cos\varphi$.

The proposed Method 2 and Method 3 to construct the fuzzy spheres considered only the combinations of the same points of the fuzzy points. While in contrast, Qiu and Zhang considered all the points on the α-cuts of the fuzzy points to define the fuzzy spheres. The constraint set in the proposed methods is a subset of that given by Qiu and Zhang. Hence, the proposed methods have less computational cost and smaller spread than that of Qiu and Zhang.

(ii) ***Fuzzy cone.*** As per the approach of [117], a fuzzy cone can be defined as

$$\mu\left((x,y,z)\middle|\widetilde{\mathscr{C}}\right) = \sup\{\alpha:\ (x,y,z)\in\mathscr{C}_\psi^\alpha,\ \text{where } \mathscr{C}_\psi^\alpha \text{ is a cone generated by a line making angle } \psi \text{ with the line joining the same points of } \widetilde{P}_1(0) \text{ and } \widetilde{P}_2(0) \text{ with the membership value } \alpha\}, \tag{4.9}$$

$\widetilde{\psi}$ is a given fuzzy number and $\mu\left(\psi\middle|\widetilde{\psi}\right)=\alpha$.

Also, as per the approach of [7], a fuzzy cone can be defined as

$$\mu\left((x,y,z)\middle|\widetilde{\mathscr{C}}\right) = \sup\{\alpha:\ (x,y,z)\in\mathscr{C}_\psi^\alpha,\ \text{where } \mathscr{C}_\psi^\alpha \text{ is a cone generated by a line making angle } \psi \text{ with the line joining the points of } \widetilde{P}_1(\alpha) \text{ and } \widetilde{P}_2(\alpha),\ \psi\in\widetilde{\psi}(0)\}, \tag{4.10}$$

and $\widetilde{\psi}$ is a given fuzzy number.

As per our discussion made previously (see p. 193), the Equations (4.9) and (4.10) are not appropriate since it contradicts the elementary property of the fuzzy sets such that $\widetilde{\mathscr{C}}(1)\nsubseteq\widetilde{\mathscr{C}}(\alpha)$, for $0\leq\alpha<1$ (depicted in Figure 4.5). Hence, none of both definitions is an appropriate idea to define a fuzzy cone. In contrast, the proposed Definition 4.3.1 does not have such type of deficiency. For a given fixed fuzzy point and a fixed fuzzy axis, Definition 4.3.1 defines a fuzzy cone as a collection of fuzzy lines making a constant angle with the fixed fuzzy axis. The proposed Definition 4.3.1 of the fuzzy cone is well defined and interpreted geometrically.

4.5 Conclusion

This paper includes three different methods to construct the fuzzy sphere. Method 1 deals with the formulation of the fuzzy sphere when the center and radius of a sphere given imprecisely. This formulation of the fuzzy sphere is based on the study that there is a fuzzy number at a predetermined distance from a given fuzzy number. Method 2 depends on the diameter of a fuzzy sphere. The diameter of the fuzzy sphere is the fuzzy line segment joining two continuous fuzzy points. We have given two methodologies to define the diameter form of a fuzzy sphere. At first, the construction of the fuzzy sphere depends on the translation of fuzzy points along the perpendicular directions passing through the core points of the fuzzy points. In second, we have extended the conventional definition of the classical diameter form of a sphere. Method 3 defines the fuzzy sphere passing through the four S-type space fuzzy points whose core points are not co-planar. Importantly, we have proved that there is a unique sphere passing through the four S-type space fuzzy points. An extensive idea of a fuzzy cone is presented thereafter. Relevantly, we have delineated the intersection of a fuzzy sphere and a fuzzy cone with a crisp plane. Sequentially, the properties of the fuzzy sphere and the fuzzy cone are also discussed.

In the future, we may also focus on a detailed analysis of the properties of the fuzzy conic sections (fuzzy parabola, fuzzy ellipse and fuzzy hyperbola). The proposed fuzzy space geometrical concepts can be used to analyze fuzzy optimization problems. We will work on this in the future.

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