Chapter 3

Analytical fuzzy space geometry II

3.1 Introduction

This paper continues our research on fuzzy space geometry [117]. In [117], some basic ideas of fuzzy space geometry, such as an S-type space fuzzy point, the theory of same and inverse points, etc., have been defined. Also, these basic ideas are used to investigate fuzzy distance and space fuzzy line segment in [117]. The ideas concerning to S-type space fuzzy point and same and inverse points are the backbone of our study in fuzzy space geometry (see [117]). In this paper, we carry on with our research in fuzzy space geometry to formulate fuzzy lines and fuzzy planes in $\mathbb{R}^3$-space.

3.1.1 Motivation and novelty

It is clear from Subsection 1.4.3 that fuzzy geometry has been successfully applied to many areas, such as fuzzy linear programming, fuzzy medical imaging, fuzzy geometrical object detection, fuzzy extrapolation or interpolation, etc.

In [1, 2, 3], basic concepts of the same and inverse points of fuzzy points, and some fuzzy plane geometrical elements have been investigated. These concepts gave a new direction to develop fuzzy plane geometry. The effect of the same and inverse points to decrease the computational cost can be observed in [87]. Further, Ghosh et al.

[117] extended the idea of same and inverse points in fuzzy space geometry. Also, in [117], some fuzzy geometrical entities, such as fuzzy distance and space fuzzy line segment, are formulated with the help of same and inverse points in $\mathbb{R}^3$. In continuation of this study, from the literature of fuzzy lines and fuzzy planes (see Subsection 1.4.1), we observe that a detailed construction and computation of fuzzy lines and planes are yet to be rigorously done. Also, from the application viewpoint, we notice that the computation of fuzzy lines and fuzzy planes is essential. Thus, a consecutive study on fuzzy space geometry is needed.

Hence, in this article, we propose a detailed construction of space fuzzy lines, the shortest distance between skew fuzzy lines, and fuzzy planes based on the concept of the same and inverse points of space fuzzy points.

The main contribution and novelty of the present study are as follows:

(i) We define a space fuzzy line passing through two continuous S-type space fuzzy points. Particularly, we also formulate symmetric fuzzy lines ($\widetilde{L}_S$). The membership value of a point $P \in \widetilde{L}_S(0)$ depends on the perpendicular distance from P to a fixed line. One can note that the α-cut of the symmetric fuzzy line is a right circular cylinder (see Figure 3.1).

(ii) We investigate the concept of skew fuzzy lines and the shortest distance between symmetric skew fuzzy lines. The proposed space fuzzy line and the shortest distance between symmetric skew fuzzy lines are based on S-type representation of fuzzy points.

(iii) With the help of the explicit expression of same and inverse points, we provide algorithms to evaluate the membership grade of a point in the space fuzzy lines (Algorithm 3.2.1), to execute the shortest distance between symmetric skew fuzzy lines (Algorithm 3.3.1), and to evaluate the membership value of a number in the shortest distance between symmetric skew fuzzy lines (Algorithm 3.3.2).

(iv) We analyze the intersection of two space fuzzy lines (see Figure 3.2).

(v) We define three different forms of fuzzy planes—a three-point form ($\widetilde{\Pi}_{3P}$), an intercept form ($\widetilde{\Pi}_I$), and a fuzzy plane passing through an S-type space fuzzy point and perpendicular to a given crisp direction ($\widetilde{\Pi}_{Pn}$).

(vi) Using the explicit expression of same and inverse points, we develop the step-wise procedure to evaluate the membership grade of a number in the three-point form of the fuzzy plane (Algorithm 3.4.1), to compute the membership value of a number in the intercept form of the fuzzy plane (Algorithm 3.4.2), and to evaluate the membership value of a number in the fuzzy plane that passes through an S-type space fuzzy point and perpendicular to a given crisp direction (Algorithm 3.4.3).

(vii) With the help of an LR-type fuzzy number and direction cosines of normal (l, m, n) to the given plane, we provide the symmetrical form of a fuzzy plane, namely, a symmetric fuzzy plane.

(viii) We introduce the angle between two fuzzy planes, and the fuzzy distance between a fuzzy point and a fuzzy plane. Table 3.7 represents the α-cuts of the angle between two fuzzy planes evaluated for Example 3.4.8. Also, Table 3.8 represents the α-cuts of the fuzzy distance between a fuzzy point and a fuzzy plane evaluated for Example 3.4.9.

In the following section, we construct space fuzzy lines and their geometric properties.

3.2 Space fuzzy line

Euclid's second postulate says that we can extend a straight line segment bi-infinitely into a straight line. Analogously, we propose that a space fuzzy line is a bi-infinite extension of the space fuzzy line segment joining two continuous space fuzzy points. To construct a space fuzzy line, we first formulate a space fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2}$ joining S-type space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$. Next, we introduce the concept of semi-infinite space fuzzy line segments $\widetilde{\widetilde{L}}_{1\infty}$ and $\widetilde{\widetilde{L}}_{2\infty}$ as in [2] (p. 88). The semi-infinite space fuzzy line segments $\widetilde{\widetilde{L}}_{1\infty}$ and $\widetilde{\widetilde{L}}_{2\infty}$ are two bunches of half-lines with varied membership values and the half-lines must be parallel to the core line. The semi-infinite space fuzzy line segments ($\widetilde{\widetilde{L}}_{i\infty},\ i = 1, 2$) is evaluated by

$$\mu\left((x, y, z)\middle|\widetilde{\widetilde{L}}_{i\infty}\right) = \sup_{(u,v,w)\in l(x,y,z)\cap\widetilde{P}_i(0)} \mu\left((u, v, w)\middle|\widetilde{P}_i\right), \tag{3.1}$$

where $l(x, y, z)$ is a line passing through (x, y, z) and the direction ratios are same as the direction ratios of the core line $\widetilde{\widetilde{L}}_{P_1P_2}(1)$, for $i = 1, 2$.

The mathematical form of the space fuzzy line $\widetilde{L}$ is

$$\widetilde{L} = \widetilde{\widetilde{L}}_{1\infty} \cup \widetilde{\widetilde{L}}_{P_1P_2} \cup \widetilde{\widetilde{L}}_{2\infty}. \tag{3.2}$$

To find the membership grade of a point $(x, y, z) \in \widetilde{L}(0)$ we have to find whether (x, y, z) belongs to $\widetilde{\widetilde{L}}_{i\infty}$ or $\widetilde{\widetilde{L}}_{P_1P_2}$, for $i = 1, 2$. Since if $(x, y, z) \in \widetilde{\widetilde{L}}_{i\infty}$, then the membership grade is given by (3.1) and if $(x, y, z) \in \widetilde{\widetilde{L}}_{P_1P_2}$, then the membership grade is given by the Definition 2.5.1. For finding the membership grade at a point in $(x, y, z) \in \widetilde{L}(0)$, a step-wise procedure is given in the following Algorithm 3.2.1.

Algorithm 3.2.1: Algorithm to find $\mu\left((x,y,z)\middle|\widetilde{L}\right)$

Input: Given two continuous S-type space fuzzy points $\widetilde{P}_i(a_i,b_i,c_i)$ with the membership functions $f_i(x-a_i,y-b_i,z-c_i)$, for $i=1,2$. We denote

$$\phi_i(\lambda_i)=f_i\left(\lambda_i\sin\varphi\cos\theta,\lambda_i\sin\varphi\sin\theta,\lambda_i\cos\varphi\right)\ \text{for}\ \lambda_i\geq 0, i=1,2.$$

The membership value $\mu\left((x,y,z)\middle|\widetilde{L}\right)$ for a given (x,y,z) has to be calculated.

Output: The membership value $\mu\left((x,y,z)\middle|\widetilde{L}\right)=\alpha_{\text{sup}}$.

```
Initialize α_sup ← 0
loop:
For α = 0 to 1; step size δα
  For θ = 0 to 2π; step size δθ
    For φ = 0 to π; step size δφ
      Compute
      φ_1^{-1}(α) = λ_1
      φ_2^{-1}(α) = λ_2
      Compute the same points
      (u^1)^α_{θφ} = (a_1 + (sin φ cos θ)φ_1^{-1}(α), b_1 + (sin φ sin θ)φ_1^{-1}(α), c_1 + (cos φ)φ_1^{-1}(α))
      (u^2)^α_{θφ} = (a_2 + (sin φ cos θ)φ_2^{-1}(α), b_2 + (sin φ sin θ)φ_2^{-1}(α), c_2 + (cos φ)φ_2^{-1}(α))
      For t = 0 to 1; step size δt
        if (x, y, z) = t (u^1)^α_{θφ} + (1 − t) (u^2)^α_{θφ} then
          if α_sup < α then
            α_sup ← α
          else
            goto loop
          end
        end
      end
      For λ'_1 = −N to 0; step size δλ'_1
        if (x, y, z) = (u^1)^α_{θφ} + λ'_1(a_1 − a_2, b_1 − b_2, c_1 − c_2) then
          if α_sup < α then
            α_sup ← α
          else
            goto loop
          end
        end
      end
```

For $\lambda_2' = 0$ to N; step size $\delta\lambda_2'$
    if $(x, y, z) = \left(u^2\right)^{\alpha}_{\theta\varphi} + \lambda_2'(a_1 - a_2, b_1 - b_2, c_1 - c_2)$ **then**
        if $\alpha_{\text{sup}} < \alpha$ **then**
            $\alpha_{\text{sup}} \leftarrow \alpha$
        else
            goto loop
        end
    end
end
end
end
end
return $\mu\left((x, y, z)\middle|\widetilde{L}\right) = \alpha_{\text{sup}}$

Next example illustrates how to evaluate the membership grade of a point on the support of a space fuzzy line by Algorithm 3.2.1.

Example 3.2.1. (*Evaluation of the membership values in* $\widetilde{L}(0)$). Let $\widetilde{P}_1(0, 0, 0)$ and $\widetilde{P}_2(3, -2, 1)$ be two continuous fuzzy points with the membership functions

$$\mu\left((x, y, z)\middle|\widetilde{P}_1(0, 0, 0)\right) = \begin{cases} 1 - \left(\left|\frac{x}{2}\right| + \left|\frac{y}{2}\right| + \left|\frac{z}{2}\right|\right) & \text{if } |x| + |y| + |z| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x, y, z)\middle|\widetilde{P}_2(3, -2, 1)\right)$$
$$= \begin{cases} 1 - \left(\left|\frac{x-3}{5}\right| + \left|\frac{y+2}{5}\right| + \left|\frac{z-1}{5}\right|\right) & \text{if } |x - 3| + |y + 2| + |z - 1| \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

The general expression of the same points on $\widetilde{P}_1(0,0,0)$ and $\widetilde{P}_2(3,-2,1)$ with $\alpha \in [0,1]$ are

$$(u^1)^{\alpha}_{\theta\varphi} : \left(\frac{2(1-\alpha)\sin\varphi\cos\theta}{R_{\theta\varphi}}, \frac{2(1-\alpha)\sin\varphi\sin\theta}{R_{\theta\varphi}}, \frac{2(1-\alpha)\cos\varphi}{R_{\theta\varphi}}\right)$$

and

$$(u^2)^{\alpha}_{\theta\varphi} : \left(3+\frac{5(1-\alpha)\sin\varphi\cos\theta}{R_{\theta\varphi}}, -2+\frac{5(1-\alpha)\sin\varphi\sin\theta}{R_{\theta\varphi}}, 1+\frac{5(1-\alpha)\cos\varphi}{R_{\theta\varphi}}\right),$$

respectively, where $R_{\theta\varphi} = |\sin\varphi\cos\theta| + |\sin\varphi\sin\theta| + |\cos\varphi|$.

Table 3.1 shows the membership values of some points in the space fuzzy line $\widetilde{L}(0)$, obtained by the Algorithm 3.2.1.

(x, y, z)	Membership value	Step sizes
$(-28.0000, -27.7750, 55.7750)$	0.7750	$\delta\alpha = 0.2250$, $\delta\theta = 1.5708$, $\delta\varphi = 0.7854$,
		$\delta t = 0.1000$, $\delta\lambda_1' = 1$, $\delta\lambda_2' = 1$ and $N = 15$
$(0.3804, -0.0000, -0.2196)$	0.7000	$\delta\alpha = 0.3000$, $\delta\theta = 2.0944$, $\delta\varphi = 1.0472$,
		$\delta t = 0.1000$, $\delta\lambda_1' = 1$, $\delta\lambda_2' = 1$ and $N = 15$
$(15.0000, 10.0000, 25.2500)$	0.5500	$\delta\alpha = 0.4500$, $\delta\theta = 3.1416$, $\delta\varphi = 6.2832$,
		$\delta t = 0.1000$, $\delta\lambda_1' = 1$, $\delta\lambda_2' = 1$ and $N = 15$

Table 3.1: Membership values of some points of $\widetilde{L}(0)$ produced by Algorithm 3.2.1 for Example 3.2.1

Theorem 3.2.1. A space fuzzy line that passes through n space fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$, $\ldots$, $\widetilde{P}_n$ is unique.

Proof. Similar to theorem 3.4.1 (see Chapter 3) in [4]. □

Theorem 3.2.2. Let $\widetilde{L}$ be a space fuzzy line and Π be a plane perpendicular to $\widetilde{L}(1)$. The intersection $\Pi \bigcap \widetilde{L}$ is a space fuzzy point on $\widetilde{L}(0)$.

Proof. Let $\mathit{\Pi}$ be a plane perpendicular to $\widetilde{L}(1)$ and represented by

$$\left\{(x, \mu\,(x|\mathit{\Pi})) : x \in \mathbb{R}^3\right\},$$

where $\mu\,(x|\mathit{\Pi})$= 1 for $x \in \mathit{\Pi}$, and 'zero' otherwise. Define a fuzzy set

$$\widetilde{P} = \left\{\left(x, \mu\left(x\middle|\widetilde{L}\right)\right) : x \in \mathbb{R}^3\right\} \bigcap \left\{(x, \mu\,(x|\mathit{\Pi})) : x \in \mathbb{R}^3\right\}, \tag{3.3}$$

where the membership value is evaluated by the t-norm as 'min'. We have to prove that $\widetilde{P}$ is a space fuzzy point on $\mathit{\Pi}$.

For $0 < \alpha \leq 1$, let $\bar{P}_\alpha$ be the plane curve which is the intersection of $\mathit{\Pi}$ with $f(x, y, z; \alpha) = 0$, where $f(x, y, z; \alpha) = 0$ is the boundary surface of $\widetilde{L}(\alpha)$. Let P be a point of intersection of $\mathit{\Pi}$ and $\widetilde{L}(1)$. Consider a set $P_{u_\alpha v_\alpha}$ which is a collection of line segments $\overline{u_\alpha v_\alpha}$ passing through the point P, where u_α, $v_\alpha \in \bar{P}_\alpha$, i.e.,

$$\begin{aligned} P_{u_\alpha v_\alpha} = \{\overline{u_\alpha v_\alpha} : \overline{u_\alpha v_\alpha} = \lambda u_\alpha + (1-\lambda)v_\alpha,\ &\text{where } u_\alpha, v_\alpha \in \bar{P}_\alpha \text{ such that } \overline{u_\alpha v_\alpha} \text{ passes} \\ &\text{through } P,\ \text{ for } 0 \leq \lambda \leq 1\}. \end{aligned} \tag{3.4}$$

Now, we will show that $\widetilde{P}(\alpha)$ is $P_{u_\alpha v_\alpha}$. Let us suppose there exist $P' = \lambda u_\alpha + (1 - \lambda)v_\alpha \in P_{u_\alpha v_\alpha}$, for some u_α, $v_\alpha \in \bar{P}_\alpha$ and $\lambda \in [0, 1]$, but $P' \notin \widetilde{P}(\alpha)$. So $\mu\left(P'\middle|\widetilde{L}\right) < \alpha$ by (3.3). Therefore, $P' \notin \widetilde{L}(\alpha)$ which arises a contradiction that α-cuts of $\widetilde{L}$ are closed convex sets and, for $0 < \alpha < \gamma \leq 1$, $\widetilde{L}(\gamma) \subseteq \widetilde{L}(\alpha)$. Thus, $\widetilde{P}(\alpha) = P_{u_\alpha v_\alpha}$ is compact and convex set and, for $0 < \alpha < \gamma \leq 1$, $\widetilde{P}(\gamma) \subseteq \widetilde{P}(\alpha)$. Therefore, $\mu\left(\cdot\middle|\widetilde{P}\right)$ is upper semi-continuous.

Now, since P is the point of intersection of $\mathit{\Pi}$ and $\widetilde{L}(1)$, then

$$\mu\left(P\middle|\widetilde{P}\right) = \min\left\{\mu\,(P|\mathit{\Pi})\,, \mu\left(P\middle|\widetilde{L}\right)\right\} = 1.$$

Thus, the fuzzy set $\widetilde{P}$ is a space fuzzy point on $\mathit{\Pi}$. □

Theorem 3.2.2 helps to visualize a space fuzzy line $\widetilde{L}$ as a set of space fuzzy points at every point of $\widetilde{L}(1)$. Thus, it may be noted that the space fuzzy lines can be observed as

(i) set of crisp points, or

(ii) union of space fuzzy points, or

(iii) union of crisp half-line segments.

The following theorem shows the unique characterization of the space fuzzy lines.

Theorem 3.2.3. A fuzzy set in $\mathbb{R}^3$ is a space fuzzy line if and only if along any plane perpendicular to the core line, there always exists a space fuzzy point on the space fuzzy line, where its core is a crisp straight line.

Proof. One side of the theorem is true by the Theorem 3.2.2. By (ii) (in p. 95), it is observed that the set of all the space fuzzy points along the plane perpendicular to the core line identifies a space fuzzy line. So, by the assumption of the theorem, the converse part is true. □

In the next subsection, the idea of symmetric fuzzy line has been initiated.

3.2.1 Symmetric fuzzy line

Given a crisp line passing through a point (α, β, γ) along the direction (l, m, n), i.e.,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}. \tag{3.5}$$

Let $P(x, y, z)$ be any point on $\mathbb{R}^3$ and the perpendicular distance of P from the given crisp line is r. The surface in which all the points are at a distance r from the crisp line (3.5) is given by

$$\{n(y-\beta)-m(z-\gamma)\}^2+\{l(z-\gamma)-n(x-\alpha)\}^2+\{m(x-\alpha)-l(y-\beta)\}^2 = r^2(l^2+m^2+n^2). \tag{3.6}$$

One can note that (3.6) is a right circular cylinder, which is symmetric about the line (3.5). With the help of (3.6) , we have given the concept of symmetric fuzzy line, denoted as $\widetilde{L}_S$. The line (3.5) is called the core line of $\widetilde{L}_S$.

The membership value of the symmetric fuzzy line $\widetilde{L}_S$ is defined below.

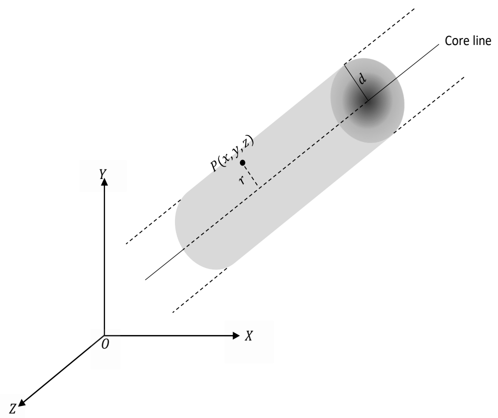

FIGURE 3.1: Symmetric fuzzy line

Definition 3.2.1. (*Symmetric fuzzy line*). Consider a crisp line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

and let $d \in \mathbb{R}$ be any given number. The symmetric fuzzy line $(\widetilde{L}_S)$ can be defined by its membership function as

$$\mu\left((x,y,z)\middle|\widetilde{L}_S\right)=\begin{cases}L\left(\frac{r}{d}\right) & \text{if } r\leq d\\ 0 & \text{if } r>d,\end{cases} \tag{3.7}$$

where r is the distance from (x,y,z) to the given crisp line, and L is the reference function.

Note that the distance of the point (x,y,z) from the line (3.5), r say, can be written as

$$r=\sqrt{\frac{\left(\{n(y-\beta)-m(z-\gamma)\}^2+\{l(z-\gamma)-n(x-\alpha)\}^2+\{m(x-\alpha)-l(y-\beta)\}^2\right)}{(l^2+m^2+n^2)}}.$$

The α-cuts of $\widetilde{L}_S$ can be formulated as

$$\widetilde{L}_S(\alpha)=\{(x,y,z):r\leq dL^{-1}(\alpha)\}. \tag{3.8}$$

According to Definition 3.2.1, a symmetric fuzzy line can be viewed as a collection of points inside or on a right circular cylinder which can be visualised by Figure 3.1. Figure 3.1 depicts that the support of $\widetilde{L}_S$ is a solid right circular cylinder of radius d. If $P(x,y,z)$ be a point in $\widetilde{L}_S(0)$ and the perpendicular distance of $P(x,y,z)$ from the given crisp line is $r\leq d$, then the membership value of $P(x,y,z)$ is evaluated by (3.7).

Example 3.2.2. (*Symmetric fuzzy line*). Let the core of the fuzzy line $(\widetilde{L}_S)$ be

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}, \tag{3.9}$$

and 5 be the radius of the support of $\widetilde{L}_S$. The membership function of $\widetilde{L}_S$ is

$$\mu\left((x,y,z)\middle|\widetilde{L}_S\right) = \begin{cases} L\left(\frac{r}{5}\right) & \text{if } r \leq 5 \\ 0 & \text{if } r > 5, \end{cases} \tag{3.10}$$

where r is the distance from (x,y,z) to the core line (3.9), and $L(x) = \max\{0, 1 - x\}$ is the reference function. Let us consider a point $P(1,1,1)$ in $\widetilde{L}_S(0)$. The perpendicular distance of $P(1,1,1)$ from the core line is 0.65. Evidently, the membership value $\mu\left((1,1,1)\middle|\widetilde{L}_S\right)$ is 0.87.

Observation 3.2.1. Let (x,y,z) be a point in the space fuzzy line $\widetilde{L} = \widetilde{\widetilde{L}}_{1\infty} \cup \widetilde{\widetilde{L}}_{P_1P_2} \cup \widetilde{\widetilde{L}}_{2\infty}$. The point (x,y,z) may belong to $\widetilde{\widetilde{L}}_{P_1P_2}$, or $\widetilde{\widetilde{L}}_{1\infty}$, or $\widetilde{\widetilde{L}}_{2\infty}$. Therefore, the membership value of (x,y,z) in $\widetilde{L}(0)$ is evaluated by the Algorithm 3.2.1 which take many steps. In contrast, to find the membership value of (x,y,z) in $\widetilde{L}_S$, we can easily calculate the distance between the point (x,y,z) and the core line of $\widetilde{L}_S$. Hence, the evaluation of the membership function in the symmetric fuzzy lines is easier than the space fuzzy lines.

The following observation is made regarding the intersection of two fuzzy lines.

Observation 3.2.2.

(i) The intersection of two space fuzzy lines may not be an S-type space fuzzy point as the intersection of the supports of two space fuzzy lines may not be a convex subset of $\mathbb{R}^3$. Figure 3.2 depicts such a case, where the intersection of $\widetilde{L}_1$ and $\widetilde{L}_2$ is not a convex subset of $\mathbb{R}^3$; $\widetilde{L}_1$ and $\widetilde{L}_2$ are space fuzzy lines that passes through two continuous fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$, and $\widetilde{P}_3$ and $\widetilde{P}_4$, respectively.

(ii) If the core lines of two symmetric fuzzy lines intersect, then their intersection is an S-type space fuzzy point since the α-cuts of symmetric fuzzy lines are closed, connected, and convex subsets of $\mathbb{R}^3$ (see Definition 3.2.1).

To illustrate (ii), let $\widetilde{L}_{S_1}$ and $\widetilde{L}_{S_2}$ be two symmetric fuzzy lines whose core lines are z-axis and y-axis, respectively. The membership functions of $\widetilde{L}_{S_i}$ are

$$\mu\left((x,y,z)\middle|\widetilde{L}_{S_i}\right) = \begin{cases} L\left(\frac{r}{d_i}\right) & \text{if } r \leq d_i \\ 0 & \text{if } r > d_i, \end{cases} \tag{3.11}$$

where r is the distance from (x, y, z) to the respective core lines, $d_i \in \mathbb{R}$, and $L(x) = \max\{0, 1 - x\}$ is the reference function, for $i = 1, 2$. The supports of the symmetric fuzzy lines $\widetilde{L}_{S_1}$ and $\widetilde{L}_{S_2}$ are

$$\{(x, y, z) : x^2 + y^2 = d_1^2\}$$

and

$$\{(x, y, z) : x^2 + z^2 = d_2^2\},$$

which are the right circular cylinders of radius d_1 and d_2, respectively. One can see that the intersection of these two symmetric fuzzy lines is a fuzzy point $\widetilde{P}$ with core at $(0, 0, 0)$. The support of $\widetilde{P}$ is bounded by the parametric curves

$$\gamma_1(t) = \left(d_1\cos(t), d_1\sin(t), \pm\sqrt{d_2^2 - d_1^2\cos^2(t)}\right)$$

and

$$\gamma_2(t) = \left(d_2\cos(t), \pm\sqrt{d_1^2 - d_2^2\cos^2(t)}, d_2\sin(t)\right),$$

where $0 \leq t \leq 2\pi$. Now, the space fuzzy point $\widetilde{P}$ is evaluated by the membership function

$$\mu\left((x,y,z)\middle|\widetilde{P}\right) = \min\left\{\mu\left((x,y,z)\middle|\widetilde{L}_{S_1}\right),\ \mu\left((x,y,z)\middle|\widetilde{L}_{S_2}\right)\right\}. \tag{3.12}$$

Apparently, let $d_1 = d_2 = 1$. Suppose we have to determine the membership grade at $(0.1000, 0.3000, 0) \in \widetilde{P}(0)$ by (3.12), where $\widetilde{P}$ is determined by the intersection of $\widetilde{L}_{S_1}$ and $\widetilde{L}_{S_2}$. The membership grade

$$\begin{aligned}\mu\left((0.1000, 0.3000, 0)\middle|\widetilde{P}\right) &= \min\left\{\mu\left((0.1000, 0.3000, 0)\middle|\widetilde{L}_{S_1}\right),\ \mu\left((0.1000, 0.3000, 0)\middle|\widetilde{L}_{S_2}\right)\right\}\\ &= \min\{0.6836, 0.9000\}\\ &= 0.6836.\end{aligned}$$

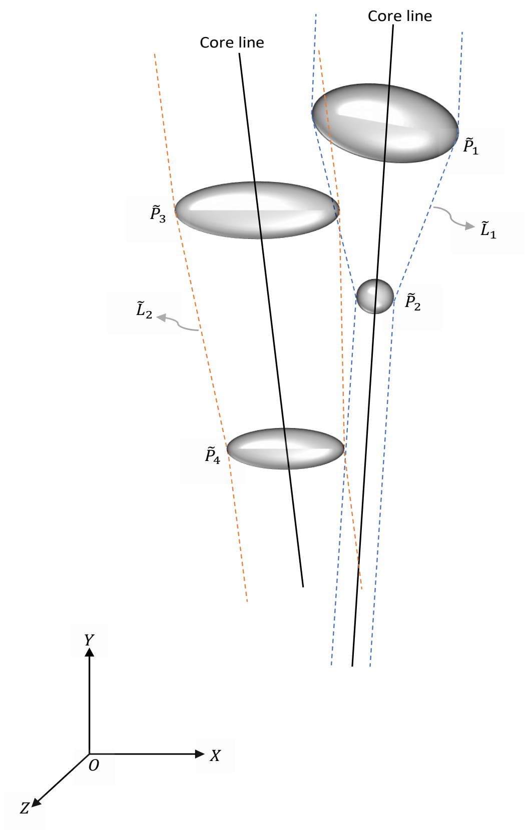

FIGURE 3.2: Intersection of two space fuzzy lines may not be a convex subset of $\mathbb{R}^3$

Note 7. In classical geometry, the intersection of two crisp lines gives a crisp point. In contrast, the intersection of two space fuzzy lines may not give an S-type space fuzzy point (see Figure 3.2).

In the next section, we give an idea to investigate skew fuzzy lines and the shortest distance between two skew fuzzy lines.

3.3 Shortest distance

We begin with a definition of skew fuzzy lines.

Definition 3.3.1. *(Skew fuzzy lines).* Two space fuzzy lines $\widetilde{L}_1$ and $\widetilde{L}_2$ passing through $\widetilde{P}_1$ to $\widetilde{P}_2$ and $\widetilde{P}_3$ to $\widetilde{P}_4$, respectively, are said to be skew fuzzy lines if the same points of $\widetilde{P}_1$, $\widetilde{P}_2$, and $\widetilde{P}_3$, $\widetilde{P}_4$ forms a tetrahedron of nonzero volume, i.e.,

$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \neq 0$, where (x_1, y_1, z_1), (x_2, y_2, z_2) are the same points of $\widetilde{P}_1$, $\widetilde{P}_2$, and (x_3, y_3, z_3), (x_4, y_4, z_4) are the same points of $\widetilde{P}_3$, $\widetilde{P}_4$.

Example 3.3.1. Consider the first pair of fuzzy points $\widetilde{P}_1(1, 0, 0)$ and $\widetilde{P}_2(0, 0, 0)$ with the membership functions

$$\mu\left((x, y, z)\middle|\widetilde{P}_1(1, 0, 0)\right) = \begin{cases} 1 - \frac{1}{0.5}\sqrt{(x-1)^2 + y^2 + z^2} & \text{if } (x-1)^2 + y^2 + z^2 \leq 0.25 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x, y, z)\middle|\widetilde{P}_2(0, 0, 0)\right) = \begin{cases} 1 - \frac{1}{0.5}\sqrt{x^2 + y^2 + z^2} & \text{if } x^2 + y^2 + z^2 \leq 0.25 \\ 0 & \text{otherwise.} \end{cases}$$

The general expressions of the same points on $\widetilde{P}_1(0)$ and $\widetilde{P}_2(0)$ are

$$(1 + 0.5(1-\alpha)\sin\varphi\cos\theta, 0.5(1-\alpha)\sin\varphi\sin\theta, 0.5(1-\alpha)\cos\varphi)$$

and

$$(0.5(1-\alpha)\sin\varphi\cos\theta, 0.5(1-\alpha)\sin\varphi\sin\theta, 0.5(1-\alpha)\cos\varphi),$$

respectively.

Consider the second pair of fuzzy points $\widetilde{P}_3(0,1,1)$ and $\widetilde{P}_4(0,1,0)$ with the membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_3(0,1,1)\right) = \begin{cases} 1-\sqrt{x^2+(y-1)^2+(z-1)^2} & \text{if } x^2+(y-1)^2+(z-1)^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_4(0,1,0)\right) = \begin{cases} 1-\frac{1}{0.5}\sqrt{x^2+(y-1)^2+z^2} & \text{if } x^2+(y-1)^2+z^2 \le 0.25 \\ 0 & \text{otherwise.} \end{cases}$$

The general expressions of the same points on $\widetilde{P}_3(0)$ and $\widetilde{P}_4(0)$ are

$$((1-\alpha)\sin\varphi\cos\theta, 1+(1-\alpha)\sin\varphi\sin\theta, 1+(1-\alpha)\cos\varphi)$$

and

$$(0.5(1-\alpha)\sin\varphi\cos\theta, 1+0.5(1-\alpha)\sin\varphi\sin\theta, 0.5(1-\alpha)\cos\varphi),$$

respectively. Let $\widetilde{L}_1$ and $\widetilde{L}_2$ be two space fuzzy lines passing through $\widetilde{P}_1$ to $\widetilde{P}_2$, and $\widetilde{P}_3$ to $\widetilde{P}_4$, respectively. The same points of $\widetilde{P}_1$, $\widetilde{P}_2$, and $\widetilde{P}_3$, $\widetilde{P}_4$ form a tetrahedron of nonzero volume, i.e.,

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$
$= -1 - \frac{(1-\alpha)\cos\varphi}{2}$

$\neq 0$, for $0 \leq \varphi \leq \pi$. Hence, $\widetilde{L}_1$ and $\widetilde{L}_2$ form a pair of skew fuzzy lines.

Now, we discuss the symmetric skew fuzzy lines. Let $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ be two fuzzy points whose supports are identical up to a translation. If $\widetilde{P}_1(a_1, b_1, c_1)$ is a fuzzy point with the membership function

$$\mu\left((x, y, z)\middle|\widetilde{P}_1\right) = f_1(x - a_1, y - b_1, z - c_1),$$

then the membership function of $\widetilde{P}_2(a_2, b_2, c_2)$ is

$$\mu\left((x, y, z)\middle|\widetilde{P}_2\right) = f_1(x - a_2, y - b_2, z - c_2).$$

Shape and size of $\widetilde{P}_1(0)$ and $\widetilde{P}_2(0)$ remains same but the core points of $\widetilde{P}_1$ and $\widetilde{P}_2$ are different. These types of fuzzy points are said to be just a photo-copy of each other.

Let $\widetilde{L}_1$ and $\widetilde{L}_2$ be space fuzzy lines passing through $\widetilde{P}_1$ to $\widetilde{P}_2$ and $\widetilde{P}_3$ to $\widetilde{P}_4$, respectively, where $\widetilde{P}_2$ is a photo-copy of $\widetilde{P}_1$ and $\widetilde{P}_4$ is a photo-copy of $\widetilde{P}_3$. By the Definition 3.3.1, if $\widetilde{L}_1$ and $\widetilde{L}_2$ are skew fuzzy lines, then $\widetilde{L}_1$ and $\widetilde{L}_2$ are called symmetric skew fuzzy lines.

Below, we discuss the shortest distance between two symmetric skew fuzzy lines.

Definition 3.3.2. (Shortest distance between two symmetric skew fuzzy lines). Let $\widetilde{P}_2$ be a photo-copy of a given fuzzy point $\widetilde{P}_1$, and $\widetilde{P}_4$ be a photo-copy of a given fuzzy point $\widetilde{P}_3$. Shortest distance, $\widetilde{D}_S$ say, between two symmetric skew fuzzy lines $\widetilde{L}_1$ and $\widetilde{L}_2$ passing through $\widetilde{P}_1$ to $\widetilde{P}_2$, and $\widetilde{P}_3$ to $\widetilde{P}_4$, respectively, can be defined by the membership function

$$\mu\left(d_s \middle| \widetilde{D}_S\right) = \sup\{\alpha : \text{where } d_s \text{ is the shortest distance between two skew lines } l_1 \text{ and } l_2 \text{ passing through the same points of } \widetilde{P}_1 \text{ and } \widetilde{P}_2, \text{ and the inverse points of } \widetilde{P}_3 \text{ and } \widetilde{P}_4 \text{ of membership value } \alpha, \text{respectively}\}.$$

Theorem 3.3.1. For two symmetric skew fuzzy lines $\widetilde{L}_1$ and $\widetilde{L}_2$, their shortest distance $\widetilde{D}_S$ is a fuzzy number in $\mathbb{R}$.

Proof. Similar to Theorem 4.1 in [1]. □

Algorithm 3.3.1 describes the procedure to obtain the shortest distance $\widetilde{D}_S$ between two symmetric skew fuzzy lines $\widetilde{L}_1$ and $\widetilde{L}_2$ passing through $\widetilde{P}_1$ to $\widetilde{P}_2$, and $\widetilde{P}_3$ to $\widetilde{P}_4$, respectively.

Algorithm 3.3.1: Algorithm to evaluate the shortest distance $\widetilde{D}_S\left(\widetilde{L}_1, \widetilde{L}_2\right)$

Input: Given four continuous S-type space fuzzy points $\widetilde{P}_i(a_i, b_i, c_i)$ with the membership functions $f_i(x - a_i, y - b_i, z - c_i)$, for $i = 1, 2, 3, 4$. The fuzzy point $\widetilde{P}_2$ is a photo-copy of $\widetilde{P}_1$, and $\widetilde{P}_4$ is a photo-copy of $\widetilde{P}_3$. We denote

$$\phi_i(\lambda_i) = f_i\left(\lambda_i \sin\varphi\cos\theta, \lambda_i \sin\varphi\sin\theta, \lambda_i\cos\varphi\right) \text{ for } \lambda_i \geq 0, i = 1, 2, 3, 4.$$

Output: The shortest distance $\widetilde{D}_S\left(\widetilde{L}_1, \widetilde{L}_2\right) = \bigvee_{\alpha\in[0,1]} \widetilde{D}_S(\alpha)$.

For $\alpha = 0$ to 1; step size $\delta\alpha$
- $(d_s)^\alpha_{\min} = M$, a very large number
- $(d_s)^\alpha_{\max} = -M$
- **For** $\theta = 0$ to 2π; step size $\delta\theta$
  - $(d_s)^\theta_{\min} = M$
  - $(d_s)^\theta_{\max} = -M$
  - **For** $\varphi = 0$ to π; step size $\delta\varphi$
    - Compute
    - $\lambda_1 = \phi_1^{-1}(\alpha)$
    - $\lambda_2 = \phi_2^{-1}(\alpha)$
    - $\lambda_3 = \phi_3^{-1}(\alpha)$
    - $\lambda_4 = \phi_4^{-1}(\alpha)$
    - Compute the same points of $\widetilde{P}_1$ and $\widetilde{P}_2$
    - $u^\alpha_{\theta\varphi} = (a_1 + (\sin\varphi\cos\theta)\phi_1^{-1}(\alpha), b_1 + (\sin\varphi\sin\theta)\phi_1^{-1}(\alpha), c_1 + (\cos\varphi)\phi_1^{-1}(\alpha))$
    - $v^\alpha_{\theta\varphi} = (a_2 + (\sin\varphi\cos\theta)\phi_2^{-1}(\alpha), b_2 + (\sin\varphi\sin\theta)\phi_2^{-1}(\alpha), c_2 + (\cos\varphi)\phi_2^{-1}(\alpha))$
    - Compute the inverse points of $\widetilde{P}_3$ and $\widetilde{P}_4$
    - $p^\alpha_{\theta\varphi} = (a_3 + (\sin\varphi\cos\theta)\phi_3^{-1}(\alpha), b_3 + (\sin\varphi\sin\theta)\phi_3^{-1}(\alpha), c_3 + (\cos\varphi)\phi_3^{-1}(\alpha))$
    - $q^\alpha_{\theta\varphi} = (a_4 - (\sin\varphi\cos\theta)\phi_4^{-1}(\alpha), b_4 - (\sin\varphi\sin\theta)\phi_4^{-1}(\alpha), c_4 - (\cos\varphi)\phi_4^{-1}(\alpha))$
    - Compute
    - $(r_1, r_2, r_3) = u^\alpha_{\theta\varphi} - v^\alpha_{\theta\varphi}$
    - $(s_1, s_2, s_3) = p^\alpha_{\theta\varphi} - q^\alpha_{\theta\varphi}$
    - We assign $(x_1, x_2, x_3) \leftarrow u^\alpha_{\theta\varphi}$ and $(y_1, y_2, y_3) \leftarrow v^\alpha_{\theta\varphi}$

Calculate the shortest distance between $\widetilde{L}_1$ and $\widetilde{L}_2$

$$(d_s)^{\alpha}_{\varphi} \leftarrow \frac{\begin{vmatrix} y_1 - x_1 & y_2 - x_2 & y_3 - x_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix}}{\sqrt{(r_2 s_3 - s_2 r_3)^2 + (r_1 s_3 - r_3 s_1)^2 + (r_1 s_2 - r_2 s_1)^2}}$$

if $(d_s)^{\alpha}_{\varphi} > (d_s)^{\theta}_{\max}$ **then**
  $(d_s)^{\theta}_{\max} \leftarrow (d_s)^{\alpha}_{\varphi}$
end
if $(d_s)^{\theta}_{\min} > (d_s)^{\alpha}_{\varphi}$ **then**
  $(d_s)^{\theta}_{\min} \leftarrow (d_s)^{\alpha}_{\varphi}$
end
if $(d_s)^{\theta}_{\max} > (d_s)^{\alpha}_{\max}$ **then**
  $(d_s)^{\alpha}_{\max} \leftarrow (d_s)^{\theta}_{\max}$
end
if $(d_s)^{\alpha}_{\min} > (d_s)^{\theta}_{\min}$ **then**
  $(d_s)^{\alpha}_{\min} \leftarrow (d_s)^{\theta}_{\min}$
end
end
end
At the end of loop, $\widetilde{D}_S(\alpha) \leftarrow [(d_s)^{\alpha}_{\min}, (d_s)^{\alpha}_{\max}]$
end
return $\widetilde{D}_S(\widetilde{L}_1, \widetilde{L}_2) = \bigvee_{\alpha \in [0,1]} \widetilde{D}_S(\alpha)$

In the following example, we employ Algorithm 3.3.1 on symmetric skew fuzzy lines to evaluate the fuzzy shortest distance.

Example 3.3.2. Consider the first pair of fuzzy points $\widetilde{P}_1(0,1,1)$ and $\widetilde{P}_2(0,1,0)$ with the membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_1(0,1,1)\right) = \begin{cases} 1 - \frac{1}{0.5}\sqrt{x^2 + (y-1)^2 + (z-1)^2} & \text{if } x^2 + (y-1)^2 + (z-1)^2 \leq 0.25 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_2(0,1,0)\right) = \begin{cases} 1-\frac{1}{0.5}\sqrt{x^2+(y-1)^2+z^2} & \text{if } x^2+(y-1)^2+z^2 \leq 0.25 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\widetilde{P}_2(0,1,0)$ is a photo-copy of $\widetilde{P}_1(0,1,1)$. For a given $\alpha \in [0,1]$, the same points on $\widetilde{P}_1(0,1,1)$ and $\widetilde{P}_2(0,1,0)$ with the membership value α are

$$\left(u^1\right)^{\alpha}_{\theta\varphi} : (0.5(1-\alpha)\sin\varphi\cos\theta, 1+0.5(1-\alpha)\sin\varphi\sin\theta, 1+0.5(1-\alpha)\cos\varphi)$$

and

$$\left(u^2\right)^{\alpha}_{\theta\varphi} : (0.5(1-\alpha)\sin\varphi\cos\theta, 1+0.5(1-\alpha)\sin\varphi\sin\theta, 0.5(1-\alpha)\cos\varphi)\,,$$

respectively.

Consider the second pair of fuzzy points $\widetilde{P}_3(0,0,0)$ and $\widetilde{P}_4(1,0,0)$ with the membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_3(0,0,0)\right) = \begin{cases} 1-\frac{1}{0.3}\sqrt{x^2+y^2+z^2} & \text{if } x^2+y^2+z^2 \leq 0.09 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_4(1,0,0)\right) = \begin{cases} 1-\frac{1}{0.3}\sqrt{(x-1)^2+y^2+z^2} & \text{if } (x-1)^2+y^2+z^2 \leq 0.09 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\widetilde{P}_4(1,0,0)$ is a photo-copy of $\widetilde{P}_3(0,0,0)$. For a given $\alpha \in [0,1]$, the inverse points on $\widetilde{P}_3(0,0,0)$ and $\widetilde{P}_4(1,0,0)$ with the membership value α are

$$\left(u^3\right)^{\alpha}_{\theta\varphi} : (0.3(1-\alpha)\sin\varphi\cos\theta, 0.3(1-\alpha)\sin\varphi\sin\theta, 0.3(1-\alpha)\cos\varphi)$$

and

$$\left(u^4\right)^{\alpha}_{\theta\varphi} : (1-0.3(1-\alpha)\sin\varphi\cos\theta, -0.3(1-\alpha)\sin\varphi\sin\theta, -0.3(1-\alpha)\cos\varphi)\,,$$

respectively. Consider $\widetilde{L}_1$ and $\widetilde{L}_2$ are two symmetric skew fuzzy lines passing through $\widetilde{P}_1$ to $\widetilde{P}_2$, and $\widetilde{P}_3$ to $\widetilde{P}_4$, respectively.

Table 3.2 gives α-cuts of the shortest distance $\widetilde{D}_S(\widetilde{L}_1,\widetilde{L}_2)$ executed by the proposed Algorithm 3.3.1 with step sizes $\delta\alpha = 0.1000$, $\delta\theta = 0.0706$ and $\delta\varphi = 0.0353$.

α	$\widetilde{D}_S(\widetilde{L}_1,\widetilde{L}_2)(\alpha)$
0.1	$[0.6618, 1.0541]$
0.2	$[0.7174, 1.0541]$
0.3	$[0.7676, 1.0541]$
0.4	$[0.8130, 1.0541]$
0.5	$[0.8540, 1.0539]$
0.6	$[0.8908, 1.0508]$
0.7	$[0.9237, 1.0437]$
0.8	$[0.9528, 1.0328]$
0.9	$[0.9782, 1.0182]$
1.0	1

TABLE 3.2: α-cuts of $\widetilde{D}_S(\widetilde{L}_1,\widetilde{L}_2)$ by Algorithm 3.3.1 for Example 3.3.2

The following Figure 3.3 displays the membership function of $\widetilde{D}_S(\widetilde{L}_1, \widetilde{L}_2)$ obtained from Table 3.2. Note that the obtained $\widetilde{D}_S$ supports Theorem 3.3.1.

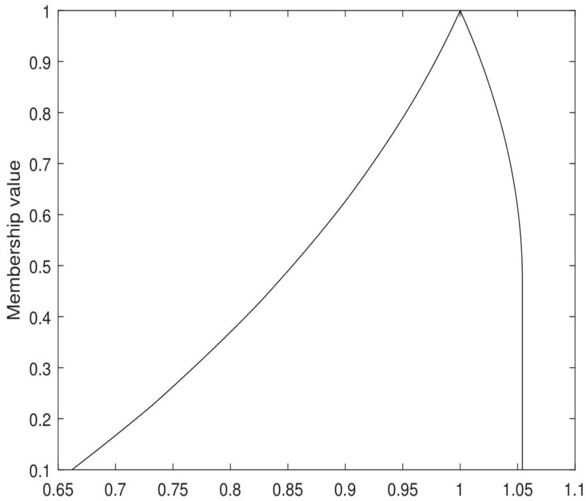

FIGURE 3.3: Shortest distance $\widetilde{D}_S(\widetilde{L}_1, \widetilde{L}_2)$ by Algorithm 3.3.1 for Example 3.3.2

Algorithm 3.3.2 provides a procedure to find the membership grade of a point in the shortest distance $\widetilde{D}_S$ between two symmetric skew fuzzy lines $\widetilde{L}_1$ and $\widetilde{L}_2$ passing through $\widetilde{P}_1$ to $\widetilde{P}_2$, and $\widetilde{P}_3$ to $\widetilde{P}_4$, respectively.

Algorithm 3.3.2: Algorithm to find $\mu(d_s|\widetilde{D}_S)$

Input: Given four continuous S-type space fuzzy points $\widetilde{P}_i(a_i, b_i, c_i)$ with the membership functions $f_i(x-a_i, y-b_i, z-c_i)$, for $i = 1, 2, 3, 4$. The fuzzy point $\widetilde{P}_2$ is a photo-copy of $\widetilde{P}_1$, and $\widetilde{P}_4$ is a photo-copy of $\widetilde{P}_3$. We denote

$$\phi_i(\lambda_i) = f_i\left(\lambda_i \sin\varphi\cos\theta, \lambda_i \sin\varphi\sin\theta, \lambda_i\cos\varphi\right) \text{ for } \lambda_i \geq 0, i = 1, 2, 3, 4.$$

We denote the shortest distance between $\widetilde{L}_1$ and $\widetilde{L}_2$ by $\widetilde{D}_S$.

Given $d_s \in \mathbb{R}$ for which the membership value $\mu\left(d_s \middle| \widetilde{D}_S\right)$ has to be calculated.

Output: The membership value $\mu(d_s|\widetilde{D}_S) = \alpha_{\text{sup}}$.

Initialize $\alpha_{\text{sup}} \leftarrow 0$

loop:

For $\alpha = 0$ to 1; step size $\delta\alpha$

For $\theta = 0$ to 2π; step size $\delta\theta$

For $\varphi = 0$ to π; step size $\delta\varphi$

Compute

$\phi_1^{-1}(\alpha) = \lambda_1$

$\phi_2^{-1}(\alpha) = \lambda_2$

$\phi_3^{-1}(\alpha) = \lambda_3$

$\phi_4^{-1}(\alpha) = \lambda_4$

Compute the same points of $\widetilde{P}_1$ and $\widetilde{P}_2$

$u_{\theta\varphi}^{\alpha} = (a_1 + (\sin\varphi\cos\theta)\phi_1^{-1}(\alpha), b_1 + (\sin\varphi\sin\theta)\phi_1^{-1}(\alpha), c_1 + (\cos\varphi)\phi_1^{-1}(\alpha))$

$v_{\theta\varphi}^{\alpha} = (a_2 + (\sin\varphi\cos\theta)\phi_2^{-1}(\alpha), b_2 + (\sin\varphi\sin\theta)\phi_2^{-1}(\alpha), c_2 + (\cos\varphi)\phi_2^{-1}(\alpha))$

Compute the inverse points of $\widetilde{P}_3$ and $\widetilde{P}_4$

$p_{\theta\varphi}^{\alpha} = (a_3 + (\sin\varphi\cos\theta)\phi_3^{-1}(\alpha), b_3 + (\sin\varphi\sin\theta)\phi_3^{-1}(\alpha), c_3 + (\cos\varphi)\phi_3^{-1}(\alpha))$

$q_{\theta\varphi}^{\alpha} = (a_4 - (\sin\varphi\cos\theta)\phi_4^{-1}(\alpha), b_4 - (\sin\varphi\sin\theta)\phi_4^{-1}(\alpha), c_4 - (\cos\varphi)\phi_4^{-1}(\alpha))$

Compute

$(r_1, r_2, r_3) = u_{\theta\varphi}^{\alpha} - v_{\theta\varphi}^{\alpha}$

$(s_1, s_2, s_3) = p_{\theta\varphi}^{\alpha} - q_{\theta\varphi}^{\alpha}$

We denote $(x_1, x_2, x_3) \leftarrow u_{\theta\varphi}^{\alpha}$ and $(y_1, y_2, y_3) \leftarrow v_{\theta\varphi}^{\alpha}$

```
            Calculate the shortest distance between L̃_1 and L̃_2
```

$$d_s^\alpha = \frac{\begin{vmatrix} y_1 - x_1 & y_2 - x_2 & y_3 - x_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix}}{\sqrt{(r_2 s_3 - s_2 r_3)^2 + (r_1 s_3 - r_3 s_1)^2 + (r_1 s_2 - r_2 s_1)^2}}$$

if $d_s = d_s^\alpha$ **then**
  if $\alpha_{\text{sup}} < \alpha$ **then**
    $\alpha_{\text{sup}} \leftarrow \alpha$
  else
    goto loop
  end
end
end
end
end
return $\mu(d_s|\widetilde{D}_S) = \alpha_{\text{sup}}$

Example 3.3.3. (*Evaluation of the membership values in* $\widetilde{D}_S\left(\widetilde{L}_1, \widetilde{L}_2\right)(0)$).

Consider the first pair of fuzzy points $\widetilde{P}_1(0,1,1)$ and $\widetilde{P}_2(0,1,0)$ as in Example 3.3.2

Consider the second pair of fuzzy points $\widetilde{P}_3(0,0,0)$ and $\widetilde{P}_4(1,0,0)$ as in Example 3.3.2.

Table 3.3 displays the membership grades of some numbers in the shortest distance $\widetilde{D}_S(\widetilde{L}_1, \widetilde{L}_2)$, obtained by Algorithm 3.3.2.

d_s	Membership value	Step sizes
0.8804	0.5696	$\delta\alpha = 0.0131$, $\delta\theta = 0.0911$ and $\delta\varphi = 0.0455$
1.0521	0.5729	$\delta\alpha = 0.0153$, $\delta\theta = 0.1065$ and $\delta\varphi = 0.0532$
0.9932	0.9658	$\delta\alpha = 0.0114$, $\delta\theta = 0.0795$ and $\delta\varphi = 0.0398$

TABLE 3.3: Membership grades of some points of $\widetilde{D}_S\left(\widetilde{L}_1, \widetilde{L}_2\right)(0)$ produced by Algorithm 3.3.2 for Example 3.3.3

3.3.1 A brief discussion of the shortest distance between non-symmetric skew fuzzy lines

First, we make attention to a particular case of it; the shortest distance between a crisp line L and a non-symmetric fuzzy line $\widetilde{L}$. Note that L and l are skew lines, where $l \in \widetilde{L}(\alpha)$, $\alpha \in [0,1]$. The shortest distance $\widetilde{D}_S\left(L, \widetilde{L}\right)$ can be defined by membership function

$$\mu\left(d_s \middle| \widetilde{D}_S\right) = \sup\{\alpha : d_s = d_s(L,l),\ \text{where } l \in \widetilde{L}(\alpha) \text{ and } d_s \text{ is the shortest distance between } L \text{ and } l\}.$$

In other words, let p be a point on $l \in \widetilde{L}(\alpha)$ that determine the shortest distance $d_s = d_s(L,l)$. The membership function of $\widetilde{D}_S\left(L, \widetilde{L}\right)$ may be defined as

$$\mu\left(d_s \middle| \widetilde{D}_S\right) = \sup\{\alpha : d_s = d_s(L,p),\ \text{where } p \in l \text{ with membership value } \alpha\}. \tag{3.13}$$

We will demonstrate that the shortest distance $\widetilde{D}_S\left(L, \widetilde{L}\right)$ (by (3.13)) may not be a fuzzy number.

Since $\widetilde{L}(\alpha)$ is non-convex set, we have $x_1 \leq \lambda x_1 + (1-\lambda)x_2 \leq x_2$ on $\widetilde{L}(0)$, for $\lambda \in (0,1)$, such that $\mu\left(\lambda x_1 + (1-\lambda)x_2\right) = \beta < \alpha$, where $x_1, x_2 \in \widetilde{L}(0)$ with membership value α. We may choose d_{s1}, $d_{s\lambda}$ and d_{s2} such a manner that $d_{s1} \leq d_{s\lambda} \leq d_{s2}$, where $d_{s1} = d_s(L, l_1)$, $d_{s\lambda} = d_s(L, l_\lambda)$, $d_{s2} = d_s(L, l_2)$, and $l_1, l_\lambda, l_2 \in \widetilde{L}(0)$. We took $x_1 \in l_1$, $\lambda x_1 + (1-\lambda)x_2 \in l_\lambda$ and $x_2 \in l_2$ as the points which determine the shortest distances $d_{s1} = d_s(L, l_1)$, $d_{s\lambda} = d_s(L, l_\lambda)$, $d_{s2} = d_s(L, l_2)$. Clearly, by (3.13), $\mu\left(d_{s\lambda} \middle| \widetilde{D}_S\right) = \beta < \alpha$. Apparently, $\widetilde{D}_S\left(L, \widetilde{L}\right)$ is not a convex set. Hence, $\widetilde{D}_S\left(L, \widetilde{L}\right)$ may not be a fuzzy number.

In a similar manner, without loss of generality, we can say that the $\widetilde{D}_S\left(\widetilde{L}, \widetilde{L'}\right)$ may not be a fuzzy number, where $\widetilde{L}$ and $\widetilde{L'}$ are non-symmetric skew fuzzy lines.

3.4 Fuzzy plane

In the three-dimensional Euclidean space, a crisp plane can be defined in many ways: a normal form of a plane, a three-point form of a plane, an intercept form of a plane, and a plane passing through a point and perpendicular to a given crisp direction. We extend analogous approaches in the fuzzy environment to define fuzzy planes. Here, an attempt is made to define a fuzzy plane in the case where positions of three points, or three intercepts, or position of a point with a given crisp direction, are given imprecisely.

There may not exist a normal form of fuzzy plane because the fuzzy linear equation $\tilde{a}\tilde{x} + \tilde{b}\tilde{y} + \tilde{c}\tilde{z} = \tilde{d}$ [118] may not have any triplet $(\tilde{x}, \tilde{y}, \tilde{z})$ that satisfy it. For instance, taking $\tilde{a} = (0.5/2/3)$, $\tilde{b} = (0/0/0)$, $\tilde{c} = (0/0/0)$ and $\tilde{d} = (1/4/5)$, and putting all these quantities in $\tilde{a}\tilde{x} + \tilde{b}\tilde{y} + \tilde{c}\tilde{z} = \tilde{d}$, where $\tilde{x} = (x - \lambda_x/x/x + \gamma_x)$, $\tilde{y} = (y - \lambda_y/y/y + \gamma_y)$ and $\tilde{z} = (z - \lambda_z/z/z + \gamma_z)$ with $\lambda_x, \lambda_y, \lambda_z, \gamma_x, \gamma_y, \gamma_z \geq 0$, we obtain $\gamma_x = -0.4$, for $\alpha = 0.5$. So it contradict that $\gamma_x \geq 0$. Thus, there does not exist any triplet $(\tilde{x}, \tilde{y}, \tilde{z})$

such that $\tilde{a}\tilde{x} + \tilde{b}\tilde{y} + \tilde{c}\tilde{z} = \tilde{d}$. Hence, we conclude that a fuzzy plane cannot be mathematically represented by a fuzzy linear equation.

In the following subsections, we propose mathematical formulations for fuzzy planes when three S-type space fuzzy points, or three intercepts, or an S-type space fuzzy point and a crisp direction are known.

3.4.1 Fuzzy plane passing through three S-type space fuzzy points $(\widetilde{\Pi}_{3P})$

Definition 3.4.1. (*Three-point form* $(\widetilde{\Pi}_{3P})$). Let $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ be three S-type space fuzzy points whose cores are not collinear. The fuzzy plane $\widetilde{\Pi}_{3P}$ that passes through the fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ can be defined by the membership function

$$\mu\left((x,y,z)\middle|\widetilde{\Pi}_{3P}\right) = \sup\{\alpha : (x,y,z) \text{ belongs to the plane passing through the same points of } \widetilde{P}_1(0),\ \widetilde{P}_2(0) \text{ and } \widetilde{P}_3(0) \text{ with membership value } \alpha\}.$$

For any $\alpha \in [0,1]$, the α-cuts of $\widetilde{\Pi}_{3P}$ is given by

$$\widetilde{\Pi}_{3P}(\alpha) = \bigvee\{\Pi : \Pi \text{ is the plane passing through the same points of } \widetilde{P}_1(0),\ \widetilde{P}_2(0) \text{ and } \widetilde{P}_3(0) \text{ with membership value } \alpha\}. \tag{3.14}$$

According to this definition of $\mu\left(\cdot\middle|\widetilde{\Pi}_{3P}\right)$, a fuzzy plane can be perceived as a collection of crisp planes passing through the same points of $\widetilde{P}_1(0)$, $\widetilde{P}_2(0)$ and $\widetilde{P}_3(0)$ with membership value $\alpha \in [0,1]$.

The membership value of a crisp plane $\mathit{\Pi}$, inside the region $\widetilde{\mathit{\Pi}}_{3P}(0)$, can be defined as

$$\mu\left(\mathit{\Pi}\middle|\widetilde{\mathit{\Pi}}_{3P}\right) = \min_{(x,y,x)\in\mathit{\Pi}} \mu\left((x,y,z)\middle|\widetilde{\mathit{\Pi}}_{3P}\right).$$

Now, we present a theorem which facilitates in finding the membership value of the plane $\mathit{\Pi}$ in $\widetilde{\mathit{\Pi}}_{3P}(0)$ by the idea of same points.

Theorem 3.4.1. Suppose that $\mathit{\Pi}$ is a plane in $\widetilde{\mathit{\Pi}}_{3P}(0)$ and there exist three same points $(x_1, y_1, z_1) \in \widetilde{P}_1(0)$, $(x_2, y_2, z_2) \in \widetilde{P}_2(0)$, and $(x_3, y_3, z_3) \in \widetilde{P}_3(0)$ with

$$\mu\left((x_1,y_1,z_1)\middle|\widetilde{\mathit{\Pi}}_{3P}\right) = \mu\left((x_2,y_2,z_2)\middle|\widetilde{\mathit{\Pi}}_{3P}\right) = \mu\left((x_3,y_3,z_3)\middle|\widetilde{\mathit{\Pi}}_{3P}\right) = \alpha$$

such that $\mathit{\Pi}$ is the plane passing through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). Then

$$\mu\left(\mathit{\Pi}\middle|\widetilde{\mathit{\Pi}}_{3P}\right) = \alpha.$$

Proof. The proof is similar to that of Theorem 3.1.1 in [2] and thus omitted. □

The Algorithm 3.4.1 demonstrates how to find the membership grade of a number in the fuzzy plane $\widetilde{\mathit{\Pi}}_{3P}$ passing through three continuous S-type space fuzzy points.

Algorithm 3.4.1: Algorithm to evaluate $\mu\left((x,y,z)\middle|\widetilde{\Pi}_{3P}\right)$

Input: Given three continuous fuzzy points $\widetilde{P}_i(a_i,b_i,c_i)$ with the membership functions $f_i(x-a_i,y-b_i,z-c_i)$, for $i=1,2,3$. We denote

$$\phi_i(\lambda_i)=f_i\left(\lambda_i\sin\varphi\cos\theta,\lambda_i\sin\varphi\sin\theta,\lambda_i\cos\varphi\right),\ \text{for } \lambda_i\geq 0, i=1,2,3.$$

Given a point (x,y,z) whose membership value in $\widetilde{\Pi}_{3P}$ is to be calculated.

Output: The membership value $\mu\left((x,y,z)\middle|\widetilde{\Pi}_{3P}\right)=\alpha_{\text{sup}}$.

Initialize $\alpha_{\text{sup}}\leftarrow 0$

loop:

For $\alpha=0$ to 1; with step size $\delta\alpha$

For $\theta=0$ to 2π; with step size $\delta\theta$

For $\varphi=0$ to π; with step size $\delta\varphi$

Compute

$\phi_1^{-1}(\alpha)=\lambda_1$

$\phi_2^{-1}(\alpha)=\lambda_2$

$\phi_3^{-1}(\alpha)=\lambda_3$

Compute the same points

$(u^1)_{\theta\varphi}^{\alpha}=$

$\big(a_1+(\sin\varphi\cos\theta)\phi_1^{-1}(\alpha),b_1+(\sin\varphi\sin\theta)\phi_1^{-1}(\alpha),c_1+(\cos\varphi)\phi_1^{-1}(\alpha)\big)$

$(u^2)_{\theta\varphi}^{\alpha}=$

$\big(a_2+(\sin\varphi\cos\theta)\phi_2^{-1}(\alpha),b_2+(\sin\varphi\sin\theta)\phi_2^{-1}(\alpha),c_2+(\cos\varphi)\phi_2^{-1}(\alpha)\big)$

$(u^3)_{\theta\varphi}^{\alpha}=$

$\big(a_3+(\sin\varphi\cos\theta)\phi_3^{-1}(\alpha),b_3+(\sin\varphi\sin\theta)\phi_3^{-1}(\alpha),c_3+(\cos\varphi)\phi_3^{-1}(\alpha)\big)$

Compute

normal = cross product of $(u^1)_{\theta\varphi}^{\alpha}-(u^2)_{\theta\varphi}^{\alpha}$ and $(u^1)_{\theta\varphi}^{\alpha}-(u^3)_{\theta\varphi}^{\alpha}$

Compute

$f_p=$ dot product of normal and $(x,y,z)-(u^1)_{\theta\varphi}^{\alpha}$

```
      if f_p = 0 then
        if α_sup < α then
          α_sup ← α
        else
          goto loop
        end
      end
    end
  end
end
return μ((x, y, z) | Π̃_3P) = α_sup
```

Example 3.4.1. (*Evaluation of the membership values in* $\widetilde{\Pi}_{3P}(0)$). Consider three fuzzy points $\widetilde{P}_1(1,0,1)$, $\widetilde{P}_2(-5,0,0)$ and $\widetilde{P}_3(4,-3,2)$ with the membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_1(1,0,1)\right) = \begin{cases} 1-\sqrt{\frac{1}{4}(x-1)^2+y^2+\frac{1}{4}(z-1)^2} & \text{if } \frac{1}{4}(x-1)^2+y^2+\frac{1}{4}(z-1)^2 \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu\left((x,y,z)\middle|\widetilde{P}_2(-5,0,0)\right) = \begin{cases} 1-\sqrt{\frac{1}{4}(x+5)^2+\frac{y^2}{4}+z^2} & \text{if } \frac{1}{4}(x+5)^2+\frac{y^2}{4}+z^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_3(4,-3,2)\right) = \begin{cases} 1-\frac{1}{4}\sqrt{(x-4)^2+(y+3)^2+(z-2)^2} & \text{if } (x-4)^2+(y+3)^2+(z-2)^2 \leq 16 \\ 0 & \text{otherwise.} \end{cases}$$

The general expression of the same points of $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ are

$$\begin{aligned}
&\left(u^1\right)^{\alpha}_{\theta\varphi} : \left(1+\tfrac{2(1-\alpha)\sin\varphi\cos\theta}{R^{\alpha}_{\theta\varphi}}, \tfrac{2(1-\alpha)\sin\varphi\sin\theta}{R^{\alpha}_{\theta\varphi}}, 1+\tfrac{2(1-\alpha)\cos\varphi}{R^{\alpha}_{\theta\varphi}}\right)\\
&\left(u^2\right)^{\alpha}_{\theta\varphi} : \left(-5+\tfrac{2(1-\alpha)\sin\varphi\cos\theta}{S^{\alpha}_{\theta\varphi}}, \tfrac{2(1-\alpha)\sin\varphi\sin\theta}{S^{\alpha}_{\theta\varphi}}, \tfrac{2(1-\alpha)\cos\varphi}{S^{\alpha}_{\theta\varphi}}\right)\\
&\left(u^3\right)^{\alpha}_{\theta\varphi} : \left(4+4(1-\alpha)\sin\varphi\cos\theta, -3+4(1-\alpha)\sin\varphi\sin\theta, 2+4(1-\alpha)\cos\varphi\right),
\end{aligned}$$

respectively, where

$$R^{\alpha}_{\theta\varphi} = \sqrt{4\sin^2\varphi\cos^2\theta + 4\sin^2\varphi\sin^2\theta + \cos^2\varphi}$$

and

$$S^{\alpha}_{\theta\varphi} = \sqrt{\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + 4\cos^2\varphi}.$$

The following Table 3.4 shows the membership values of some points in the fuzzy plane $\widetilde{\Pi}_{3P}$ by execution of the Algorithm 3.4.1.

(x,y,z)	Membership Value	Step size
$(4,-3,4.7)$	0.3250	$\delta\alpha = 0.225$, $\delta\theta = 1.5708$ and $\delta\varphi = 0.7854$
$(-5,0,0.225)$	0.7750	$\delta\alpha = 0.225$, $\delta\theta = 1.5708$ and $\delta\varphi = 0.7854$
$(-1,-2,1)$	1	$\delta\alpha = 0.225$, $\delta\theta = 1.5708$ and $\delta\varphi = 0.7854$

TABLE 3.4: Membership values of some points $(x,y,z) \in \widetilde{\Pi}_{3P}(0)$ produced by Algorithm 3.4.1 for Example 3.4.1

In our next result, we show that a fuzzy plane is uniquely determined by three fuzzy points similar to that a fuzzy line [4] has been uniquely determined by two fuzzy points (see Theorem 3.4.1 in [4]).

Theorem 3.4.2. Given three S-type space fuzzy points whose core points are not collinear. There exists a unique fuzzy plane that passes through these S-type space fuzzy points.

Proof. The proof of the theorem is easily followed from the proof of the Theorem 3.4.1 in [4]. □

In order to understand the intercept form of a fuzzy plane, it is important to familiarize ourselves with the intercept of the fuzzy plane $\left(\widetilde{\Pi}_{3P}\right)$. Importantly, to define the intercept of $\widetilde{\Pi}_{3P}$, it should pass through the coordinate axis.

As earlier, by the Definition 3.4.1, it is noted that the fuzzy plane can be constructed by taking the union of the crisp planes with varied membership values. Thus, the x-intercept of the fuzzy plane can be a collection of the x-intercepts of the crisp planes that lie on the support of $\widetilde{\Pi}_{3P}$. The x-intercept of the fuzzy plane $\left(\widetilde{\Pi}_{3P}\right)$ can be written by the membership function as follows. Similarly, the y-intercept and the z-intercept can be found.

Definition 3.4.2. (x*-intercept of* $\widetilde{\Pi}_{3P}$). Let $\widetilde{\Pi}_{3P}$ be a fuzzy plane passing through three fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$. The x-intercept of $\widetilde{\Pi}_{3P}$, denoted as $\tilde{a}$, can be formulated as

$$\mu(a|\tilde{a}) = \begin{cases} \mu\left((a,0,0)\middle|\widetilde{\Pi}_{3P}\right) & \text{if } (a,0,0) \in \widetilde{\Pi}_{3P}(0) \bigcap (x\text{-axis}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\tilde{a}$ may not be a fuzzy number. Here, we discuss two cases:

(i) $\widetilde{\Pi}_{3P}(1) \bigcap (x\text{-axis}) = \emptyset$.

(ii) $\widetilde{\Pi}_{3P}(1) \bigcap (x\text{-axis}) \neq \emptyset$.

In the first case, there does not exist a point $(a, 0, 0) \in \widetilde{\Pi}_{3P}(1) \bigcap (x\text{-axis})$ such that $\mu\left((a, 0, 0)\middle|\widetilde{\Pi}_{3P}\right) = 1$. More explicitly, in this case, the fuzzy set $\tilde{a}$ cannot be a fuzzy number since a fuzzy number should be a normal fuzzy set. In the second case, the fuzzy set $\tilde{a}$ is a fuzzy number. The following theorem supports this case.

Theorem 3.4.3. If $\widetilde{\Pi}_{3P}(1) \bigcap (x\text{-axis}) \neq \emptyset$, then the x-intercept of $\widetilde{\Pi}_{3P}$ that passes through $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ is a fuzzy number with a singleton core.

Proof. Let $\tilde{a}$ be x-intercept of the fuzzy plane passing through $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$. Let

$$(a, 0, 0) \in \widetilde{\Pi}_{3P}(1) \bigcap (x\text{-axis}).$$

Then, obviously $\mu((a, 0, 0)|\tilde{a}) = 1$.

For $\alpha \in [0, 1]$, let the least upper bound and the greatest lower bound of the x-intercepts of the crisp planes in $\widetilde{\Pi}_{3P}(\alpha)$ be $a_1(\alpha)$ and $a_2(\alpha)$, respectively. It can be easily seen that $\tilde{a} = \widetilde{\Pi}_{3P} \bigcap (x\text{-axis})$ represents a fuzzy number since $\widetilde{\Pi}_{3P}(\alpha)$ is closed and bounded across the core plane $\widetilde{\Pi}_{3P}(1)$, and $\tilde{a}(\alpha) = [a_1(\alpha), a_2(\alpha)]$. Now $\tilde{a}(\alpha) \subseteq \tilde{a}(\beta)$, that is, $[a_1(\alpha), a_2(\alpha)] \subseteq [a_1(\beta), a_2(\beta)]$ since $\widetilde{\Pi}_{3P}(\alpha) \subseteq \widetilde{\Pi}_{3P}(\beta)$, for $0 < \beta < \alpha \leq 1$. Hence, $\tilde{a}$ is a fuzzy number with a singleton core. □

3.4.2 Intercept form ($\widetilde{\Pi}_I$)

Let $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ be three fuzzy numbers. Suppose that a fuzzy line $\widetilde{\Pi}_I$ has to be constructed. Consider $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ as the x-intercept, y-intercept and z-intercept, respectively, of the fuzzy plane. To formulate the fuzzy plane $\widetilde{\Pi}_I$, we can consider that $\widetilde{\Pi}_I$ passes through $\widetilde{P}_1$ (on the x-axis), $\widetilde{P}_2$ (on the y-axis) and $\widetilde{P}_3$ (on the z-axis). Evidently, $\widetilde{P}_1$ is at a distance $\tilde{a}$ from the origin, $\widetilde{P}_2$ is at a distance $\tilde{b}$ from the origin, and $\widetilde{P}_3$ is at a distance $\tilde{c}$ from the origin of the $\mathbb{R}^3$ space. Though, there are many such continuous fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$. Consider all the fuzzy planes $\widetilde{\Pi}_{3P}$ passing through all possible such continuous $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$.

We construct $\widetilde{\Pi}_I$ as the fuzzy plane that passes through $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ such that the points $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ have the smallest possible support. If Δ_1, Δ_2 and Δ_3 are the volumes of $\widetilde{P}_1(0)$, $\widetilde{P}_2(0)$ and $\widetilde{P}_3(0)$, respectively, then the fuzzy plane $\widetilde{\Pi}_I$ can be obtained as

$$\widetilde{\Pi}_I = \lim_{\substack{\Delta_1 \to 0\\ \Delta_2 \to 0\\ \Delta_3 \to 0}} \left\{ \widetilde{\Pi}_{3P} : \widetilde{\Pi}_{3P} \text{ passes through } \widetilde{P}_1,\ \widetilde{P}_2 \text{ and } \widetilde{P}_3 \right\} \tag{3.15}$$

since zero is the smallest possible value of Δ_1, Δ_2 and Δ_3.

Note 8. We mean by the smallest possible support of the fuzzy point as the points on the support when $\theta \to 0, \varphi \to 0$, where $\theta \in [0, 2\pi], \varphi \in [0, \pi]$. In simplest way, if the support of the fuzzy point $\widetilde{P}(0, 0, c)$ is described by a parametric representation

$$(x, y, z) = (r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, c + r\cos\varphi),$$

for $r > 0$, $\theta \in [0, 2\pi], \varphi \in [0, \pi]$. Then, for $\theta \to 0, \varphi \to 0$, the point (x, y, z) tends to the $(0, 0, c + r)$. For the smallest possible support $(\theta \to 0, \varphi \to 0)$, the support of fuzzy point $\widetilde{P}(0, 0, c)$ is contained along any coordinate axis (say, z-axis). Here, the notation $\Delta_i \to 0$ for the volumes of $\widetilde{P}_i(0)$ refers to the smallest possible support $(\theta \to 0, \varphi \to 0)$, for $i = 1, 2, 3$.

The S-type space fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ are continuous in (3.15). Since there are many continuous fuzzy points $\widetilde{P}_1$ (on the x-axis), $\widetilde{P}_2$ (on the y-axis) and $\widetilde{P}_3$ (on

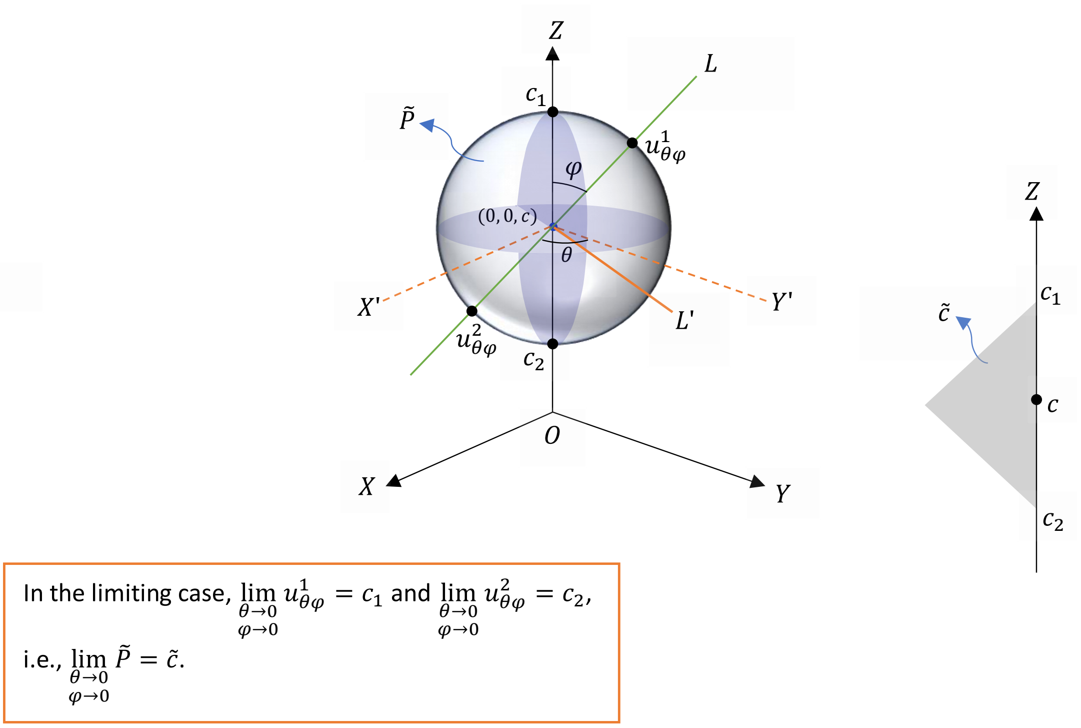

FIGURE 3.4: In the limiting case, the fuzzy set on L converges to $\tilde{c}$

the z-axis), consider all possible $\widetilde{\Pi}_{3P}$ that can be formulated by passing through the same points of $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ (see Definition 3.4.1).

When $\widetilde{P}_1 = \tilde{a}$ (i.e., $\Delta_1 \to 0$), all the fuzzy numbers along the different directions on $\widetilde{P}_1(0)$ will converge to $\tilde{a}$. The same phenomena happens for $\widetilde{P}_2$ and $\widetilde{P}_3$.

This phenomena is described as follows.

Let $\widetilde{P}(0, 0, c)$ be an S-type space fuzzy point located at distance $\tilde{c}$ from O (see Figure 3.4). Let L be a line passing through $(0, 0, c)$ on the support of $\widetilde{P}$, and $u^1_{\theta\varphi}$ and $u^2_{\theta\varphi}$ be two points on the line L (depicted in Figure 3.4). In the limiting case, we attempt to decrease the volume of $\widetilde{P}$ such that the distance between O and $\widetilde{P}$ will remain unchanged ($\tilde{c}$). Apparently, when $\theta \to 0, \varphi \to 0$, the points $u^1_{\theta\varphi}$ and $u^2_{\theta\varphi}$ will tend to the points c_1 and c_2, respectively, such that,

$$\lim_{\substack{\theta\to 0\\ \varphi\to 0}} u^1_{\theta\varphi} = c_1 \text{ and } \lim_{\substack{\theta\to 0\\ \varphi\to 0}} u^2_{\theta\varphi} = c_2.$$

Thus, we have the following definition of $\widetilde{\Pi}_I$ by (3.15).

Definition 3.4.3. (*Intercept form* ($\widetilde{\Pi}_I$)). Let $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ be three fuzzy numbers that are x-intercept, y-intercept and z-intercept, respectively, of a fuzzy plane $\widetilde{\Pi}_I$. The fuzzy plane $\widetilde{\Pi}_I$ can be formulated as $\widetilde{\Pi}_{3P}$, where $\widetilde{P}_1(a, 0, 0) = \tilde{a}$ along the x-axis, $\widetilde{P}_2(0, b, 0) = \tilde{b}$ along the y-axis, and $\widetilde{P}_3(0, 0, c) = \tilde{c}$ along the z-axis.

Thus, if $a \in \tilde{a}(0)$, $b \in \tilde{b}(0)$ and $c \in \tilde{c}(0)$ are the three same points with $\alpha \in [0, 1]$, then the membership value of a point in $\widetilde{\Pi}_I(0)$ can be formulated as

$$\begin{aligned}\mu\left((x, y, z)\middle|\widetilde{\Pi}_I\right) = \ & \sup\{\alpha : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \text{where } a \in \tilde{a}(0),\ b \in \tilde{b}(0) \text{ and} \\ & c \in \tilde{c}(0) \text{ are three same points with membership value } \alpha\}.\end{aligned}$$

For any $\alpha \in [0, 1]$, the α-cuts of $\widetilde{\Pi}_I$ is given by

$$\begin{aligned}\widetilde{\Pi}_I(\alpha) = \bigvee\{\Pi : \Pi &\text{ is the plane with } x\text{-intercept } a \in \tilde{a}(\alpha),\ y\text{-intercept } b \in \tilde{b}(\alpha) \\ &\text{and } z\text{-intercept } c \in \tilde{c}(\alpha),\ \text{and } a, b \text{ and } c \text{ are the same points}\}. \quad (3.16)\end{aligned}$$

Example 3.4.2. (*Intercept form* ($\widetilde{\Pi}_I$)). Let $\tilde{a} = \tilde{b} = \tilde{c} = (3/4/5)$ be three fuzzy numbers. The points '$3+\alpha$, $3+\alpha$ and $3+\alpha$', or '$5-\alpha$, $5-\alpha$ and $5-\alpha$' are the same points of $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$, respectively, with the membership value $\alpha \in [0, 1]$. The support of $\widetilde{\Pi}_I$ is the collection of crisp planes that pass through the points $(3 + \alpha, 0, 0)$, $(0, 3 + \alpha, 0)$ and $(0, 0, 3 + \alpha)$, or $(5 - \alpha, 0, 0)$, $(0, 5 - \alpha, 0)$ and $(0, 0, 5 - \alpha)$ of $\tilde{a}(0)$, $\tilde{b}(0)$ and $\tilde{c}(0)$, respectively. More precisely,

$$\widetilde{\Pi}_I(0) = \bigcup_{\alpha\in[0,1]} \left\{(x, y, z) : \frac{x}{3+\alpha} + \frac{y}{3+\alpha} + \frac{z}{3+\alpha} = 1, \text{or } \frac{x}{5-\alpha} + \frac{y}{5-\alpha} + \frac{z}{5-\alpha} = 1\right\}$$
$$= \left\{(x, y, z) : \frac{x}{3} + \frac{y}{3} + \frac{z}{3} \geq 1,\ \frac{x}{5} + \frac{y}{5} + \frac{z}{5} \leq 1,\ x,\ y,\ z \geq 0\right\},$$

and $\widetilde{\Pi}_I(1) : \frac{x}{4} + \frac{y}{4} + \frac{z}{4} = 1.$

The membership value of a crisp plane $\Pi \in \widetilde{\Pi}_I(0)$ is evaluated by the following theorem.

Theorem 3.4.4. Let $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ be three fuzzy numbers that are x-intercept, y-intercept and z-intercept, respectively, of the fuzzy plane $\widetilde{\Pi}_I$. Let Π be a crisp plane in $\widetilde{\Pi}_I(0)$ and there be three same points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ with respect to $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$, respectively, with

$$\mu\left((a, 0, 0)\middle|\widetilde{\Pi}_I\right) = \mu\left((0, b, 0)\middle|\widetilde{\Pi}_I\right) = \mu\left((0, 0, c)\middle|\widetilde{\Pi}_I\right) = \alpha$$

such that $\Pi : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Then

$$\mu\left(\Pi\middle|\widetilde{\Pi}_I\right) = \alpha.$$

Proof. The proof is similar to that of Theorem 3.4.1. □

The following Algorithm 3.4.2 demonstrates how to find the membership grade of a point in the fuzzy plane $\widetilde{\Pi}_I$.

Algorithm 3.4.2: Algorithm to evaluate $\mu\left((x,y,z)\middle|\widetilde{\Pi}_I\right)$

Input: Given three LR-type fuzzy numbers $\tilde{a} = (m_1 - \ell_1/m_1/m_1 + r_1)_{L_1R_1}$, $\tilde{b} = (m_2 - \ell_2/m_2/m_2 + r_2)_{L_2R_2}$ and $\tilde{c} = (m_3 - \ell_3/m_3/m_3 + r_3)_{L_3R_3}$ that are x-intercept, y-intercept and z-intercept, respectively, of the fuzzy plane $\widetilde{\Pi}_I$, where L_1, L_2, L_3, R_1, R_2 and R_3 are the reference functions.

The membership value $\mu\left((x,y,z)\middle|\widetilde{\Pi}_I\right)$ for a given (x,y,z) has to be calculated.

Output: The membership value $\mu\left((x,y,z)\middle|\widetilde{\Pi}_I\right) = \alpha_{\sup}$.

Initialize $\alpha_{\sup} \leftarrow 0$

loop:

for $\alpha = 0$ to 1; step size $\delta\alpha$ **do**

Compute the same points

$u_1^\alpha = m_1 - \ell_1 L_1^{-1}(\alpha)$, $u_2^\alpha = m_2 - \ell_2 L_2^{-1}(\alpha)$, $u_3^\alpha = m_3 - \ell_3 L_3^{-1}(\alpha)$

and

$v_1^\alpha = m_1 + r_1 R_1^{-1}(\alpha)$, $v_2^\alpha = m_2 + r_2 R_2^{-1}(\alpha)$, $v_3^\alpha = m_3 + r_3 R_3^{-1}(\alpha)$

Compute

$f_{p1} = \frac{x}{u_1^\alpha} + \frac{y}{u_2^\alpha} + \frac{z}{u_3^\alpha}$

$f_{p2} = \frac{x}{v_1^\alpha} + \frac{y}{v_2^\alpha} + \frac{z}{v_3^\alpha}$

if $f_{p1} = 1$ or $f_{p2} = 1$ **then**

if $\alpha_{\sup} < \alpha$ **then**

$\alpha_{\sup} \leftarrow \alpha$

else

goto loop

end

end

end

return $\mu\left((x,y,z)\middle|\widetilde{\Pi}_I\right) = \alpha_{\sup}$

Example 3.4.3. (*Evaluation of the membership values in* $\widetilde{\Pi}_I(0)$).

Let $\tilde{a} = (2-\ell_1/2/2+r_1)_{L_1R_1}$, $\tilde{b} = (3-\ell_2/3/3+r_2)_{L_2R_2}$ and $\tilde{c} = (4-\ell_3/4/4+r_3)_{L_3R_3}$

be three fuzzy numbers with $\ell_1 = r_1 = 2$, $\ell_2 = 1$, $r_2 = 2$, $\ell_3 = 3$ and $r_3 = 5$, where the reference functions are

$$L_1(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ \sqrt{1+x} & \text{if } -1 \leq x \leq 0, \end{cases}$$

$R_1(x) = \max\{0, 1-x\}$, $L_2(x) = R_2(x) = \max\{0, 1-x\}$, and $L_3(x) = R_3(x) = \max\{0, 1-x^2\}$.

The general expression of the same points of $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ are

$$u_1^\alpha : 2 + 2(1-\alpha^2), u_2^\alpha : 3 + (1-\alpha), u_3^\alpha : 4 + 3(\sqrt{1-\alpha})$$

or

$$v_1^\alpha : 2 - 2(1-\alpha), v_2^\alpha : 3 - 2(1-\alpha), v_3^\alpha : 4 - 5(\sqrt{1-\alpha}),$$

respectively. The following Table 3.5 displays the membership grades of some numbers in the fuzzy plane $\widetilde{\Pi}_I$ by execution of the Algorithm 3.4.2.

(x, y, z)	Membership value	Step size
$(3.9800, 0, 0)$	0.1000	$\delta\alpha = 0.0474$
$(0.9450, 1.0790, 1.8195)$	0.7632	$\delta\alpha = 0.0474$
$(2, 0, 0)$	1	$\delta\alpha = 0.0474$

Table 3.5: Membership grades of some numbers of $\widetilde{\Pi}_I$ produced by Algorithm 3.4.2 for Example 3.4.3

Observation 3.4.1. Let $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ be x-intercept, y-intercept and z-intercept, of the fuzzy plane $\widetilde{\Pi}_{I_1}$, respectively. Let $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ be x-intercept, y-intercept and z-intercept, of the fuzzy plane $\widetilde{\Pi}_{I_2}$, respectively. For $\alpha \in [0, 1]$, consider the α-cuts of $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ as $\tilde{a}(\alpha) = [a_1(\alpha), a_2(\alpha)]$, $\tilde{b}(\alpha) = [b_1(\alpha), b_2(\alpha)]$, $\tilde{c}(\alpha) = [c_1(\alpha), c_2(\alpha)]$, $\tilde{p}(\alpha) = [p_1(\alpha), p_2(\alpha)]$, $\tilde{q}(\alpha) = [q_1(\alpha), q_2(\alpha)]$ and $\tilde{r}(\alpha) = [r_1(\alpha), r_2(\alpha)]$, respectively. Note that the core of $\widetilde{\Pi}_{I_1}$ and $\widetilde{\Pi}_{I_2}$ intersect with each other. Let

$$\frac{x}{a_1(\alpha)} + \frac{y}{b_1(\alpha)} + \frac{z}{c_1(\alpha)} = 1 \text{ and } \frac{x}{a_2(\alpha)} + \frac{y}{b_2(\alpha)} + \frac{z}{c_2(\alpha)} = 1$$

be the boundary of α-cut of $\widetilde{\Pi}_{I_1}$, for some $\alpha \in [0, 1]$.

Let

$$\frac{x}{p_1(\alpha)} + \frac{y}{q_1(\alpha)} + \frac{z}{r_1(\alpha)} = 1 \text{ and } \frac{x}{p_2(\alpha)} + \frac{y}{q_2(\alpha)} + \frac{z}{r_2(\alpha)} = 1$$

be the boundary of α-cut of $\widetilde{\Pi}_{I_2}$, for some $\alpha \in [0, 1]$.

Let

$$\gamma_1^\alpha : \left\{ (x, y, z) : \frac{x}{a_1(\alpha)} + \frac{y}{b_1(\alpha)} + \frac{z}{c_1(\alpha)} = 1, \frac{x}{p_1(\alpha)} + \frac{y}{q_1(\alpha)} + \frac{z}{r_1(\alpha)} = 1 \right\},$$

$$\gamma_2^\alpha : \left\{ (x, y, z) : \frac{x}{a_1(\alpha)} + \frac{y}{b_1(\alpha)} + \frac{z}{c_1(\alpha)} = 1, \frac{x}{p_2(\alpha)} + \frac{y}{q_2(\alpha)} + \frac{z}{r_2(\alpha)} = 1 \right\},$$

$$\gamma_3^\alpha : \left\{ (x, y, z) : \frac{x}{a_2(\alpha)} + \frac{y}{b_2(\alpha)} + \frac{z}{c_2(\alpha)} = 1, \frac{x}{p_1(\alpha)} + \frac{y}{q_1(\alpha)} + \frac{z}{r_1(\alpha)} = 1 \right\},$$

and

$$\gamma_4^\alpha : \left\{ (x, y, z) : \frac{x}{a_2(\alpha)} + \frac{y}{b_2(\alpha)} + \frac{z}{c_2(\alpha)} = 1, \frac{x}{p_2(\alpha)} + \frac{y}{q_2(\alpha)} + \frac{z}{r_2(\alpha)} = 1 \right\}$$

be the lines of intersection of boundaries of α-cuts of $\widetilde{\Pi}_{I_1}$ and $\widetilde{\Pi}_{I_2}$. One can see that the intersection of the boundaries of α-cuts of these two fuzzy planes is the surface S which is bounded by the surfaces

$$S_1^\alpha(\lambda) = \{(x, y, z) : (x, y, z) = \lambda x_1 + (1 - \lambda)x_2, \text{ where } x_1 \in \gamma_1^\alpha \text{ and } x_2 \in \gamma_2^\alpha \text{ for } 0 \le \lambda \le 1\},$$

$$S_2^\alpha(\lambda) = \{(x, y, z) : (x, y, z) = \lambda x_3 + (1 - \lambda)x_4, \text{ where } x_3 \in \gamma_3^\alpha \text{ and } x_4 \in \gamma_4^\alpha \text{ for } 0 \le \lambda \le 1\},$$

$$S_3^\alpha(\lambda) = \{(x,y,z) : (x,y,z) = \lambda x_1 + (1-\lambda)x_3, \text{ where } x_1 \in \gamma_1^\alpha \text{ and } x_3 \in \gamma_3^\alpha \text{ for } 0 \le \lambda \le 1\},$$

and

$$S_4^\alpha(\lambda) = \{(x,y,z) : (x,y,z) = \lambda x_2 + (1-\lambda)x_4, \text{ where } x_2 \in \gamma_2^\alpha \text{ and } x_4 \in \gamma_4^\alpha \text{ for } 0 \le \lambda \le 1\}.$$

The surface S can be perceived as the boundary of α-cuts of a space fuzzy line $\widetilde{L}$ since it is a closed and connected subset of $\mathbb{R}^3$, for $\alpha \in [0,1]$.

Now, the space fuzzy line $\widetilde{L}$ is evaluated by the membership function

$$\mu\left((x,y,z)\Big|\widetilde{L}\right) = \min\left\{\mu\left((x,y,z)\Big|\widetilde{\varPi}_{I_1}\right),\ \mu\left((x,y,z)\Big|\widetilde{\varPi}_{I_2}\right)\right\}. \tag{3.17}$$

The core of the $\widetilde{L}$ is the straight line

$$\gamma : \left\{(x,y,z) : \frac{x}{\tilde{a}(1)} + \frac{y}{\tilde{b}(1)} + \frac{z}{\tilde{c}(1)} = 1,\ \frac{x}{\tilde{p}(1)} + \frac{y}{\tilde{q}(1)} + \frac{z}{\tilde{r}(1)} = 1\right\}.$$

To illustrate, let us consider the fuzzy plane $\widetilde{\varPi}_{I_1}$ as in Example 3.4.3. The x-intercept, y-intercept and z-intercept of $\widetilde{\varPi}_{I_1}$ are the fuzzy numbers $\tilde{a} = (2-\ell_1/2/2+r_1)_{L_1R_1}$, $\tilde{b} = (3-\ell_2/3/3+r_2)_{L_2R_2}$ and $\tilde{c} = (4-\ell_3/4/4+r_3)_{L_3R_3}$, respectively. Let $\widetilde{p} = \widetilde{q} = \widetilde{r} = (2/3/4)$ be three fuzzy numbers along x-axis, y-axis and z-axis, respectively. Let $\widetilde{\varPi}_{I_2}$ be the fuzzy plane whose x-intercept, y-intercept and z-intercept are $\widetilde{p}$, $\widetilde{q}$ and $\widetilde{r}$, respectively. For $\alpha \in [0,1]$, $\tilde{a}(\alpha) = [2-2(1-\alpha), 2+2(1-\alpha^2)]$, $\tilde{b}(\alpha) = [3-2(1-\alpha), 3+(1-\alpha)]$, $\tilde{c}(\alpha) = [4-5(\sqrt{1-\alpha}), 4+3(\sqrt{1-\alpha})]$, $\tilde{p}(\alpha) = [2+\alpha, 4-\alpha]$, $\tilde{q}(\alpha) = [2+\alpha, 4-\alpha]$ and $\tilde{r}(\alpha) = [2+\alpha, 4-\alpha]$. Note that the intersection of $\widetilde{\varPi}_{I_1}(1)$ and $\widetilde{\varPi}_{I_2}(1)$ is $\{(x,y,z) : x = \lambda, y = 3-3\lambda, z = 2\lambda, \text{ where } \lambda \in \mathbb{R}\}$.

Suppose we have to determine the membership grade at $(3.9800, 0, 0) \in \widetilde{L}(0)$ by (3.17), where $\widetilde{L}$ is determined by the intersection of $\widetilde{\varPi}_{I_1}$ and $\widetilde{\varPi}_{I_2}$. The membership grade is given by

$$\begin{aligned}\mu\left((3.9800,0,0)\middle|\widetilde{L}\right) &= \min\left\{\mu\left((3.9800,0,0)\middle|\widetilde{\Pi}_{I_1}\right),\ \mu\left((3.9800,0,0)\middle|\widetilde{\Pi}_{I_2}\right)\right\}\\ &= \min\{0.1000, 0.0200\}\\ &= 0.0200.\end{aligned}$$

The following subsection illustrates the construction of a fuzzy plane that passes through a given S-type space fuzzy point and perpendicular to a given crisp direction.

3.4.3 Fuzzy plane passing through an S-type space fuzzy point and perpendicular to a given crisp direction ($\widetilde{\Pi}_{Pn}$)

Let an S-type space fuzzy point $\widetilde{P}$ and a crisp direction be given. A fuzzy plane, $\widetilde{\Pi}_{Pn}$ say, is to be formulated that passes through $\widetilde{P}$. We can visualize the fuzzy plane $\widetilde{\Pi}_{Pn}$ as the collection of the crisp planes that pass through a point in $\widetilde{P}(0)$ and perpendicular to the given crisp direction.

Definition 3.4.4. (*Fuzzy plane* ($\widetilde{\Pi}_{Pn}$)). Let $\widetilde{P}(a,b,c)$ be an S-type space fuzzy point, and (n_1,n_2,n_3) be a given crisp direction. The membership value of $(x,y,z) \in \widetilde{\Pi}_{Pn}(0)$ is formulated by

$$\begin{aligned}\mu\left((x,y,z)\middle|\widetilde{\Pi}_{Pn}\right) = \sup\{\alpha :\ &(x,y,z) \text{ lies on the plane passing through}\\ &(p,q,r) \in \widetilde{P}(\alpha) \text{ and perpendicular to the given crisp}\\ &\text{direction } (n_1,n_2,n_3)\}.\end{aligned}$$

More precisely,

$$\mu\left((x,y,z)\middle|\widetilde{\Pi}_{Pn}\right) = \sup\{\alpha : (x,y,z) \text{ belongs to the plane } n_1(x-p)+n_2(y-q)+n_3(z-r)=0, \text{ where } (p,q,r)\in\widetilde{P}(\alpha)\}.$$

For any $\alpha \in [0,1]$, the α-cuts of $\widetilde{\Pi}_{Pn}$ is given by

$$\widetilde{\Pi}_{Pn}(\alpha) = \bigvee\{\Pi : \Pi \text{ is the plane passing through } (p,q,r)\in\widetilde{P}(\alpha) \text{ and perpendicular to the given crisp direction } (n_1,n_2,n_3)\}. \quad (3.18)$$

Example 3.4.4. (*Fuzzy plane* ($\widetilde{\Pi}_{Pn}$)). Let $\widetilde{P}(1,2,3)$ be an S-type space fuzzy point with the membership function

$$\mu\left((x,y,z)\middle|\widetilde{P}(1,2,3)\right) = \begin{cases} 1-\sqrt{\frac{(x-1)^2}{4}+\frac{(y-2)^2}{4}+\frac{(z-3)^2}{4}} & \text{if } \frac{(x-1)^2}{4}+\frac{(y-2)^2}{4}+\frac{(z-3)^2}{4}\le 1 \\ 0 & \text{otherwise,} \end{cases}$$

and let $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ be a given crisp direction. The support of $\widetilde{P}$ is

$$\left\{(x,y,z) : (x-1)^2+(y-2)^2+(z-3)^2 \le 4\right\}.$$

A generic point of the membership value $\alpha \in [0,1]$ in $\widetilde{P}$ is

$$(1+2(1-\alpha)\sin\varphi\cos\theta, 2+2(1-\alpha)\sin\varphi\sin\theta, 3+2(1-\alpha)\cos\varphi),$$

$\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$. Then, the support of the fuzzy plane $\widetilde{\Pi}_{Pn}$ is

$$\widetilde{\Pi}_{Pn}(0) = \bigcup_{\alpha\in[0,1]} \bigcup_{\varphi\in[0,\pi]} \bigcup_{\theta\in[0,2\pi]} \left\{(x, y, z) : \frac{1}{\sqrt{3}}(x - (1 + 2(1 - \alpha)\sin\varphi\cos\theta)) \right.$$
$$\left. + \frac{1}{\sqrt{3}}(y - (2 + 2(1 - \alpha)\sin\varphi\sin\theta)) + \frac{1}{\sqrt{3}}(z - (3 + 2(1 - \alpha)\cos\varphi)) = 0 \right\}.$$

Proposition 3.4.1. Let $\widetilde{P}(a, b, c)$ be an S-type space fuzzy point, and (n_1, n_2, n_3) be a given crisp direction. Let $\widetilde{\Pi}_{Pn}$ be a fuzzy plane that passes through $\widetilde{P}$ and perpendicular to the given crisp direction (n_1, n_2, n_3). Let $(x_0, y_0, z_0) \in \widetilde{\Pi}_{Pn}(0)$, and Π be a plane passing through (x_0, y_0, z_0) and perpendicular to the given crisp direction (n_1, n_2, n_3). Let (x_c, y_c, z_c) be the closest point on $\widetilde{P} \cap \Pi$ from the core point of $\widetilde{P}$. Then,

$$\mu\left((x_0, y_0, z_0)\middle|\widetilde{\Pi}_{Pn}\right) = \mu\left((x_c, y_c, z_c)\middle|\widetilde{P}\right).$$

Proof. Define a fuzzy set $\widetilde{D}$ with membership value as

$$\mu\left(d\middle|\widetilde{D}\right) = \sup\{\alpha : d = d((x, y, z), (a, b, c))\}, \tag{3.19}$$

where $(x, y, z) \in \widetilde{P} \cap \Pi$, $\mu\left((x, y, z)\middle|\widetilde{P}\right) = \alpha$ and 'd' is the Euclidean distance. The fuzzy set $\widetilde{D}$ is convex since $\widetilde{P}$ is convex. Let d is the Euclidean distance between the points (x_c, y_c, z_c) and (a, b, c), i.e., $d = d((x_c, y_c, z_c), (a, b, c))$. Choose an arbitrary point $(x', y', z') \in \widetilde{P} \cap \Pi$ and let $d' = d((x', y', z'), (a, b, c))$. Since (x_c, y_c, z_c) is the closest point on the $\widetilde{P} \cap \Pi$ from the core point of $\widetilde{P}$, we have $d' \geq d \geq 0$. As $\widetilde{D}$ is convex, we have

$$\mu\left(d\middle|\widetilde{D}\right) \geq \min\{\mu\left(d'\middle|\widetilde{D}\right), \mu\left(0\middle|\widetilde{D}\right)\},$$

i.e., $\mu\left(d\middle|\widetilde{D}\right) \geq \left(d'\middle|\widetilde{D}\right)$. Clearly by (3.19), $\mu\left((x_c, y_c, z_c)\middle|\widetilde{P}\right) \geq \mu\left((x', y', z')\middle|\widetilde{P}\right)$. So, by Definition 3.4.4, we have

$$\mu\left((x_0, y_0, z_0)\middle|\widetilde{\Pi}_{Pn}\right) = \sup_{(x,y,z)\in\widetilde{P}(0)\cap\Pi} \mu\left((x, y, z)\middle|\widetilde{P}\right) = \mu\left((x_c, y_c, z_c)\middle|\widetilde{P}\right).$$

□

Example 3.4.5. Consider the fuzzy point $\widetilde{P}(1, 2, 3)$ as in Example 3.4.4, and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is the given crisp direction. Suppose that the membership value at $(0, 2, 0) \in \widetilde{\Pi}_{Pn}(0)$ has to be obtained. Let

$$\Pi : \frac{x}{\sqrt{3}} + \frac{(y-2)}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 0$$

be the plane passing through the point $(0, 2, 0)$ and perpendicular to the given direction $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. The point $\left(-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right)$ on $\widetilde{P}(0) \cap \Pi$ is closest to the core point $(1, 2, 3)$. Hence, by the Proposition 3.4.1, the membership value of $(0, 2, 0)$ is

$$\mu\left((0, 2, 0)\middle|\widetilde{\Pi}_{Pn}\right) = \mu\left(\left(-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right)\middle|\widetilde{P}\right) = 0.089.$$

Here, as per the notations of the Proposition 3.4.1, $(x_0, y_0, z_0) \equiv (0, 2, 0)$ and $(x_c, y_c, z_c) \equiv \left(-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right)$.

The following Algorithm 3.4.3 demonstrates how to find the membership grade of a number in the fuzzy plane $\widetilde{\Pi}_{Pn}$.

Algorithm 3.4.3: Algorithm to evaluate $\mu\left((x_0, y_0, z_0)\middle|\widetilde{\Pi}_{Pn}\right)$

Input: Given a continuous fuzzy point $\widetilde{P}(a, b, c)$ whose membership function is strictly decreasing along the rays emanated from the core point, and $n = (n_1, n_2, n_3)$ is a given crisp direction.

Given a point (x_0, y_0, z_0) whose membership value in $\widetilde{\Pi}_{Pn}$ is to be calculated.

Output: The membership value $\mu\left((x_0, y_0, z_0)\middle|\widetilde{\Pi}_{Pn}\right) = \alpha$.

Compute

$k = \dfrac{n_1 a + n_2 b + n_3 c - n_1 x_0 - n_2 y_0 - n_3 z_0}{n_1^2 + n_2^2 + n_3^2}$

$(x_c, y_c, z_c) = (a + kn_1, b + kn_2, c + kn_3)$

$\alpha = \mu\left((x_c, y_c, z_c)\middle|\widetilde{P}\right)$

return $\mu\left((x_0, y_0, z_0)\middle|\widetilde{\Pi}_{Pn}\right) = \alpha$

Example 3.4.6. (*Evaluation of the membership values in* $\widetilde{\Pi}_{Pn}(0)$). Let $\widetilde{P}(0,0,0)$ be an S-type space fuzzy point with the membership function

$$\mu\left((x, y, z)\middle|\widetilde{P}(0,0,0)\right) = \begin{cases} 1 - \sqrt{x^2 + y^2 + z^2} & \text{if } x^2 + y^2 + z^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The support of $\widetilde{P}$ is

$$\left\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\right\},$$

and let $(0, 0, 1)$ be a given crisp direction. The following Table 3.6 displays the membership grades of some numbers in the fuzzy plane $\widetilde{\Pi}_{Pn}$ by execution of the Algorithm 3.4.3.

(x, y, z)	Membership Value
$(0, 0, 0.5)$	0.5000
$(1, 0.2, 0.25)$	0.7500
$(0, 1, 0.93)$	0.0700
$(0, 0.2, 0.8)$	0.2000

TABLE 3.6: Membership grades of some numbers of $\widetilde{\mathit{\Pi}}_{Pn}$ produced by Algorithm 3.4.3 for Example 3.4.6

3.4.4 Symmetric fuzzy plane ($\widetilde{\mathit{\Pi}}_S$)

In classical Euclidean geometry, the equation of a plane in normal form is $lx+my+nz=p$, where (l, m, n) is the direction cosine of the normal to the plane and p is the perpendicular distance from the origin to the plane. If we vary the distance p of the plane $lx+my+nz=p$ from the origin, it gives another plane parallel to the plane $lx+my+nz=p$. Let $\widetilde{p}$ be a fuzzy number. For each $p \in \widetilde{p}(0)$, there is a plane parallel to the plane $lx+my+nz=p$. Hence, to define a symmetric fuzzy plane, denoted as $\widetilde{\mathit{\Pi}}_S$, we need only a fuzzy number $\tilde{p}$, and the direction cosine (l, m, n) of the normal to the plane.

Definition 3.4.5. (*Symmetric fuzzy plane* ($\widetilde{\mathit{\Pi}}_S$)). Let $lx+my+nz=p$ be a plane and $\widetilde{p} = (p-\ell/p/p+r)_{LR}$ be a fuzzy number. A symmetric fuzzy plane, denoted as $\widetilde{\mathit{\Pi}}_S$, can be defined by the membership value

$$\mu\left((x, y, z)\middle|\widetilde{\mathit{\Pi}}_S\right) = \mu(p|\widetilde{p}),$$

where $p = lx+my+nz$.

For any $\alpha \in [0, 1]$, the α-cuts of the $\widetilde{\Pi}_S$ is given by

$$\widetilde{\Pi}_S(\alpha) = \bigvee \{(x, y, z) : lx + my + nz = p, \text{ where } p \in \widetilde{p}(\alpha)\}.$$

Example 3.4.7. (*Symmetric fuzzy plane* ($\widetilde{\Pi}_S$)). Let $\frac{1}{\sqrt{14}}x + \frac{2}{\sqrt{14}}y + \frac{3}{\sqrt{14}}z = 8$ be a plane. Let $\widetilde{8} = (8 - \ell/8/8 + r)_{LR}$ be a fuzzy number with $\ell = 3$, $r = 5$ whose reference functions are

$$L(x) = R(x) = \begin{cases} \sqrt{1-x} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1. \end{cases}$$

Accordingly, the membership function of $\widetilde{8}$ is

$$\mu(p|\widetilde{8}) = \begin{cases} \sqrt{1 - \left(\dfrac{8-p}{3}\right)} & \text{if } 5 \leq p \leq 8 \\ \sqrt{1 - \left(\dfrac{p-8}{5}\right)} & \text{if } 8 \leq p \leq 13. \end{cases} \tag{3.20}$$

By Definition 3.4.5, the membership function of $\widetilde{\Pi}_S$ is

$$\mu\left((x, y, z)\middle|\widetilde{\Pi}_S\right) = \mu(p|\widetilde{8}) = \begin{cases} \sqrt{1 - \left(\dfrac{8-p}{3}\right)} & \text{if } 5 \leq p \leq 8 \\ \sqrt{1 - \left(\dfrac{p-8}{5}\right)} & \text{if } 8 \leq p \leq 13, \end{cases}$$

where $\frac{1}{\sqrt{14}}x + \frac{2}{\sqrt{14}}y + \frac{3}{\sqrt{14}}z = p$. It can be easily seen that $\mu\left((8, 5, 3)\middle|\widetilde{\Pi}_S\right) = 0.8594$.

The following defines the notion of the angle between two fuzzy planes.

Definition 3.4.6. (*Angle between two fuzzy planes*). Let $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ be three S-type space fuzzy points whose cores are not collinear and let $\widetilde{\Pi}_{3P}$ be the fuzzy plane that passes through $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$. Again, let $\widetilde{Q}_1$, $\widetilde{Q}_2$ and $\widetilde{Q}_3$ be three S-type space fuzzy points whose cores are not collinear and let $\widetilde{\Pi}_{3Q}$ be the fuzzy plane that passes through $\widetilde{Q}_1$, $\widetilde{Q}_2$ and $\widetilde{Q}_3$. The angle between $\widetilde{\Pi}_{3P}$ and $\widetilde{\Pi}_{3Q}$, $\widetilde{\theta}$ say, can be defined as

$$\mu\left(\theta\middle|\widetilde{\theta}\right) = \sup\{\alpha : \theta \text{ is the angle between the planes } \Pi_{3P} \text{ and } \Pi_{3Q} \text{ that passes through the same points of } \widetilde{P}_1(0),\ \widetilde{P}_2(0),\ \widetilde{P}_3(0), \text{and } \widetilde{Q}_1(0),\ \widetilde{Q}_2(0),\ \widetilde{Q}_3(0),\ \text{respectively, with membership value } \alpha\}.$$

Theorem 3.4.5. For two fuzzy planes $\widetilde{\Pi}_{3P}$ and $\widetilde{\Pi}_{3Q}$ that passes through $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$, and $\widetilde{Q}_1$, $\widetilde{Q}_2$, $\widetilde{Q}_3$, respectively,

(i) $\widetilde{\theta}(\alpha) = \{\theta : \theta \text{ is the angle between the planes } \Pi_{3P} \text{ and } \Pi_{3Q} \text{ that passes through the same points of} \widetilde{P}_1(0),\ \widetilde{P}_2(0),\ \widetilde{P}_3(0),\ \text{and } \widetilde{Q}_1(0),\ \widetilde{Q}_2(0),\ \widetilde{Q}_3(0),\ \text{respectively, with membership value } \alpha\}$.

(ii) $\widetilde{\theta}$ is a fuzzy number in $\mathbb{R}$.

Proof. The proof is similar to that of Theorem 4.1 in [117]. □

Example 3.4.8. (*Angle between two fuzzy planes*). Consider three fuzzy points $\widetilde{P}_1(0,0,0)$, $\widetilde{P}_2(0,1,0)$ and $\widetilde{P}_3(0,0,1)$ with membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_1(0,0,0)\right) = \begin{cases} 1-\sqrt{x^2+y^2+z^2} & \text{if } x^2+y^2+z^2 \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu\left((x,y,z)\middle|\widetilde{P}_2(0,1,0)\right)=\begin{cases}1-\frac{1}{2}\sqrt{x^2+(y-1)^2+z^2} & \text{if } x^2+(y-1)^2+z^2\leq 4\\ 0 & \text{otherwise}\end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_3(0,0,1)\right)=\begin{cases}1-\sqrt{x^2+y^2+(z-1)^2} & \text{if } x^2+y^2+(z-1)^2\leq 1\\ 0 & \text{otherwise.}\end{cases}$$

The general expression of the same points of $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ are

$$\begin{aligned}&\left(u^1\right)^{\alpha}_{\theta\varphi}:((1-\alpha)\sin\varphi\cos\theta,(1-\alpha)\sin\varphi\sin\theta,(1-\alpha)\cos\varphi)\,,\\&\left(u^2\right)^{\alpha}_{\theta\varphi}:(2(1-\alpha)\sin\varphi\cos\theta,1+2(1-\alpha)\sin\varphi\sin\theta,2(1-\alpha)\cos\varphi)\,,\\&\left(u^3\right)^{\alpha}_{\theta\varphi}:((1-\alpha)\sin\varphi\cos\theta,(1-\alpha)\sin\varphi\sin\theta,1+(1-\alpha)\cos\varphi)\,,\end{aligned}$$

respectively.

Consider another three fuzzy points $\widetilde{Q}_1(1,0,0)$, $\widetilde{Q}_2(0,1,0)$ and $\widetilde{Q}_3(0,0,1)$ with membership functions

$$\mu\left((x,y,z)\middle|\widetilde{Q}_1(1,0,0)\right)=\begin{cases}1-\sqrt{(x-1)^2+y^2+z^2} & \text{if } (x-1)^2+y^2+z^2\leq 1\\ 0 & \text{otherwise,}\end{cases}$$

$$\mu\left((x,y,z)\middle|\widetilde{Q}_2(0,1,0)\right)=\begin{cases}1-\sqrt{x^2+(y-1)^2+z^2} & \text{if } x^2+(y-1)^2+z^2\leq 1\\ 0 & \text{otherwise}\end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{Q}_3(0,0,1)\right)=\begin{cases}1-\frac{1}{2}\sqrt{x^2+y^2+(z-1)^2} & \text{if } x^2+y^2+(z-1)^2\leq 4\\ 0 & \text{otherwise.}\end{cases}$$

The general expression of the same points of $\widetilde{Q}_1$, $\widetilde{Q}_2$ and $\widetilde{Q}_3$ are

$$\begin{aligned}
&\left(v^1\right)^{\alpha}_{\theta\varphi} : \left(1+(1-\alpha)\sin\varphi\cos\theta,(1-\alpha)\sin\varphi\sin\theta,(1-\alpha)\cos\varphi\right),\\
&\left(v^2\right)^{\alpha}_{\theta\varphi} : \left((1-\alpha)\sin\varphi\cos\theta,1+(1-\alpha)\sin\varphi\sin\theta,(1-\alpha)\cos\varphi\right),\\
&\left(v^3\right)^{\alpha}_{\theta\varphi} : \left(2(1-\alpha)\sin\varphi\cos\theta,2(1-\alpha)\sin\varphi\sin\theta,1+2(1-\alpha)\cos\varphi\right),
\end{aligned}$$

respectively.

Let $\widetilde{\Pi}_{3P}$ and $\widetilde{\Pi}_{3Q}$ be two fuzzy planes passing through $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{P}_3$, and $\widetilde{Q}_1$, $\widetilde{Q}_2$, $\widetilde{Q}_3$, respectively.

The following Table 3.7 and Figure 3.5 show the α-cuts and the membership function of the angle between two fuzzy planes $\widetilde{\Pi}_{3P}$ and $\widetilde{\Pi}_{3Q}$.

α	$\widetilde{\theta}(\alpha)$
0.1	$[35.3499, 101.0759]$
0.2	$[36.8108, 94.9898]$
0.3	$[38.4282, 88.7758]$
0.4	$[40.2120, 82.8473]$
0.5	$[42.1710, 77.0350]$
0.6	$[44.3122, 71.5421]$
0.7	$[46.4788, 66.5157]$
0.8	$[48.8802, 62.0344]$
0.9	$[51.6328, 58.1239]$
1	54.7356

TABLE 3.7: α-cuts of the angle between two fuzzy planes $\widetilde{\Pi}_{3P}$ and $\widetilde{\Pi}_{3Q}$ for Example 3.4.8

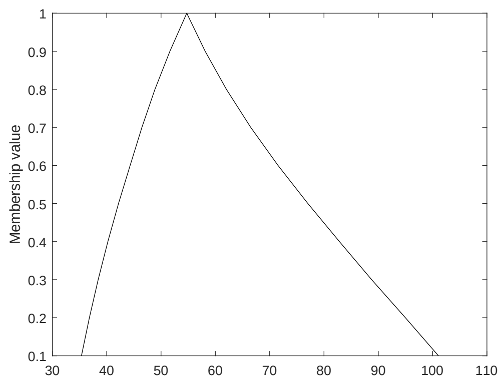

FIGURE 3.5: Angle between two fuzzy planes $\widetilde{\Pi}_{3P}$ and $\widetilde{\Pi}_{3Q}$ for Example 3.4.8

The following defines the distance between an S-type space fuzzy point and a fuzzy plane.

Definition 3.4.7. (*Distance between an S-type space fuzzy point and a fuzzy plane*). Let $\widetilde{P}$ be an S-type space fuzzy point and let $\widetilde{\Pi}$ be a fuzzy plane. The distance between $\widetilde{P}$ and $\widetilde{\Pi}$, denoted as $\widetilde{D}$, can be defined by

$$\widetilde{D} = \bigvee_{\alpha \in [0,1]} [d_l^\alpha, d_u^\alpha], \tag{3.21}$$

where

$$d_l^\alpha = \inf \left\{ d(u,v) : \ u \in \widetilde{P}(\alpha) \text{ and } v \in \widetilde{\Pi}(\alpha) \right\}$$

and

$$d_u^\alpha = \sup \left\{ d(u,v) : \ u \in \widetilde{P}(\alpha) \text{ and } v \in \widetilde{\Pi}(\alpha) \right\}.$$

The notion d_l^α and d_u^α are adjoined with the membership value α.

Theorem 3.4.6. The distance $\widetilde{D}$ between a fuzzy point and a fuzzy plane is a fuzzy number.

Proof. The proof is similar to that of Theorem 4.1 in [117]. □

Example 3.4.9. (*Distance between an S-type space fuzzy point and a fuzzy plane*). Consider three fuzzy points $\widetilde{P}_1(1,0,0)$, $\widetilde{P}_2(0,1,0)$ and $\widetilde{P}_3(0,0,1)$ as in Example 3.4.8. Let $\widetilde{\Pi}_{3P}$ be the fuzzy plane that passes through the fuzzy points $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$, and let $\widetilde{P}(0,0,0)$ be a fuzzy point whose core does not lie in the core plane of $\widetilde{\Pi}_{3P}$. The membership function of $\widetilde{P}(0,0,0)$ is

$$\mu\left((x,y,z)\middle|\widetilde{P}(0,0,0)\right) = \begin{cases} 1 - \sqrt{x^2+y^2+z^2} & \text{if } x^2+y^2+z^2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

The following Table 3.8 and Figure 3.6 show the α-cuts and the membership function of the distance between the fuzzy point $\widetilde{P}$ and the fuzzy plane $\widetilde{\Pi}_{3P}$.

α	$\widetilde{D}(\alpha)$
0.1	$[0.0990, 0.7070]$
0.2	$[0.1751, 0.7071]$
0.3	$[0.2423, 0.7070]$
0.4	$[0.3069, 0.7031]$
0.5	$[0.3666, 0.6935]$
0.6	$[0.4154, 0.6784]$
0.7	$[0.4617, 0.6588]$
0.8	$[0.5046, 0.6360]$
0.9	$[0.5433, 0.6087]$
1	0.5774

TABLE 3.8: α-cuts of the distance between the fuzzy point $\widetilde{P}$ and the fuzzy plane $\widetilde{\Pi}_{3P}$ for Example 3.4.9

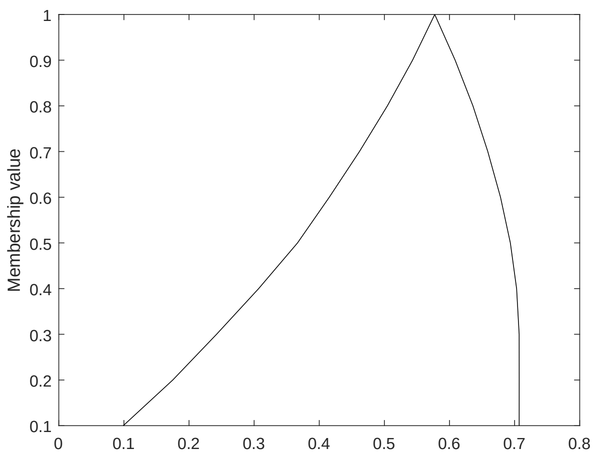

FIGURE 3.6: Distance between the fuzzy point $\widetilde{P}$ and the fuzzy plane $\widetilde{\Pi}_{3P}$ for Example 3.4.9

The following section compares the proposed formulations of the space fuzzy lines, the shortest distance between symmetric skew fuzzy lines, and the fuzzy planes with existing formulations [43, 47, 38, 44, 46, 22, 9, 37, 7, 21, 45].

3.5 Discussion and comparison

In the proposed formulation of the space fuzzy lines and the fuzzy planes, we observe the following properties:

(i) Space fuzzy line segments are extended bi-infinitely to form the space fuzzy lines (see Section 3.2).

(ii) The support of the particular form of space fuzzy line (symmetric fuzzy line) is a right circular cylinder. The axis of this cylinder is the core line $\widetilde{L}_S(1)$ (see Definition 3.2.1).

(iii) The α-cuts of the space fuzzy lines are closed, connected, and an arc-wise connected subset of $\mathbb{R}^3$ but are not necessarily convex. However, the α-cuts of the symmetric fuzzy lines are closed, connected, arc-wise connected, and convex subset of $\mathbb{R}^3$ (see (3.8)).

(iv) Space fuzzy lines are always normal fuzzy sets.

(v) The intersection of a plane perpendicular to the core line of the space fuzzy line is a fuzzy point along that plane (see Theorem 3.2.2).

(vi) The intersection of two space fuzzy lines may not be an S-type space fuzzy point (see Figure 3.2), but the intersection of two symmetric fuzzy lines is an S-type space fuzzy point (see Observation 3.2.2).

(vii) The mathematical expressions of the membership functions of fuzzy planes (see Definitions 3.4.1, 3.4.3, 3.4.4) demonstrate that the fuzzy plane is a collection of crisp points with varied membership grades, or a collection of crisp planes joining same points with different membership values.

(viii) A fuzzy plane is always a normal fuzzy set, and its core is a crisp plane.

(ix) The α-cuts of fuzzy planes are closed, connected and convex subsets of $\mathbb{R}^3$ (see (3.14), (3.16), (3.18)).

(x) The intersection of two fuzzy planes is a space fuzzy line (see Observation 3.4.1).

We have made a remark based on points (vi) and (x) (in p. 143) regarding the intersection of two space fuzzy lines and two fuzzy planes as per the approach of [89].

Remark 3.5.1. (i) As per the approach of [89], the fuzzy lines depicted in Figure 3.2 are fuzzily intersecting fuzzy lines since their cores do not intersect even when extended. Still, some portions of the fuzzy lines intersect in $\mathbb{R}^3$. In this case, as per the notions of [89], the measure of the intersection of two fuzzy lines ($v = 1 - \rho$) lies between 0 and 1. Here, $\rho = 1 - \lambda$ gives the measure of parallelness, and

$$\lambda = \sup_{(x,y,z)\in\mathbb{R}^3} \left\{ \min \left(\mu \left((x,y,z) \middle| \widetilde{L_1} \right),\ \mu \left((x,y,z) \middle| \widetilde{L_2} \right) \right) \right\} \tag{3.22}$$

gives the height of the fuzzy intersection region. Explicitly, we can see that $0 < v < 1$ since the cores of fuzzy lines never intersect ($0 < \lambda < 1$ by (3.22)).

(ii) As per the approach of [89], the considered fuzzy lines in Observation 3.2.2 (ii) are completely intersecting fuzzy lines since their cores intersect. In this case, as per the notions of [89], the measure of the intersection of two fuzzy lines is $v = 1$. This is because the hight of fuzzy intersection region is $\lambda = 1$ by (3.22).

(iii) As per the approach of [89], the considered fuzzy planes in Observation 3.4.1 are completely intersecting fuzzy planes since their cores intersect. The hight of fuzzy intersection region is $\lambda = 1$ by (3.22). Hence, the measure of intersection of two fuzzy planes is $v = 1$ since $\rho = 1 - \lambda = 0$.

Below, we give a point-wise comparison of space fuzzy lines and then of fuzzy planes.

Note 9. All the comparisons of the proposed ideas for space fuzzy lines and the shortest distance between skew fuzzy lines are made with the formulations of [38, 21, 7] and [7, 38, 47, 43, 44, 46, 45], respectively, when the concepts of [43, 47, 38, 44, 46, 21, 7, 45] are extended in the same way in the fuzzy space geometry.

The point-wise comparison of space fuzzy lines and the shortest distance between skew fuzzy lines are as follows:

- ***Space fuzzy line.*** As per the approach of [38, 21, 7], the definition and deficiency of space fuzzy line segments (in space-$\mathbb{R}^3$) is given in [117]. It is noted in [117] that the space fuzzy line segment in [117] is more appropriate than [38, 21, 7], and the proposed space fuzzy line is a union of the space fuzzy line segments (see Definition 2.5.1 and (3.2) and (3.1)). Hence, the proposed space fuzzy line is better than the formulations in [38, 21, 7].

- ***Shortest distance between two skew fuzzy lines.*** In [117], it is noted that the fuzzy distance between pair of fuzzy sets proposed in [117] is more appropriate than the existing formulations [7, 38, 47, 43, 44, 46, 45]. This is because, in [117], the concept of the same and inverse point has been used. In a similar manner, the proposed shortest distance between skew fuzzy lines is also based on the same and inverse point theory. Hence, the proposed formulations is highly effective than that of [7, 38, 47, 43, 44, 46, 45].

Next, we present interrelations between the proposed different forms of fuzzy planes. In general, $\widetilde{\Pi}_{3P}$ or $\widetilde{\Pi}_I$ cannot be equivalent to $\widetilde{\Pi}_{Pn}$ since all the crisp planes are parallel in the support of $\widetilde{\Pi}_{Pn}$. However, $\widetilde{\Pi}_{3P}$ is an $\widetilde{\Pi}_I$ and vice versa.

Theorem 3.5.1. $\widetilde{\Pi}_I$ is an $\widetilde{\Pi}_{3P}$.

Proof. The proof is similar to that of Theorem 4.0.3 in [2]. □

In this paper, we have given three approaches to construct the fuzzy plane. The approach of the fuzzy plane passing through an S-type space fuzzy point and perpendicular to a given crisp direction is one of them. In this approach, a question may arise: why a crisp direction is being taken rather than the fuzzy direction (see Definition 3.4.4). To answer this question, from the (iii) (p. 148), we have observed that if we take a fuzzy direction instead of a crisp direction, the support of the fuzzy plane may be unbounded. Hence, we have neglected that case in this article.

Let us compare the proposed methodological analysis with existing formulations of fuzzy planes in [22, 9, 37, 7].

In the literature, Rosenfeld [37] and Ghosh and Chakraborty [9] introduced some concepts regarding the fuzzy half-plane. The fuzzy half-plane in [37] is either the entire plane $\mathbb{R}^2$, or half-plane bounded by a line, or empty. According to [9], a fuzzy half-plane can be determined when the fuzzy line $\widetilde{L}$ fuzzily separates the whole $\mathbb{R}^2$-plane into two parts. The points on the support of fuzzy (closed) half-plane in [9] are either on one side of $\widetilde{L}(1)$ or on it, but no points on the other side. A fuzzy plane, according to [22], is a thin planer shell with variable thickness containing the family of crisp planes. The crisp planes are the extension of the family of crisp lines representing a fuzzy line. All these proposed fuzzy half-planes in [9, 37] and fuzzy planes in [22] referred to as the fuzzy line in $\mathbb{R}^2$-plane. Thus, for $\alpha \in [0, 1]$, the α-cuts of the fuzzy half-planes or fuzzy planes in [22, 9, 37] is a subset of $\mathbb{R}^2$-plane, or entire $\mathbb{R}^2$-plane. Therefore, these definitions do not coincide with the conventional plane definitions in the Euclidean space $\mathbb{R}^3$. Whereas a fuzzy plane should be a fuzzy set in $\mathbb{R}^3$, and the core must coincide with a crisp plane's customary definitions.

After analyzing all these deficiencies of fuzzy half-planes or fuzzy planes in [22, 9, 37], we have proposed three different forms of the fuzzy planes (see Definitions 3.4.1, 3.4.3, 3.4.4) by extending the definitions of the Euclidean plane (in $\mathbb{R}^3$-space) in the fuzzy environment. Prior to the proposed work, only Qiu and Zhang [7] demonstrated the fuzzy plane using extension principle [15] by extending the known ideas in [5, 6]. Additionally, the fuzzy plane, described in [7], follows the definitions of the Euclidean plane when its attributes are imprecise. However, one can visualize that three proposed constructions for fuzzy planes are either easier to evaluate the membership functions or have less spread when compared to those of Qiu and Zhang in [7]. This is because the proposed fuzzy planes are based on the theory of same and inverse points (see Definitions 3.4.1, 3.4.3, 3.4.4). A point-wise comparison of these two different approaches of the fuzzy plane is realized as mentioned below.

(i) ***(Three-point form).***

As per the approach of [7], a fuzzy plane may be defined as

$$\widetilde{\Pi}_{3P} = \bigvee\{(x,y,z) : (x,y,z) \text{ lies on the plane passing through the points in } \widetilde{P}_1(0),\ \widetilde{P}_2(0) \text{ and } \widetilde{P}_3(0)\}.$$

However, we have investigated the fuzzy plane as

$$\widetilde{\Pi}_{3P} = \bigvee\{(x,y,z) : (x,y,z) \text{ lies on the plane passing through the same points of } \widetilde{P}_1(0),\ \widetilde{P}_2(0) \text{ and } \widetilde{P}_3(0) \text{ with membership value } \alpha\}.$$

The proposed Definition 3.4.1 shows the fuzzy plane as

$$\widetilde{\Pi}_{3P} = \bigvee_{\alpha\in[0,1]} \widetilde{\Pi}_{3P}(\alpha),$$

where

$$\widetilde{\Pi}_{3P}(\alpha) = \{(x, y, z) : n_1(x-u) + n_2(y-v) + n_3(z-w) = 0, \text{ where } u \in \widetilde{P}_1(\alpha), \\ v \in \widetilde{P}_2(\alpha) \text{ and } w \in \widetilde{P}_3(\alpha) \text{ are the same points, and } (n_1, n_2, n_3) = \\ \text{cross}(u-v, u-w)\}.$$

In fact, the support of the proposed fuzzy plane $\widetilde{\Pi}_{3P}$ is a subset of the support of a fuzzy plane in [7]. This is because in the proposed method we considers only the combination of the same points. Hence, the proposed fuzzy plane has less imprecision than that in [7].

In addition, the evaluation of the membership function for $\widetilde{\Pi}_{3P}$ becomes easier than that in [7] since the explicit expression of the same points is used in the proposed method.

(ii) ***(Intercept form).***

Intercept form of the fuzzy plane $\widetilde{\Pi}_I$ is defined (Definition 3.4.3) as the limiting case of the three-point form of the fuzzy plane. According to the proposed approach, the fuzzy plane $\widetilde{\Pi}_I$ is a collection of crisp planes whose intercepts are the same points of the given $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$. The Definition 3.4.3 evaluates the fuzzy plane $\widetilde{\Pi}_I$ as

$$\widetilde{\Pi}_I = \bigvee_{\alpha \in [0,1]} \widetilde{\Pi}_I(\alpha),$$

where

$$\widetilde{\Pi}_I(\alpha) = \{(x, y, z) : \frac{x}{u_1^\alpha} + \frac{y}{u_2^\alpha} + \frac{z}{u_3^\alpha} = 1 \text{ or } \frac{x}{v_1^\alpha} + \frac{y}{v_2^\alpha} + \frac{z}{v_3^\alpha} = 1, \text{ where } u_1^\alpha,\ u_2^\alpha, \\ u_3^\alpha \text{ and } v_1^\alpha,\ v_2^\alpha,\ v_3^\alpha \text{ are the same points of } \tilde{a},\ \tilde{b} \text{ and } \tilde{c}, \text{ respectively}\}.$$

However, Qiu and Zhang [7] considered all the possible crisp planes whose intercepts are arbitrary points of the given $\tilde{a}(0)$, $\tilde{b}(0)$ and $\tilde{c}(0)$. Clearly, in the proposed consideration of $\widetilde{\Pi}_I$, there is a narrower spread than such construction as in Qiu and Zhang [7]. In the proposed analysis, we find that the proposed $\widetilde{\Pi}_I$ across the core plane $\widetilde{\Pi}_I(1)$ has a bounded imprecise part. In contrast, the fuzzy plane $\widetilde{\Pi}_I$ defined in [7] has unbounded support on either side of $\widetilde{\Pi}_I(1)$. The membership grade evaluation of a point in $\widetilde{\Pi}_I(0)$ is easier than that in [7]. This is because the constraint set which we use in the optimization problem (see Definition 3.4.3) to get the membership value of $\widetilde{\Pi}_I$ is a proper subset of that used by Qiu and Zhang [7].

(iii) ***(A fuzzy plane passing through an*** S***-type space fuzzy point*** $\widetilde{P}$ ***and perpendicular to a given crisp direction*** n **(**$\widetilde{\Pi}_{Pn}$**)).**

In the proposed $\widetilde{\Pi}_{Pn}$ (Definition 3.4.4), we consider the only crisp direction, and an S-type space fuzzy point. By Definition 3.4.4, $\widetilde{\Pi}_{Pn}$ is a collection of the crisp planes parallel to the core plane. Thus, the proposed $\widetilde{\Pi}_{Pn}$ has a bounded imprecise part across the core plane $\widetilde{\Pi}_{Pn}(1)$.

In contrast, the fuzzy plane passing through a space fuzzy point $\widetilde{P}$ and perpendicular to a given *fuzzy* direction $\widetilde{N}$, $\widetilde{\Pi}_{PN}$ say, (see definition by Qiu and Zhang [7]) has unbounded support since a fuzzy direction $\widetilde{N}$ is a collection of crisp directions. Also, for each $n \in \widetilde{N}(0)$, there exists a fuzzy plane passing through a space fuzzy point $\widetilde{P}$ and perpendicular to a given crisp direction n ($\widetilde{\Pi}_{Pn}$). More precisely,

$$\widetilde{\Pi}_{PN} = \bigcup_{n \in \widetilde{N}(0)} \widetilde{\Pi}_{Pn},$$

where the membership function of $\widetilde{\Pi}_{Pn}$ is given by the Definition 3.4.4. If the fuzzy direction $\widetilde{N}$ reduces to a crisp direction n, then $\widetilde{\Pi}_{Pn} = \widetilde{\Pi}_{PN}$.

The alternative approaches to construct the fuzzy plane depend on what information is known about it. The three-point form of the fuzzy plane must be preferred when the three S-type space fuzzy points are known. It is preferable to use the intercept form of a fuzzy plane when its three intercepts are known. When an S-type space fuzzy point and a crisp direction are known, the third approach of the fuzzy plane (Definition 3.4.4) is preferred. Therefore, the use of different formulations of fuzzy planes is favoured according to the information available for fuzzy planes in our study.

3.6 Conclusion

In this paper, we have discussed space fuzzy lines and three different forms of a fuzzy plane in $\mathbb{R}^3$. Notably, new ideas of the skew fuzzy lines and the shortest distance between skew fuzzy lines are presented. Moreover, the three different forms of fuzzy planes, namely, three-point form, intercept form, and a fuzzy plane passing through an S-type space fuzzy point and perpendicular to a given crisp direction, have been proposed. We have also developed the algorithms for finding the membership values of all the proposed formulations of the space fuzzy lines and the fuzzy planes and added suitable numerical examples. The geometric properties of all these proposed forms of fuzzy planes and their interrelations are also investigated. The intersection of two space fuzzy lines may not be a space fuzzy point as the α-cuts of space fuzzy line may not be convex (see Figure 3.2). However, the intersection of two symmetric fuzzy lines is a fuzzy point (see Observation 3.2.2). Also, the interrelations between the three-point form and intercept form are found to be equivalent. In a sequel, we have defined the notion of the angle between two fuzzy planes. Theorem 3.4.5 shows that the angle between two fuzzy planes is a fuzzy number. Also, we have defined the distance between an S-type space fuzzy point and a fuzzy plane, which is a fuzzy number (see Theorem 3.4.6). A brief analysis of the shortest distance between two non-symmetric skew fuzzy lines has been given in this paper. However, a detailed study of the shortest distance between two non-symmetric skew fuzzy lines will be discussed in the future.

In a future study, the fuzzy space geometrical objects such as fuzzy cones, fuzzy spheres, fuzzy ellipsoids, and their properties will be formulated in detail. All the ideas can be extended to an n-dimensional space, $n \geq 4$. Future work can focus on such an extension.

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