Chapter 2

Analytical fuzzy space geometry I

2.1 Introduction

This chapter continues the study on fuzzy geometry [1, 2, 3, 4, 35]. In [1, 2, 3, 4, 35], Ghosh and Chakraborty have investigated the basic concepts of fuzzy plane geometry using the theory of *same and inverse points*. In this article, we study a few basic concepts on fuzzy space geometry. Throughout the chapter, by *space* we mean the three-dimensional Euclidean space ($\mathbb{R}^3$-space).

The basis on which the entire construction of fuzzy plane geometry by Ghosh and Chakraborty [1, 2, 3, 4, 35] is laid is the concept of *same-and-inverse points*. The idea of same and inverse points with respect to a pair of fuzzy points was figured out in [1] by finding a fuzzy point as a collection of normal and convex fuzzy sets along the lines passing through the core of the fuzzy points. It is reported in [1] that if the fuzzy sets along the same direction are combined by the extension principle, then two types of combinations can be found: effective combinations (combinations of same or inverse points) and redundant combinations. After identifying redundant combinations, the concepts of the same points and inverse points have been introduced in [1].

2.2 Motivation and Contributions

It is observed in [1, 2, 3] that the ideas of same and inverse points are truly effective to construct fuzzy plane geometrical elements. Furthermore, in [87], it is reported that the theory of the same points can massively reduce the computational cost. Moreover, fuzzy geometry has been successfully applied in many realistic fields (see Subsection). Thus, there is a need to advance the theory of fuzzy geometry. Due to this need, in this paper, we attempt to develop fuzzy space geometrical elements with the help of the same and inverse points.

The main contribution and novelty of this chapter are as follows.

(i) We introduce a three-variable reference function (see Subsection 2.3.1) by which space fuzzy points are represented in a unified way (Theorem 2.3.1). Accordingly, we define an S-type space fuzzy point (Definition 2.3.2) to represent a fuzzy point with the help of a reference function. Importantly, it is shown that on the omission of two variables, a three variable reference function S reduces to an LR-reference function for fuzzy numbers (Note 2). Also, by extending the number of variables from three to n, the reference function will act as a reference function for a fuzzy point in $\mathbb{R}^n$.

(ii) With the help of S-type representation of space fuzzy points, we give an explicit expression of the same and inverse points (see Subsections 2.3.3.4 and 2.3.3.5) for two general S-type space fuzzy points. As the concepts of same and inverse points are the basis for this study, the identification of explicit expressions facilitates the computations of membership functions of the proposed fuzzy geometrical entities and fuzzy distance.

(iii) With the help of the same and inverse point theory, the concepts of fuzzy line segments and fuzzy distance for fuzzy space geometry are introduced. It is noteworthy that the proposed space fuzzy line segment and fuzzy distance are based on S-type representation fuzzy points. In contrast, the fuzzy line segment and fuzzy distance in the $\mathbb{R}^2$-plane are not based on reference functions. Thus, the proposed ideas are not a straightforward extension of the existing two-dimensional entities.

(iv) With the help of the explicit expressions of same and inverse points, we provide the explicit step-wise procedure to find the fuzzy distance between two fuzzy points (Algorithm 2.4.1), to compute the membership value of a point in the fuzzy distance (Algorithm 2.4.2), to execute space fuzzy line segment joining two fuzzy points (Algorithm 2.5.1), and the evaluation of the membership value of a point in space fuzzy line segment (Algorithm 2.5.2).

2.3 Space fuzzy point

In this section, we propose a reference function of three variables to represent space fuzzy points. We further define the concepts of the same and inverse points for space fuzzy points. Next, we give general expressions of the same and inverse points with the help of three-variable reference functions.

Definition 2.3.1. (*Space fuzzy point*). A space fuzzy point at $(a, b, c) \in \mathbb{R}^3$, denoted $\widetilde{P}(a, b, c)$, is a fuzzy set in $\mathbb{R}^3$ whose membership function has the following properties:

(i) $\mu\left((x, y, z)\middle|\widetilde{P}(a, b, c)\right) = 1$ only at $(x, y, z) = (a, b, c)$, and

(ii) $\widetilde{P}(a, b, c)(\alpha)$ is a compact and convex subset of $\mathbb{R}^3$, for all α in $[0, 1]$.

Example 2.3.1. The fuzzy set $\widetilde{P}(a,b,c)$ in $\mathbb{R}^3$ that has the membership function

$$\mu\left((x,y,z)\middle|\widetilde{P}(a,b,c)\right) = \begin{cases} 1-\sqrt{\left(\frac{x-a}{p}\right)^2+\left(\frac{y-b}{q}\right)^2+\left(\frac{z-c}{r}\right)^2} & \text{if } \left(\frac{x-a}{p}\right)^2+\left(\frac{y-b}{q}\right)^2+\left(\frac{z-c}{r}\right)^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

is a space fuzzy point. For any $\alpha \in (0,1]$, the α-cut of the space fuzzy point $\widetilde{P}(a,b,c)$ is the ellipsoid

$$\widetilde{P}(a,b,c)(\alpha) = \left\{(x,y,z) \in \mathbb{R}^3 : \left(\frac{x-a}{p}\right)^2+\left(\frac{y-b}{q}\right)^2+\left(\frac{z-c}{r}\right)^2 \leq (1-\alpha)^2\right\},$$

which is evidently a convex and compact subset of $\mathbb{R}^3$.

Space fuzzy points are basic elements to develop fuzzy space geometry. For a unified representation of space fuzzy points, we construct a reference function in $\mathbb{R}^3$. After that, we formulate a space fuzzy point by a reference function.

2.3.1 A reference function of three variables

A function $S : \mathbb{R}^3 \to [0,1]$ which

1. is non-increasing along any direction (d_1, d_2, d_3) emanated from $(0,0,0)$,
2. is symmetric about $(0,0,0)$ and about the axes on any line $L : \frac{x}{d_1} = \frac{y}{d_2} = \frac{z}{d_3}$ passing through origin, i.e., for any $\lambda \in \mathbb{R}$:

$$\begin{aligned} S(\lambda d_1, \lambda d_2, \lambda d_3) &= S(-\lambda d_1, -\lambda d_2, -\lambda d_3) \\ &= S(-\lambda d_1, \lambda d_2, \lambda d_3) = S(\lambda d_1, -\lambda d_2, \lambda d_3) = S(\lambda d_1, \lambda d_2, -\lambda d_3), \text{and} \end{aligned}$$

3. satisfies either of the following two conditions:

(i) $S(0,0,0) = 1$ and there exists a norm $\|\cdot\| : \mathbb{R}^3 \to [0,\infty)$ such that on the unit sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : \|(x,y,z)\| = 1\}$, the function value of S is zero, i.e., $S(\mathbb{S}^2) = \{S(x,y,z) : (x,y,z) \in \mathbb{S}^2\} = \{0\}$, or

(ii) $S(0,0,0) = 1$, $S(x,y,z) > 0$ for all (x,y,z) in $\mathbb{R}^3$, and $\lim\limits_{x\to+\infty} S(x,y,z) = \lim\limits_{y\to+\infty} S(x,y,z) = \lim\limits_{z\to+\infty} S(x,y,z) = 0$

is called a *reference function* of three variables.

Few examples of three-variable reference functions are

(a) $S(x,y,z) = \max\{0, 1 - (|x| + |y| + |z|)^p\}$ with $p > 0$,

(b) $S(x,y,z) = \max\left\{0, 1 - (|x|^p + |y|^p + |z|^p)^{\frac{1}{p}}\right\}$ with $p > 0$,

(c) $S(x,y,z) = \max\{0, 1 - \|(x,y,z)\|\}$ where $\|\cdot\|$ is any norm in $\mathbb{R}^3$,

(d) $S(x,y,z) = \exp(-(|x| + |y| + |z|))$,

(e) $S(x,y,z) = \exp(-(x^2 + y^2 + z^2))$, etc.

Note 2. We observe that in the omission of any two of the three variables x, y, and z, the definition of the *reference function* of three variables reduces to the definition of *reference function for a fuzzy number* (Definition 1.3.3). For instance, if we omit y and z in the definition of the reference function S, then S becomes non-increasing in $[0,\infty)$, symmetric about '0' and the conditions (3i) and (3ii) for S reduce to the following conditions:

(i) $S(0) = 1$ and $S(1) = 0$, or

(ii) $S(0) = 1$, $S(x) > 0$ for all x, and $\lim\limits_{x\to+\infty} S(x) = 0$, respectively. Hence, S qualifies all the conditions of Definition 1.3.3. Thus, the definition of *reference function of three-variables* is a true generalization of the Definition 1.3.3 of *reference function for fuzzy numbers*. Furthermore, it is noticeable that just by extending the number of variables from three to n, we will get an n-variable reference function for a fuzzy point in $\mathbb{R}^n$.

2.3.2 Representation of a space fuzzy point by a reference function

Definition 2.3.2. (*S-type space fuzzy point*). A space fuzzy point $\widetilde{P}$ is called an S-type space fuzzy point at (a, b, c) if there exists a three-variable reference function S and a homeomorphism $T_\alpha : \partial\bar{P}(\alpha) \to \mathbb{S}^2(\alpha)$ with $T_\alpha(0, 0, 0) = (0, 0, 0)$ for all $\alpha \in [0, 1]$ such that

$$\mu\left((x, y, z)\middle|\widetilde{P}\right) = \alpha = S(T_\alpha(x - a, y - b, z - c)),$$

where $\partial\bar{P}(\alpha)$ is the set of all boundary points of the convex set

$$\bar{P}(\alpha) = \left\{(x - a, y - b, z - c) : (x, y, z) \in \widetilde{P}(\alpha)\right\}$$

and $\mathbb{S}^2(\alpha)$ is the sphere $\{(x, y, z) \in \mathbb{R}^3 : \|(x, y, z)\| = \alpha\}$ with respect to a norm $\|\cdot\|$ on $\mathbb{R}^3$.

Example 2.3.2. (*Ellipsoid base S-type space fuzzy point*). Consider the fuzzy point $\widetilde{P}(0,1,0)$ whose membership function is given by

$$\mu\left((x,y,z)\Big|\widetilde{P}(0,1,0)\right) = \begin{cases} 1-\left\{\left(\frac{x}{2}\right)^2+(y-1)^2+\left(\frac{z}{3}\right)^2\right\} & \text{if } \left(\frac{x}{2}\right)^2+(y-1)^2+\left(\frac{z}{3}\right)^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

We note that

(i) $\partial\widetilde{P}(0,1,0)(\alpha) = \{(x,y,z) \in \mathbb{R}^3 : \left(\frac{x}{2}\right)^2+(y-1)^2+\left(\frac{z}{3}\right)^2 = (1-\alpha)^2\}$,

(ii) the unit sphere with respect to the usual Euclidean norm is $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$,

(iii) for each $\alpha \in [0,1)$, the translated $\widetilde{P}(\alpha)$ by the translation $(x,y,z) \to (x,y,z) - (0,1,0)$ is

$$\bar{P}(\alpha) = \{(x,y,z) : \left(\tfrac{x}{2}\right)^2+(y)^2+\left(\tfrac{z}{3}\right)^2 \leq (1-\alpha)^2\},$$

(iv) the map $T_\alpha(x,y,z) = \left(\frac{x\alpha}{2(1-\alpha)}, \frac{y\alpha}{1-\alpha}, \frac{z\alpha}{3(1-\alpha)}\right)$ is a homeomorphism from $\partial\bar{P}(\alpha)$ to $\mathbb{S}^2(\alpha)$, and

(v) with the reference function $S(x,y,z) = \max\{0, 1-(x^2+y^2+z^2)\}$,

$$\mu\left((x,y,z)\Big|\widetilde{P}(0,1,0)\right) = S(T_\alpha(x,y-1,z)).$$

Thus, $\widetilde{P}(0,1,0)$ is an S-type space fuzzy point.

Theorem 2.3.1. A fuzzy set $\widetilde{P}$ on $\mathbb{R}^3$ is a space fuzzy point if and only if there exists a three-variable reference function $S : \mathbb{R}^3 \to [0,1]$ and a homeomorphism

$T : \partial\bar{P}(\alpha) \to \mathbb{S}^2(\alpha)$ with $T_\alpha(0,0,0) = (0,0,0)$ for all $\alpha \in [0,1)$ and

$$\mu\left((x,y,z)\middle|\widetilde{P}\right) = S(T_\alpha(x-a, y-b, z-c)), \tag{2.1}$$

where

(i) $\partial\bar{P}(\alpha)$ is the set of all boundary points of $\bar{P}(\alpha)$, the translated $\widetilde{P}(\alpha)$ by the translation $(x,y,z) \to (x,y,z) - (a,b,c)$,

(ii) S is a monotonic non-increasing function and is continuous from right on any ray emanated from (a,b,c) and

(iii) $\mathbb{S}^2(\alpha)$ is the sphere $\{(x,y,z) \in \mathbb{R}^3 : \|(x,y,z)\| = \alpha\}$ of $\mathbb{R}^3$ with respect to a norm $\|\cdot\| : \mathbb{R}^3 \to [0,\infty)$.

Proof. The result is followed from Definition 2.3.1 and the definition of three-variable reference function, and from the facts that

(i) $\widetilde{P}(\alpha)$ is a closed convex set for all $\alpha \in [0,1]$,

(ii) $\mathbb{S}^2(\alpha)$ is the set of all boundary points of the closed convex set $\{(x,y,z) \in \mathbb{R}^3 : \|(x,y,z)\| \leq \alpha\}$, and

(iii) boundaries of two closed convex sets in $\mathbb{R}^3$ with nonempty interiors are homeomorphic.

□

Note 3. Theorem 2.3.1 proves that any fuzzy point in $\mathbb{R}^3$ is an S-type fuzzy point. Thus, in the rest of the articles, we use the term 'S-type fuzzy point' to mean a fuzzy point in $\mathbb{R}^3$.

Note 4. If we apply Theorem 2.3.1 on a continuous fuzzy point $\widetilde{P}(a,b,c)$ whose membership function is monotonically strictly decreasing along any ray emanated from (a,b,c), then the membership function μ of $\widetilde{P}$ can be represented by

$$\mu\left((x,y,z)\middle|\widetilde{P}(a,b,c)\right) = f(x-a, y-b, z-c),$$

where $f = S \circ T$ is a bijective map. The function f is bijective since so is S along any ray emanated from (a,b,c) and T is a homeomorphism.

2.3.3 Same and inverse points

To define the concepts of the same and inverse points, we need the following idea of a fuzzy number along a line passing through the core of an S-type space fuzzy point.

2.3.3.1 Fuzzy numbers along a line

Let $\widetilde{N}$ be a fuzzy number. In $\mathbb{R}^3$, the z-axis can be imagined as the real line. Considering the z-axis as the universal set, the membership function of the fuzzy number $\widetilde{N}$ can be presented by

$$\mu\left((x,y,z)\middle|\widetilde{N}\right) = \begin{cases} \mu\left(z\middle|\widetilde{N}\right) & \text{if } x = 0 \text{ and } y = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Next, we define a fuzzy number along a line passing through the core of an S-type space fuzzy point. This formulation will help us to visualize an S-type space fuzzy point as a collection of normal and convex fuzzy sets along different lines passing through the core of the S-type space fuzzy point.

Definition 2.3.3. (*Fuzzy number along a line passing through the core point of an S-type space fuzzy point*). Let $\widetilde{P}(a,b,c)$ be an S-type space fuzzy point by a reference function S. Let L_1 be a line passing through the core point (a,b,c) with direction cosines (l_x, l_y, l_z), i.e.,

$$L_1 : \tfrac{x-a}{l_x} = \tfrac{y-b}{l_y} = \tfrac{z-c}{l_z}.$$

Evidently, the intersection $\widetilde{P}(a,b,c) \bigcap L_1$ is a normal and convex fuzzy set on L_1. We refer this normal and convex fuzzy set as a fuzzy number along L_1. The membership function of $\widetilde{N}_{L_1} = \widetilde{P}(a,b,c) \bigcap L_1$ is given by

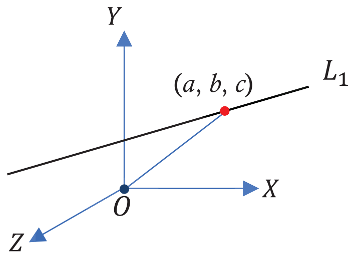
$$\mu\left((x,y,z)\middle|\widetilde{N}_{L_1}\right) = \begin{cases} \mu\left((x,y,z)\middle|\widetilde{P}(a,b,c)\right) & \text{if } (x,y,z) \in L_1 \\ 0 & \text{elsewhere.} \end{cases}$$

In the following, we show that corresponding to the fuzzy number $\widetilde{N}_{L_1}$, there is a fuzzy number on the z-axis.

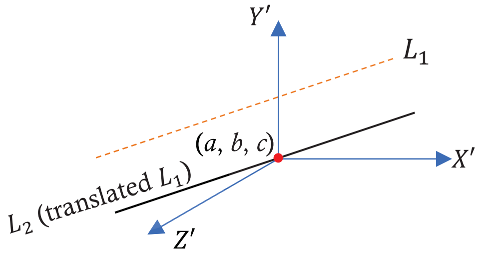
We apply the following transformations on the line L_1 so that it coincides with the z-axis.

Step 1: (Translate the origin to (a,b,c)). Apply the translation matrix $T_{(a,b,c)}$ (say) on L_1 and get $L_2 = T_{(a,b,c)}L_1$ (see (b) in Figure 2.1), where $T_{(a,b,c)} = \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{pmatrix}$. By representing a point (x,y,z) in L_1 by the column vector $(x,y,z,1)^T$, we note that the translation matrix $T_{(a,b,c)}$ translates a point $(a+kl_x, b+kl_y, c+kl_z)$ on L_1 to (kl_x, kl_y, kl_z), where k is a constant.

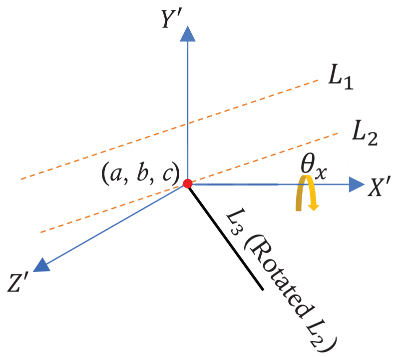
Step 2: (Apply rotations to make the line L_2 coincident with the z-axis).



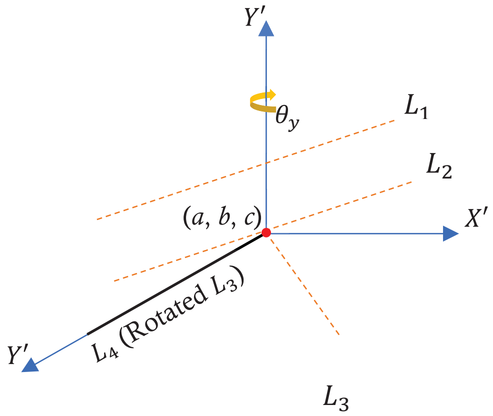
(a) L_1 is a line passing through (a, b, c)



(b) Translation of the origin to (a, b, c)



(c) Rotation about x-axis



(d) Rotation about y-axis

FIGURE 2.1: The steps for the transformation of the line L_1 to z-axis

(i) Let

$$R_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\sin\theta_x = \frac{l_y}{\sqrt{l_y^2+l_z^2}}$ and $\cos\theta_x = \frac{l_z}{\sqrt{l_y^2+l_z^2}}$ (Figure 2.2). Note that R_x gives a rotation with respect to x-axis by an angle θ_x. Thus, by R_x, a

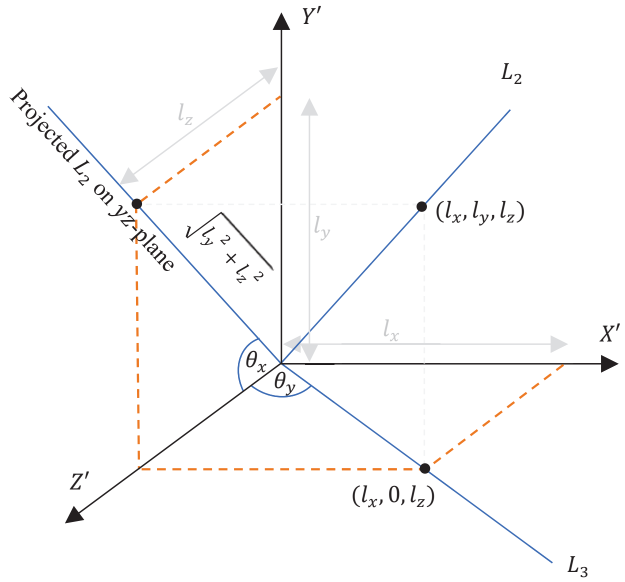


FIGURE 2.2: Angle of rotations about x-axis and y-axis

point (kl_x, kl_y, kl_z) on L_2 is mapped to the point

$$\left(kl_x, 0, k\left(\frac{l_y^2}{\sqrt{l_y^2+l_z^2}} + \frac{l_z^2}{\sqrt{l_y^2+l_z^2}}\right)\right).$$

Accordingly, by the rotation R_x, L_2 is transformed to

$$L_3 : \frac{x}{l_x} = \frac{z}{\left(\frac{l_y^2}{\sqrt{l_y^2+l_z^2}}+\frac{l_z^2}{\sqrt{l_y^2+l_z^2}}\right)}, y = 0$$

(see (c) in Figure 2.1).

(ii) On L_3 apply the rotation

$$R_y = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\sin\theta_y = l_x, \;\; \cos\theta_y = \sqrt{l_y^2 + l_z^2}$ (refer to Figure 2.2).

Note that R_y gives a rotation with respect to y-axis by an angle θ_y. Thus, by R_y, a point

$$\left(kl_x, 0, k\left(\frac{l_y^2}{\sqrt{l_y^2+l_z^2}} + \frac{l_z^2}{\sqrt{l_y^2+l_z^2}}\right)\right)$$

on L_3 is transformed to $(0, 0, k)$. Here the rotated line $L_4 = R_y L_3$ coincides with z-axis (see (d) in Figure 2.1).

Let $\widetilde{N}_{L_1}^z$ be the fuzzy set on z-axis that is obtained by applying the above three transformations ($T_{(a,b,c)}$, R_y and R_x) on $\widetilde{N}_{L_1}$. Then, we note that membership function of $\widetilde{N}_{L_1}^z$ is defined by

$$\mu\left((0,0,z)\middle|\widetilde{N}_{L_1}^z\right) = \begin{cases} \mu\left((u,v,w)\middle|\widetilde{N}_{L_1}\right) & \text{if } (0,0,z,1) = R_y R_x T_{(a,b,c)}(u,v,w,1), \\ & \frac{u-a}{l_x} = \frac{v-b}{l_y} = \frac{w-c}{l_z}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

As $\widetilde{N}_{L_1}$ is a normal and convex fuzzy set, the fuzzy set $\widetilde{N}_{L_1}^z$ on z-axis, is also so. Hence, $\widetilde{N}_{L_1}^z$ is a fuzzy number on z-axis. We note that by varying L_1 to another line L that passes through (a, b, c), we will get a fuzzy number $\widetilde{N}_L = \widetilde{P}(a,b,c) \bigcap L$ that corresponds a fuzzy number, $\widetilde{N}_L^z$ say, on z-axis. Accordingly, the fuzzy point $\widetilde{P}(a, b, c)$ can be observed as a collection of fuzzy numbers $\widetilde{N}_L$'s along different lines L's, i.e.,

$$\widetilde{P}(a,b,c) = \bigcup \{\widetilde{N}_L : L \text{ is a line passing through } (a,b,c)\}.$$

Example 2.3.3. Let $\widetilde{P}(0, 0, 0)$ be an S-type space fuzzy point with the membership function

$$\mu\left((x,y,z)\middle|\widetilde{P}(0,0,0)\right) = \begin{cases} 1 - \sqrt{x^2+y^2+z^2} & \text{if } x^2+y^2+z^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the line

$$L_1 : \frac{x}{\frac{1}{\sqrt{3}}} = \frac{y}{\frac{1}{\sqrt{3}}} = \frac{z}{\frac{1}{\sqrt{3}}}$$

that passes through the core point of $\widetilde{P}$.

Let $T_{(0,0,0)} = I_4$, the identity matrix of order 4,

$$R_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } R_y = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$R_y R_x T_{(0,0,0)} = \begin{pmatrix} \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $R_y R_x T_{(0,0,0)}$ transforms

(i) L_1 to the z-axis, and

(ii) the fuzzy number $\widetilde{N}_{L_1} = \widetilde{P}(0,0,0) \bigcap L_1$ to the fuzzy number $\widetilde{N}_L^z$ with the following membership function

$$\mu\left((0,0,z)\middle|\widetilde{N}_L^z\right) = \mu\left((u,v,w)\middle|\widetilde{N}_{L_1}\right) = \begin{cases} 1 - z \text{ if } (0,0,z,1) = R_y R_x(u,v,w,1), \frac{u}{\frac{1}{\sqrt{3}}} = \frac{v}{\frac{1}{\sqrt{3}}} = \frac{w}{\frac{1}{\sqrt{3}}}, \\ 0 \qquad \text{otherwise.} \end{cases}$$

2.3.3.2 Addition operation of two S-type space fuzzy points

Let $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ be two S-type space fuzzy points. For a given $\theta \in [0, 2\pi]$ and $\varphi \in \left[0, \frac{\pi}{2}\right]$, consider a line that passes through (a_1, b_1, c_1):

$$L^1_{\theta\varphi} : \tfrac{x-a_1}{\sin\varphi\cos\theta} = \tfrac{y-b_1}{\sin\varphi\sin\theta} = \tfrac{z-c_1}{\cos\varphi}.$$

We note that the intersection of $\widetilde{P}_1(a_1, b_1, c_1)$ and $L^1_{\theta\varphi}$ constitutes a fuzzy number along $L^1_{\theta\varphi}$. Thus, the fuzzy point $\widetilde{P}_1(a_1, b_1, c_1)$ can be viewed as a collection of fuzzy numbers along the lines $L^1_{\theta\varphi}$ for varied $\theta \in [0, 2\pi]$ and $\varphi \in \left[0, \frac{\pi}{2}\right]$ as follows:

$$\widetilde{P}_1(a_1, b_1, c_1) = \bigcup_{\theta\in[0,2\pi]} \bigcup_{\varphi\in\left[0,\frac{\pi}{2}\right]} \widetilde{N}^1_{\theta\varphi},$$

where $\widetilde{N}^1_{\theta\varphi}$ is the fuzzy number $\widetilde{P}_1(a_1, b_1, c_1) \bigcap L^1_{\theta\varphi}$ along $L^1_{\theta\varphi}$. We refer here the Figure 2.3. In Figure 2.3, φ is the angle between the positive z'-axis and the line $L^1_{\theta\varphi}$, and θ is the angle between the projected line $L^1_{\theta\varphi}$ on the $x'y'$-plane and the positive x'-axis.

Similarly,

$$\widetilde{P}_2(a_2, b_2, c_2) = \bigcup_{\theta\in[0,2\pi]} \bigcup_{\varphi\in\left[0,\frac{\pi}{2}\right]} \widetilde{N}^2_{\theta\varphi},$$

where $\widetilde{N}^2_{\theta\varphi}$ is the fuzzy number $\widetilde{P}_2(a_2, b_2, c_2) \bigcap L^2_{\theta\varphi}$ along the line $L^2_{\theta\varphi} : \frac{x-a_2}{\sin\varphi\cos\theta} = \frac{y-b_2}{\sin\varphi\sin\theta} = \frac{z-c_2}{\cos\varphi}$.

We define an addition of two S-type space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ by

$$\widetilde{P}_1 + \widetilde{P}_2 = \bigcup_{\theta\in[0,2\pi]} \bigcup_{\varphi\in[0,\frac{\pi}{2}]} \left(\widetilde{N}^1_{\theta\varphi} \oplus \widetilde{N}^2_{\theta\varphi}\right). \tag{2.3}$$

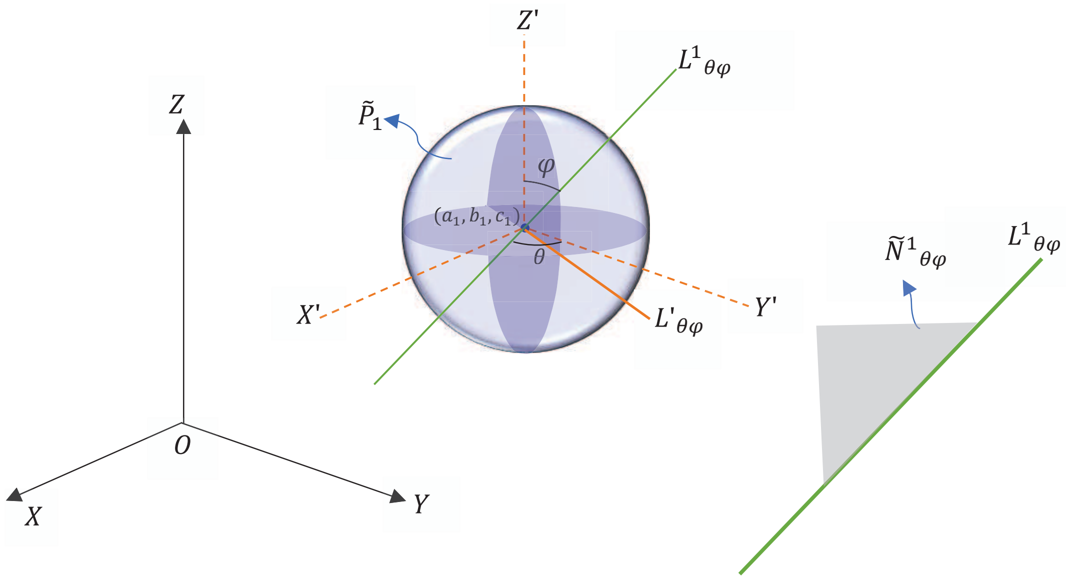


FIGURE 2.3: Fuzzy number $\widetilde{N}^1_{\theta\varphi}$ along the line $L^1_{\theta\varphi}$ on the support of an S-type space fuzzy point

Ghosh and Chakraborty [1] provided a work on effective and redundant combinations for efficiently computing an addition of two fuzzy numbers and addition of two fuzzy points in $\mathbb{R}^2$. In next subsection, we give a study for identifying the effective and redundant combinations for the addition (2.3) of two S-type space fuzzy points.

2.3.3.3 Separation of effective combinations for the addition of two S-type space fuzzy points

The following Lemma 2.3.1 and Theorem 2.3.2 are useful to separate out the effective and redundant combinations to compute $\widetilde{P}_1 + \widetilde{P}_2$ in (2.3).

Lemma 2.3.1. Let $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ be two S-type space continuous fuzzy points whose membership functions are strictly monotonic along the rays emanated from their respective core points. Let

$$L^1_{\theta\varphi} : \frac{x-a_1}{\sin\varphi\cos\theta} = \frac{y-b_1}{\sin\varphi\sin\theta} = \frac{z-c_1}{\cos\varphi} = \lambda_1 \geq 0$$

and

$$L^2_{\theta\varphi} : \frac{x-a_2}{\sin\varphi\cos\theta} = \frac{y-b_2}{\sin\varphi\sin\theta} = \frac{z-c_2}{\cos\varphi} = \lambda_2 \geq 0$$

be two rays emanated from (a_1, b_1, c_1) and (a_2, b_2, c_2), respectively. If (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points in $L^1_{\theta\varphi} \bigcap \widetilde{P}_1(0)$ and $L^2_{\theta\varphi} \bigcap \widetilde{P}_2(0)$, respectively, such that $\mu\left((x_1, y_1, z_1)\Big|\widetilde{P}_1\right) = \mu\left((x_2, y_2, z_2)\Big|\widetilde{P}_2\right) = \alpha$, then

$$\mu\left((x_1 + x_2, y_1 + y_2, z_1 + z_2)\Big|\widetilde{P}_1 + \widetilde{P}_2\right) = \alpha.$$

Proof. The point (x_1, y_1, z_1) and (x_2, y_2, z_2) can be represented by

$$(x_1, y_1, z_1) = (a_1 + \lambda_1 \sin\varphi\cos\theta, b_1 + \lambda_1 \sin\varphi\sin\theta, c_1 + \lambda_1 \cos\varphi)$$

and

$$(x_2, y_2, z_2) = (a_2 + \lambda_2 \sin\varphi\cos\theta, b_2 + \lambda_2 \sin\varphi\sin\theta, c_2 + \lambda_2 \cos\varphi),$$

for some constants λ_1 and λ_2.

Consider two points

$$(x'_1, y'_1, z'_1) = (a_1 + \lambda'_1 \sin\varphi\cos\theta, b_1 + \lambda'_1 \sin\varphi\sin\theta, c_1 + \lambda'_1 \cos\varphi)$$

and

$$(x'_2, y'_2, z'_2) = (a_2 + \lambda'_2 \sin\varphi\cos\theta, b_2 + \lambda'_2 \sin\varphi\sin\theta, c_2 + \lambda'_2 \cos\varphi)$$

in $L^1_{\theta\varphi} \bigcap \widetilde{P}_1(0)$ and $L^2_{\theta\varphi} \bigcap \widetilde{P}_2(0)$, respectively, such that

$$(x'_1, y'_1, z'_1) \neq (x_1, y_1, z_1) \text{ and } (x'_2, y'_2, z'_2) \neq (x_2, y_2, z_2)$$

but

$$(x_1' + x_2', y_1' + y_2', z_1' + z_2') = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

Thus,

$$x_1' + x_2' = x_1 + x_2, \quad y_1' + y_2' = y_1 + y_2, \; z_1' + z_2' = z_1 + z_2.$$

This implies $\lambda_1' + \lambda_2' = \lambda_1 + \lambda_2$. Here, two cases may arise:

(i) $\lambda_1 \geq \lambda_1'$, i.e., $\lambda_2 \leq \lambda_2'$ or

(ii) $\lambda_1 \leq \lambda_1'$, i.e., $\lambda_2 \geq \lambda_2'$.

In each of the cases, either $\lambda_1' \geq \lambda_1$ or $\lambda_2' \geq \lambda_2$. Since

$$\mu\left((\lambda_1 \sin\varphi\cos\theta, \lambda_1 \sin\varphi\sin\theta, \lambda_1\cos\varphi)\middle|\widetilde{P}_1\right)$$

and

$$\mu\left((\lambda_2 \sin\varphi\cos\theta, \lambda_2 \sin\varphi\sin\theta, \lambda_2\cos\varphi)\middle|\widetilde{P}_2\right)$$

are non-increasing along $L^1_{\theta\varphi}$ and $L^2_{\theta\varphi}$, respectively, in any of the cases: either $\lambda_1' \geq \lambda_1$ or $\lambda_2' \geq \lambda_2$, we have

$$\begin{aligned}
& \mu\left((x_1 + x_2, y_1 + y_2, z_1 + z_2)\middle|\widetilde{P}_1 + \widetilde{P}_2\right) \\
= & \sup_{\lambda_1' + \lambda_2' = \lambda_1 + \lambda_2} \min\Big\{\mu\left((\lambda_1' \sin\varphi\cos\theta, \lambda_1' \sin\varphi\sin\theta, \lambda_1'\cos\varphi)\middle|\widetilde{P}_1\right), \\
& \qquad\qquad \mu\left((\lambda_2' \sin\varphi\cos\theta, \lambda_2' \sin\varphi\sin\theta, \lambda_2'\cos\varphi)\middle|\widetilde{P}_2\right)\Big\} \\
\leq & \; \alpha
\end{aligned}$$

and the maximum is attained for (x_1, y_1, z_1) and (x_2, y_2, z_2). This yields the result that

$$\mu\left((x_1 + x_2, y_1 + y_2, z_1 + z_2)\Big|\widetilde{P}_1 + \widetilde{P}_2\right) = \alpha.$$

□

Theorem 2.3.2. Let $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ be two continuous S-type space fuzzy points whose membership functions are strictly monotonic along the rays emanated from their respective core points. Let

$$L^1_{\theta\varphi} : \tfrac{x-a_1}{\sin\varphi\cos\theta} = \tfrac{y-b_1}{\sin\varphi\sin\theta} = \tfrac{z-c_1}{\cos\varphi} = \lambda_1 \geq 0$$

and

$$L^2_{\theta\varphi} : \tfrac{x-a_2}{\sin\varphi\cos\theta} = \tfrac{y-b_2}{\sin\varphi\sin\theta} = \tfrac{z-c_2}{\cos\varphi} = \lambda_2 \geq 0$$

be two rays emanated from (a_1, b_1, c_1) and (a_2, b_2, c_2) on the supports of the S-type space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively. If

$$(x_1, y_1, z_1) = (a_1 + \lambda_1 \sin\varphi\cos\theta, b_1 + \lambda_1 \sin\varphi\sin\theta, c_1 + \lambda_1 \cos\varphi)$$

and

$$(x_2, y_2, z_2) = (a_2 + \lambda_2 \sin\varphi\cos\theta, b_2 + \lambda_2 \sin\varphi\sin\theta, c_2 + \lambda_2 \cos\varphi)$$

are two points in $L^1_{\theta\varphi} \bigcap \widetilde{P}_1(0)$ and $L^2_{\theta\varphi} \bigcap \widetilde{P}_2(0)$, respectively, then exists two points

$$(x_1^\star, y_1^\star, z_1^\star) = (a_1 + \lambda_1' \sin\varphi\cos\theta, b_1 + \lambda_1' \sin\varphi\sin\theta, c_1 + \lambda_1' \cos\varphi)$$

and

$$(x_2^\star, y_2^\star, z_2^\star) = (a_2 + \lambda_2' \sin\varphi\cos\theta, b_2 + \lambda_2' \sin\varphi\sin\theta, c_2 + \lambda_2' \cos\varphi)$$

in $L^1_{\theta\varphi} \bigcap \widetilde{P}_1(0)$ and $L^2_{\theta\varphi} \bigcap \widetilde{P}_2(0)$, respectively, where λ_1', λ_2' are constants such that:

(i) $\mu\left((x_1^\star, y_1^\star, z_1^\star)\Big|\widetilde{P}_1\right) = \mu\left((x_2^\star, y_2^\star, z_2^\star)\Big|\widetilde{P}_2\right)$,

(ii) $x_1 + x_2 = x_1^\star + x_2^\star,\ y_1 + y_2 = y_1^\star + y_2^\star,\ z_1 + z_2 = z_1^\star + z_2^\star$, and

(iii) $\mu\left((x_1 + x_2, y_1 + y_2, z_1 + z_2)\Big|\widetilde{P}_1 + \widetilde{P}_2\right) = \mu\left((x_1^\star, y_1^\star, z_1^\star)\Big|\widetilde{P}_1\right) = \mu\left((x_2^\star, y_2^\star, z_2^\star)\Big|\widetilde{P}_2\right)$.

Proof. We note that on $L^1_{\theta\varphi}$, the membership function of $\widetilde{P}_1$ reduces to

$$\phi_1(\lambda_1) := \mu\left((\lambda_1 \sin\varphi\cos\theta, \lambda_1 \sin\varphi\sin\theta, \lambda_1\cos\varphi)\Big|\widetilde{P}_1\right), \lambda_1 \geq 0,$$

which is a monotonic strictly decreasing function.

Similarly,

$$\phi_2(\lambda_2) := \mu\left((\lambda_2 \sin\varphi\cos\theta, \lambda_2 \sin\varphi\sin\theta, \lambda_2\cos\varphi)\Big|\widetilde{P}_2\right), \lambda_2 \geq 0,$$

is a monotonic strictly decreasing function.

We note that both ϕ_1 and ϕ_2 are bijective and hence ϕ_1^{-1} and ϕ_2^{-1} exist and they are continuous and strictly non-increasing on $[0, 1]$.

Consider the function $f = \phi_1^{-1} + \phi_2^{-1}$. Then, obviously, f is strictly non-increasing and continuous on $[0, 1]$.

Let $k = \lambda_1 + \lambda_2$ and α be the value of $f^{-1}(k)$. For this α, we consider the constants λ_1' and λ_2' defined by $\lambda_1' = \phi_1^{-1}(\alpha)$ and $\lambda_2' = \phi_2^{-1}(\alpha)$.

Addition of these constants is

$$\lambda_1' + \lambda_2' = f(\alpha) = k.$$

As ϕ_1^{-1} is strictly non-increasing on $[0,1]$ and $\phi_1^{-1}(0) = (a_1, b_1, c_1)$, we note that $\lambda_1' \geq 0$. Similarly, $\lambda_2' \geq 0$.

Corresponding to these λ_1' and λ_2',

$$(x_1^\star, y_1^\star, z_1^\star) = (a_1 + \lambda_1' \sin\varphi\cos\theta, b_1 + \lambda_1' \sin\varphi\sin\theta, c_1 + \lambda_1' \cos\varphi)$$

and

$$(x_2^\star, y_2^\star, z_2^\star) = (a_2 + \lambda_2' \sin\varphi\cos\theta, b_2 + \lambda_2' \sin\varphi\sin\theta, c_2 + \lambda_2' \cos\varphi)$$

are two points on $L_{\theta\varphi}^1$ and $L_{\theta\varphi}^2$, respectively. According to Lemma 2.3.1, we obtain

$$\mu\left((x_1+x_2, y_1+y_2, z_1+z_2)\middle|\widetilde{P}_1+\widetilde{P}_2\right) = \mu\left((x_1^\star, y_1^\star, z_1^\star)\middle|\widetilde{P}_1\right) = \mu\left((x_2^\star, y_2^\star, z_2^\star)\middle|\widetilde{P}_2\right) = \alpha,$$

which proves the theorem. □

Example 2.3.4. Let $\widetilde{P}_1(3,3,3)$ and $\widetilde{P}_2(1,1,1)$ be two continuous S-type space fuzzy points with the reference functions

$$S_1(x,y,z) = S_2(x,y,z) = \max\{0, 1 - \sqrt{x^2+y^2+z^2}\}$$

and membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_1(3,3,3)\right) = \begin{cases} 1 - \sqrt{\frac{1}{3}\{(x-3)^2+(y-3)^2+(z-3)^2\}} & \text{if } (x-3)^2+(y-3)^2+(z-3)^2 \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\Big|\widetilde{P}_2(1,1,1)\right)$$
$$=\begin{cases}1-\sqrt{\frac{3}{5}(x-1)^2+\frac{1}{5}(y-1)^2+\frac{1}{5}(z-1)^2} & \text{if } 3(x-1)^2+(y-1)^2+(z-1)^2\leq 5\\ 0 & \text{otherwise.}\end{cases}$$

Consider $\theta = 45°$ and $\varphi = 54.73°$, and accordingly the rays

$$L^1_{\theta\varphi}: \tfrac{x-3}{-0.5} = \tfrac{y-3}{0.5} = \tfrac{z-3}{-0.5} = \lambda_1 \geq 0$$

and

$$L^2_{\theta\varphi}: \tfrac{x-1}{-0.5} = \tfrac{y-1}{0.5} = \tfrac{z-1}{-0.5} = \lambda_2 \geq 0$$

which pass through the cores of the fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively.

Consider two points

$$(x_1, y_1, z_1) \equiv (2.45, 3.55, 2.45) = (3 - 0.5\lambda_1, 3 + 0.5\lambda_1, 3 - 0.5\lambda_1)$$

and

$$(x_2, y_2, z_2) \equiv (0.55, 1.45, 0.55) = (1 - 0.5\lambda_2, 1 + 0.5\lambda_2, 1 - 0.5\lambda_2)$$

in $L^1_{\theta\varphi} \bigcap \widetilde{P}_1(0)$ and $L^2_{\theta\varphi} \bigcap \widetilde{P}_2(0)$, respectively, for $\lambda_1 = 1.1$ and $\lambda_2 = 0.9$.

Here, $\lambda_1 + \lambda_2 = 2 > 0$. We observe that there exist

$$(2.5, 3.5, 2.5) = (3 - 0.5\lambda'_1, 3 + 0.5\lambda'_1, 3 - 0.5\lambda'_1)$$

and

$$(0.5, 1.5, 0.5) = (1 - 0.5\lambda_2', 1 + 0.5\lambda_2', 1 - 0.5\lambda_2')$$

in $L^1_{\theta\varphi} \bigcap \widetilde{P}_1(0)$ and $L^2_{\theta\varphi} \bigcap \widetilde{P}_2(0)$, respectively, for $\lambda_1' = 1$ and $\lambda_2' = 1$ such that:

(i) $\mu\left((2.5, 3.5, 2.5)\middle|\widetilde{P}_1\right) = \mu\left((0.5, 1.5, 0.5)\middle|\widetilde{P}_2\right) = 0.5,$

(ii) $2.45 + 0.55 = 2.5 + 0.5,\ 3.55 + 1.45 = 3.5 + 1.5,\ 2.45 + 0.55 = 2.5 + 0.5,$ and

(iii) $\mu\left((3, 5, 3)\middle|\widetilde{P}_1 + \widetilde{P}_2\right) = \mu\left((2.5, 3.5, 2.5)\middle|\widetilde{P}_1\right) = \mu\left((0.5, 1.5, 0.5)\middle|\widetilde{P}_2\right) = 0.5.$

Here, according to the notations of Theorem 2.3.2, $(x_1^\star, y_1^\star, z_1^\star) \equiv (2.5, 3.5, 2.5)$ and $(x_2^\star, y_2^\star, z_2^\star) \equiv (0.5, 1.5, 0.5)$.

Note 5. Theorem 2.3.2 describes that in computing the addition $\widetilde{P}_1 + \widetilde{P}_2$ of two continuous S-type space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$, to find the membership value of a point $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$, there are many possible pair of points

$$(x_1', y_1', z_1') = (a_1 + \lambda_1' \sin\varphi\cos\theta, b_1 + \lambda_1' \sin\varphi\sin\theta, c_1 + \lambda_1' \cos\varphi)$$

and

$$(x_2', y_2', z_2') = (a_2 + \lambda_2' \sin\varphi\cos\theta, b_2 + \lambda_2' \sin\varphi\sin\theta, c_2 + \lambda_2' \cos\varphi)$$

in $L^1_{\theta\varphi} \bigcap \widetilde{P}_1(0)$ and $L^2_{\theta\varphi} \bigcap \widetilde{P}_2(0)$, respectively, for which

$$(x_1' + x_2', y_1' + y_2', z_1' + z_2') = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

The collection of pairs which satisfy either of the following conditions:

(i) $\mu\left((x_1', y_1', z_1')\middle|\widetilde{P}_1\right) \neq \mu\left((x_2', y_2', z_2')\middle|\widetilde{P}_2\right)$ or

(ii) $\lambda_1' \leq 0,\ \lambda_2' \geq 0$ or $\lambda_1' \geq 0,\ \lambda_2' \leq 0$

are *redundant pairs* to find the membership value of $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$ in $\widetilde{P}_1 + \widetilde{P}_2$. Hence, we call the pair of points $(x_1^\star, y_1^\star, z_1^\star)$ and $(x_2^\star, y_2^\star, z_2^\star)$ as in Theorem 2.3.2 as the *effective pairs* with respect to the S-type space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$. In the addition of two fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$:

$$\widetilde{P}_1 + \widetilde{P}_2 = \bigcup_{\theta_1,\theta_2 \in [0,2\pi]} \ \bigcup_{\varphi_1,\varphi_2 \in \left[0,\frac{\pi}{2}\right]} \left(\widetilde{N}^1_{\theta_1\varphi_1} \oplus \widetilde{N}^2_{\theta_2\varphi_2}\right).$$

We observe that we need to calculate $\left(\widetilde{N}^1_{\theta_1\varphi_1} \oplus \widetilde{N}^2_{\theta_2\varphi_2}\right)$ for various $\theta_1, \theta_2 \in [0, 2\pi]$ and $\varphi_1, \varphi_2 \in [0, \frac{\pi}{2}]$. However, $\left(\widetilde{N}^1_{\theta_1\varphi_1} \oplus \widetilde{N}^2_{\theta_2\varphi_2}\right)$ is not a fuzzy number along a line if $\theta_1 \neq \theta_2$ or $\varphi_1 \neq \varphi_2$.

An S-type space fuzzy point can be viewed as a collection of fuzzy sets along any line passing through the core of the S-type space fuzzy point. To view addition of two S-type space fuzzy points as an S-type space fuzzy point, we considered only the combinations $\left(\widetilde{N}^1_{\theta\varphi} \oplus \widetilde{N}^2_{\theta\varphi}\right)$, $\theta \in [0, 2\pi]$ and $\varphi \in \left[0, \frac{\pi}{2}\right]$, i.e., we take

$$\widetilde{P}_1 + \widetilde{P}_2 = \bigcup_{\theta \in [0,2\pi]} \ \bigcup_{\varphi \in \left[0,\frac{\pi}{2}\right]} \left(\widetilde{N}^1_{\theta\varphi} \oplus \widetilde{N}^2_{\theta\varphi}\right).$$

According to Theorem 2.3.2 and Note 5, to calculate

$$\bigcup_{\theta \in [0,2\pi]} \ \bigcup_{\varphi \in \left[0,\frac{\pi}{2}\right]} \left(\widetilde{N}^1_{\theta\varphi} \oplus \widetilde{N}^2_{\theta\varphi}\right),$$

only the effective combinations are to be combined. This observation moderates to define *same and inverse points* with respect to continuous fuzzy points in $\mathbb{R}^3$.

Definition 2.3.4. (*Same points*). Let $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ be two S-type space fuzzy points with continuous membership functions which are strictly decreasing along the rays emanated from their respective core points. Consider two points (x_1, y_1, z_1) and (x_2, y_2, z_2) on the supports of $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively. Let

$$L_1 : \tfrac{x-a_1}{p_1} = \tfrac{y-b_1}{q_1} = \tfrac{z-c_1}{r_1}$$

be the line joining (x_1, y_1, z_1) and (a_1, b_1, c_1), and

$$L_2 : \tfrac{x-a_2}{p_2} = \tfrac{y-b_2}{q_2} = \tfrac{z-c_2}{r_2}$$

be the line joining (x_2, y_2, z_2) and (a_2, b_2, c_2). The points

$$(x_1, y_1, z_1) = (a_1 + \lambda_1 p_1, b_1 + \lambda_1 q_1, c_1 + \lambda_1 r_1)$$

and

$$(x_2, y_2, z_2) = (a_2 + \lambda_2 p_2, b_2 + \lambda_2 q_2, c_2 + \lambda_2 r_2),$$

where λ_1, λ_2 are constants, are said to be same points with respect to $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$, respectively, if:

(i) $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$, or $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$,

(ii) $\mu_1\left((x_1, y_1, z_1)\middle|\widetilde{P}_1\right) = \mu_2\left((x_2, y_2, z_2)\middle|\widetilde{P}_2\right)$, and

(iii) the direction ratios of L_1 and L_2 are identical, i.e., $\frac{p_1}{p_2} = \frac{q_1}{q_2} = \frac{r_1}{r_2}$.

Example 2.3.5. (*Same points*). Let us consider the fuzzy points $\widetilde{P}_1(3, 3, 3)$ and $\widetilde{P}_2(1, 1, 1)$ in Example 2.3.4. Consider a pair of points $(2.5, 3.5, 2.5)$ and $(0.5, 1.5, 0.5)$

from $\widetilde{P}_1(0)$ and $\widetilde{P}_2(0)$, respectively. The line joining $(2.5, 3.5, 2.5)$ and $(3, 3, 3)$ is

$$L_1 : \tfrac{x-3}{0.5} = \tfrac{y-3}{-0.5} = \tfrac{z-3}{0.5}.$$

The line joining $(0.5, 1.5, 0.5)$ and $(1, 1, 1)$ is

$$L_2 : \tfrac{x-1}{0.5} = \tfrac{y-1}{-0.5} = \tfrac{z-1}{0.5}.$$

The points

$$(2.5, 3.5, 2.5) = (3 + 0.5\lambda_1, 3 - 0.5\lambda_1, 3 + 0.5\lambda_1)$$

and

$$(0.5, 1.5, 0.5) = (1 + 0.5\lambda_2, 1 - 0.5\lambda_2, 1 + 0.5\lambda_2)$$

on L_1 and L_2, respectively, are such that

(i) $\lambda_1 = -1 \leq 0$ and $\lambda_2 = -1 \leq 0$,

(ii) $\mu\left((2.5, 3.5, 2.5)\middle|\widetilde{P}_1(3, 3, 3)\right) = \mu\left((0.5, 1.5, 0.5)\middle|\widetilde{P}_2(1, 1, 1)\right) = 0.5$, and

(iii) direction ratios of L_1 and L_2 are identical.

Therefore, the points $(2.5, 3.5, 2.5)$ and $(0.5, 1.5, 0.5)$ are same points with respect to the $\widetilde{P}_1(3, 3, 3)$ and $\widetilde{P}_2(1, 1, 1)$.

2.3.3.4 General expression of same points with respect to two continuous S-type space fuzzy points

Let $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ be two continuous S-type space fuzzy points with continuous membership functions which are strictly decreasing along the rays emanated from their respective core points. Then, according to Note 4, there exists two bijective functions f_1 and f_2 such that

$$\mu\left((x,y,z)\middle|\widetilde{P}_1\right) = f_1(x-a_1, y-b_1, z-c_1) \text{ and } \mu\left((x,y,z)\middle|\widetilde{P}_2\right) = f_2(x-a_2, y-b_2, z-c_2).$$

Let $\alpha \in [0,1]$ and (x_1, y_1, z_1) and (x_2, y_2, z_2) be two same points of membership value α with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively. Let the line joining (x_1, y_1, z_1) and (a_1, b_1, c_1) be $L^1_{\theta\varphi}$, and the line joining (x_2, y_2, z_2) and (a_2, b_2, c_2) be $L^2_{\theta\varphi}$. As (x_1, y_1, z_1) and (x_2, y_2, z_2) are same points, we have $\theta_1 = \theta_2$ and $\varphi_1 = \varphi_2$. Let $\theta = \theta_1 = \theta_2$, $\varphi = \varphi_1 = \varphi_2$, and the equations of $L^1_{\theta\varphi}$ and $L^2_{\theta\varphi}$ be

$$L^1_{\theta\varphi} : (x,y,z) = (a_1, b_1, c_1) + \lambda_1(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

and

$$L^2_{\theta\varphi} : (x,y,z) = (a_2, b_2, c_2) + \lambda_2(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi),$$

respectively. As (x_1, y_1, z_1) be a point of the membership value α, we have

$$\phi_1(\lambda_1) := f_1\left(x_1 - a_1, y_1 - b_1, z_1 - c_1\right) = f_1\left(\lambda_1 \sin\varphi\cos\theta, \lambda_1\sin\varphi\sin\theta, \lambda_1\cos\varphi\right) = \alpha.$$

Then, evidently, ϕ_1 is bijective for $\lambda_1 \geq 0$ and $\lambda_1 = \phi_1^{-1}(\alpha)$. This yields that the point (x_1, y_1, z_1) with membership value α on $\widetilde{P}_1$ can be represented as

$$(u^1)^\alpha_{\theta\varphi} : \left(a_1 + \phi_1^{-1}(\alpha)(\sin\varphi\cos\theta), b_1 + \phi_1^{-1}(\alpha)(\sin\varphi\sin\theta), c_1 + \phi_1^{-1}(\alpha)(\cos\varphi)\right). \tag{2.4}$$

Similarly, (x_2, y_2, z_2) on $\widetilde{P}_2$ can be expressed by

$$(u^2)^\alpha_{\theta\varphi} : \left(a_2 + \phi_2^{-1}(\alpha)(\sin\varphi\cos\theta), b_2 + \phi_2^{-1}(\alpha)(\sin\varphi\sin\theta), c_2 + \phi_2^{-1}(\alpha)(\cos\varphi)\right), \tag{2.5}$$

where

$$\phi_2(\lambda_2) := f_2\left(\lambda_2 \sin\varphi\cos\theta, \lambda_2\sin\varphi\sin\theta, \lambda_2\cos\varphi\right).$$

Example 2.3.6. Let $\widetilde{P}_1(0,0,0)$ and $\widetilde{P}_2(0,1,0)$ be two S-type space fuzzy points with membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_1(0,0,0)\right) = \begin{cases} 1-\sqrt{\left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2+z^2} & \text{if } \left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2+z^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_2(0,1,0)\right)$$
$$= \begin{cases} 1-\sqrt{(\frac{x}{2})^2+(y-1)^2+\left(\frac{z}{\frac{2}{3}}\right)^2} & \text{if } (\frac{x}{2})^2+(y-1)^2+\left(\frac{z}{\frac{2}{3}}\right)^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The lines $L^1_{\theta\varphi}: \frac{x}{1}=\frac{y}{2}=\frac{z}{1}$ and $L^2_{\theta\varphi}: \frac{x}{1}=\frac{y-1}{2}=\frac{z}{1}$ are the lines that pass through the core points of $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively, for $\theta = 63.44°$ and $\varphi = 65.90°$. The points

$$\left(u^1\right)^{\alpha}_{\theta\varphi}: \left(\tfrac{2}{3}\sqrt{(1-\alpha)}, \tfrac{4}{3}\sqrt{(1-\alpha)}, \tfrac{2}{3}\sqrt{(1-\alpha)}\right)$$

and

$$\left(u^2\right)^{\alpha}_{\theta\varphi}: \left(\sqrt{\tfrac{2}{13}(1-\alpha)}, 1+2\sqrt{\tfrac{2}{13}(1-\alpha)}, \sqrt{\tfrac{2}{13}(1-\alpha)}\right)$$

are the same points with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively, with membership value α. One can note that

$$\phi_1(\lambda_1) = 1-\tfrac{9\lambda^2}{4} \text{ and } \phi_2(\lambda_2) = 1-\tfrac{13\lambda^2}{2}.$$

Example 2.3.7. (*Same points on two fuzzy points with ellipsoidal base*). Let $\widetilde{P}_1(a_1,b_1,c_1)$ and $\widetilde{P}_2(a_2,b_2,c_2)$ be two space fuzzy points with bases

$$\{(x,y,z): \left(\tfrac{x-a_1}{p_1}\right)^2+\left(\tfrac{y-b_1}{q_1}\right)^2+\left(\tfrac{z-c_1}{r_1}\right)^2 \leq 1\}$$

and

$$\{(x,y,z) : \left(\tfrac{x-a_2}{p_2}\right)^2 + \left(\tfrac{y-b_2}{q_2}\right)^2 + \left(\tfrac{z-c_2}{r_2}\right)^2 \le 1\},$$

respectively. Let the membership functions of $\widetilde{P}_1$ and $\widetilde{P}_2$ be

$$\mu\left((x,y,z)\middle|\widetilde{P}_1(a_1,b_1,c_1)\right) = \begin{cases} 1-\sqrt{\left(\tfrac{x-a_1}{p_1}\right)^2 + \left(\tfrac{y-b_1}{q_1}\right)^2 + \left(\tfrac{z-c_1}{r_1}\right)^2} & \text{if } \left(\tfrac{x-a_1}{p_1}\right)^2 + \left(\tfrac{y-b_1}{q_1}\right)^2 + \left(\tfrac{z-c_1}{r_1}\right)^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_2(a_2,b_2,c_2)\right) = \begin{cases} 1-\sqrt{\left(\tfrac{x-a_2}{p_2}\right)^2 + \left(\tfrac{y-b_2}{q_2}\right)^2 + \left(\tfrac{z-c_2}{r_2}\right)^2} & \text{if } \left(\tfrac{x-a_2}{p_2}\right)^2 + \left(\tfrac{y-b_2}{q_2}\right)^2 + \left(\tfrac{z-c_2}{r_2}\right)^2 \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

respectively. The expressions of same points with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$ are

$$\left(u^1\right)^{\alpha}_{\theta\varphi} : \left(a_1 + \tfrac{(1-\alpha)A_{\theta\varphi}}{R_{\theta\varphi}}, b_1 + \tfrac{(1-\alpha)B_{\theta\varphi}}{R_{\theta\varphi}}, c_1 + \tfrac{(1-\alpha)C_{\theta\varphi}}{R_{\theta\varphi}}\right)$$

and

$$\left(u^2\right)^{\alpha}_{\theta\varphi} : \left(a_2 + \tfrac{(1-\alpha)A_{\theta\varphi}}{S_{\theta\varphi}}, b_2 + \tfrac{(1-\alpha)B_{\theta\varphi}}{S_{\theta\varphi}}, c_2 + \tfrac{(1-\alpha)C_{\theta\varphi}}{S_{\theta\varphi}}\right),$$

respectively, where $A_{\theta\varphi} = \sin\varphi\cos\theta, B_{\theta\varphi} = \sin\varphi\sin\theta$, $C_{\theta\varphi} = \cos\varphi$,

$$R_{\theta\varphi} = \sqrt{\tfrac{\sin^2\varphi\cos^2\theta}{p_1^2} + \tfrac{\sin^2\varphi\sin^2\theta}{q_1^2} + \tfrac{\cos^2\varphi}{r_1^2}}$$

and

$$S_{\theta\varphi} = \sqrt{\tfrac{\sin^2\varphi\cos^2\theta}{p_2^2} + \tfrac{\sin^2\varphi\sin^2\theta}{q_2^2} + \tfrac{\cos^2\varphi}{r_2^2}}.$$

Figure 2.4 depicts the locations of $(u^1)^\alpha_{\theta\varphi}$ and $(u^2)^\alpha_{\theta\varphi}$.

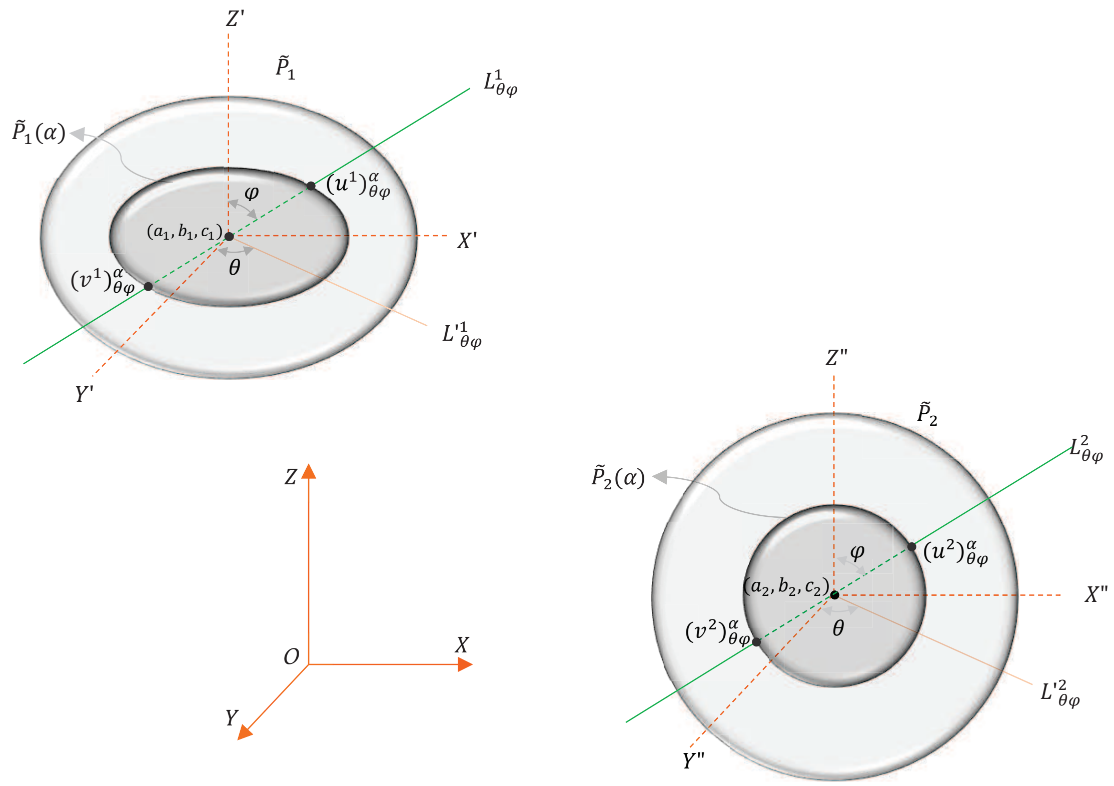


FIGURE 2.4: Same and inverse points for two continuous space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$

Definition 2.3.5. (*Addition of two S-type space fuzzy points*). Let $\widetilde{P}_1$ and $\widetilde{P}_2$ be two S-type space fuzzy points. Addition of these two S-type space fuzzy points is denoted by $\widetilde{P}_1 + \widetilde{P}_2$ and its membership function is defined by

$$\mu\left((x,y,z)\middle|\widetilde{P}_1 + \widetilde{P}_2\right) = \sup\big\{\alpha : (x,y,z) = (x_1+x_2, y_1+y_2, z_1+z_2),\ \text{where } (x_1,y_1,z_1) \in \widetilde{P}_1(0)$$
$$\text{and } (x_2,y_2,z_2) \in \widetilde{P}_2(0) \text{ are same points with membership value } \alpha\big\}.$$

Definition 2.3.6. (*Scalar multiplication of an S-type space fuzzy point*). Let $\lambda \in \mathbb{R}$. The scalar multiplication of an S-type space fuzzy point $\widetilde{P}(a,b,c)$ by λ, denoted

$\lambda\widetilde{P}(a,b,c)$, is defined by the membership function:

$$\mu\left((x,y,z)\middle|\lambda\widetilde{P}(a,b,c)\right)$$
$$= \begin{cases} \mu\left(\left(\frac{x}{\lambda},\frac{y}{\lambda},\frac{z}{\lambda}\right)\middle|\widetilde{P}(a,b,c)\right) & \text{if } \lambda \neq 0, \\ \sup\limits_{(u,v,w)\in\mathbb{R}^3} \mu\left((u,v,w)\middle|\widetilde{P}(a,b,c)\right) & \text{if } \lambda = 0, (x,y,z) = (0,0,0), \\ 0 & \text{if } \lambda = 0, (x,y,z) \neq (0,0,0). \end{cases}$$

Theorem 2.3.3. If $\widetilde{P}_1$ and $\widetilde{P}_2$ are two continuous S-type space fuzzy points, then

(i) $\lambda\widetilde{P}_1$ is an S-type space fuzzy point for any $\lambda \in \mathbb{R}$,

(ii) $\widetilde{P}_1 + \widetilde{P}_2$ is an S-type space fuzzy point, and

(iii) the linear combination $\lambda_1\widetilde{P}_1 + \lambda_2\widetilde{P}_2$ is also an S-type space fuzzy point, where $\lambda_1, \lambda_2 \in \mathbb{R}$.

Proof. Similar to Theorem 3.1 in [1]. □

From Theorem 2.3.3 and Definition 2.3.5, we notice that for a given pair of S-type space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$, the computation

$$\widetilde{P}_1 - \widetilde{P}_2 = \lambda_1\widetilde{P}_1 + \lambda_2\widetilde{P}_2 \ \text{ with } \lambda_1 = 1 \text{ and } \lambda_2 = -1$$

can be done by the same points of $\widetilde{P}_1$ and $-\widetilde{P}_2$. In the below the same points of $\widetilde{P}_1$ and $-\widetilde{P}_2$ are referred to inverse points. This idea of inverse points is used later, in Section 2.4, to evaluate the distance between a pair of fuzzy points.

Definition 2.3.7. (*Inverse points with respect to continuous S-type space fuzzy points*). Let $\widetilde{P}_1(a_1,b_1,c_1)$ and $\widetilde{P}_2(a_2,b_2,c_2)$ be two continuous S-type space fuzzy points with the reference functions S_1 and S_2, respectively, for $i = 1, 2$. Two points (x_1, y_1, z_1)

and (x_2, y_2, z_2) on the supports of $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively, are called inverse points with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$ if (x_1, y_1, z_1) and $(-x_2, -y_2, -z_2)$ are same points with respect to $\widetilde{P}_1(a_1, b_1, c_1)$ and $-\widetilde{P}_2(a_2, b_2, c_2)$, where $-\widetilde{P}_2$ is $\lambda\widetilde{P}_2$, with $\lambda = -1$.

Example 2.3.8. (*Inverse points*). Let us consider the fuzzy points $\widetilde{P}_1(3, 3, 3)$ and $\widetilde{P}_2(1, 1, 1)$ in Example 2.3.4. Consider the points $(2.5, 3.5, 2.5)$ and $(-1.5, -0.5, -1.5)$ from $\widetilde{P}_1(3, 3, 3)(0)$ and $-\widetilde{P}_2(1, 1, 1)(0)$, respectively.

The line joining $(2.5, 3.5, 2.5)$ and $(3, 3, 3)$ is

$$L_1 : \tfrac{x-3}{0.5} = \tfrac{y-3}{-0.5} = \tfrac{z-3}{0.5}.$$

The line joining $(-1.5, -0.5, -1.5)$ and $(-1, -1, -1)$ is

$$L_2 : \tfrac{x+1}{0.5} = \tfrac{y+1}{-0.5} = \tfrac{z+1}{0.5}.$$

Note that

$$(2.5, 3.5, 2.5) = (3 + 0.5\lambda_1, 3 - 0.5\lambda_1, 3 + 0.5\lambda_1) \text{ for } \lambda_1 = -1$$

and

$$(-1.5, -0.5, -1.5) = (-1 + 0.5\lambda_2, -1 - 0.5\lambda_2, -1 + 0.5\lambda_2) \text{ for } \lambda_2 = -1.$$

The points $(2.5, 3.5, 2.5)$ and $(-1.5, -0.5, -1.5)$ are same points with respect to $\widetilde{P}_1$ and $-\widetilde{P}_2$ since

(i) $\lambda_1 = -1 \leq 0$ and $\lambda_2 = -1 \leq 0$,

(ii) $\mu\left((2.5, 3.5, 2.5)\middle|\widetilde{P}_1(3, 3, 3)\right) = \mu\left((-1.5, -0.5, -1.5)\middle|-\widetilde{P}_2(1, 1, 1)\right) = 0.5$, and

(iii) direction ratios of L_1 and L_2 are identical.

Hence, by the Definition 2.3.7, the points $(2.5, 3.5, 2.5)$ and $(1.5, 0.5, 1.5)$ are inverse points with respect to the $\widetilde{P}_1(3, 3, 3)$ and $\widetilde{P}_2(1, 1, 1)$.

2.3.3.5 General expression of inverse points with respect to two continuous S-type space fuzzy points

Consider two continuous fuzzy points $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ with continuous membership functions which are strictly decreasing along the rays emanated from their respective core points.

The expression of inverse points with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$ are

$$\left(u^1\right)^{\alpha}_{\theta\varphi} : \left(a_1 + \phi_1^{-1}(\alpha)(\sin\varphi\cos\theta), b_1 + \phi_1^{-1}(\alpha)(\sin\varphi\sin\theta), c_1 + \phi_1^{-1}(\alpha)(\cos\varphi)\right)$$

and

$$\left(v^2\right)^{\alpha}_{\theta\varphi} : \left(a_2 - \phi_2^{-1}(\alpha)(\sin\varphi\cos\theta), b_2 - \phi_2^{-1}(\alpha)(\sin\varphi\sin\theta), c_2 - \phi_2^{-1}(\alpha)(\cos\varphi)\right),$$

respectively , where ϕ_1 and ϕ_2 are the functions as described in the first paragraph of the Subsection 2.3.3.4.

Example 2.3.9. (*Inverse points on two fuzzy points with ellipsoidal base*). Let us consider two fuzzy points $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ in Example 2.3.7. The expressions of inverse points with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$ are

$$\left(u^1\right)^{\alpha}_{\theta\varphi} : \left(a_1 + \tfrac{(1-\alpha)A_{\theta\varphi}}{R_{\theta\varphi}}, b_1 + \tfrac{(1-\alpha)B_{\theta\varphi}}{R_{\theta\varphi}}, c_1 + \tfrac{(1-\alpha)C_{\theta\varphi}}{R_{\theta\varphi}}\right)$$

and

$$\left(v^2\right)^{\alpha}_{\theta\varphi} : \left(a_2 - \tfrac{(1-\alpha)A_{\theta\varphi}}{S_{\theta\varphi}}, b_2 - \tfrac{(1-\alpha)B_{\theta\varphi}}{S_{\theta\varphi}}, c_2 - \tfrac{(1-\alpha)C_{\theta\varphi}}{S_{\theta\varphi}}\right),$$

respectively, where $A_{\theta\varphi} = \sin\varphi\cos\theta$, $B_{\theta\varphi} = \sin\varphi\sin\theta$, $C_{\theta\varphi} = \cos\varphi$,

$$R_{\theta\varphi} = \sqrt{\frac{\sin^2\varphi\cos^2\theta}{p_1^2} + \frac{\sin^2\varphi\sin^2\theta}{q_1^2} + \frac{\cos^2\varphi}{r_1^2}}$$

and

$$S_{\theta\varphi} = \sqrt{\frac{\sin^2\varphi\cos^2\theta}{p_2^2} + \frac{\sin^2\varphi\sin^2\theta}{q_2^2} + \frac{\cos^2\varphi}{r_2^2}}.$$

Figure 2.4 depicts the locations of $(u^1)^\alpha_{\theta\varphi}$ and $(v^2)^\alpha_{\theta\varphi}$. In Figure 2.4, the points $(v^1)^\alpha_{\theta\varphi}$ and $(u^2)^\alpha_{\theta\varphi}$ are also inverse points with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$.

Example 2.3.10. Consider two fuzzy points $\widetilde{P}_1(0,0,0)$ and $\widetilde{P}_2(0,1,0)$ with the reference functions

$$S_1(x,y,z) = S_2(x,y,z) = \max\{0, 1-(x^2+y^2+z^2)\}$$

in Example 2.3.6. For any given $\alpha \in [0,1]$, the points

$$(u^1)^\alpha_{\theta\varphi} : \left(\tfrac{2}{3}\sqrt{(1-\alpha)}, \tfrac{4}{3}\sqrt{(1-\alpha)}, \tfrac{2}{3}\sqrt{(1-\alpha)}\right)$$

and

$$(v^2)^\alpha_{\theta\varphi} : \left(-\sqrt{\tfrac{2}{13}(1-\alpha)}, 1-2\sqrt{\tfrac{2}{13}(1-\alpha)}, -\sqrt{\tfrac{2}{13}(1-\alpha)}\right)$$

are inverse points of the $\widetilde{P}_1$ and $\widetilde{P}_2$, respectively, with membership value α; here, $\theta = 63.44°$ and $\varphi = 65.90°$.

2.4 Fuzzy distance

In this section, we define the fuzzy distance between two continuous S-type space fuzzy points and the coincidence of two S-type space fuzzy points by the idea of the same and inverse points.

Definition 2.4.1. (*Fuzzy distance between two continuous S-type space fuzzy points*). The fuzzy distance between two continuous S-type space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$, denoted $\widetilde{D}(\widetilde{P_1}, \widetilde{P_2})$, is defined by the membership function:

$$\mu\left(d\middle|\widetilde{D}(\widetilde{P_1}, \widetilde{P_2})\right) = \sup\{\alpha : \text{where } d = d(u,v),\ u \in \widetilde{P}_1(0) \text{ and } v \in \widetilde{P}_2(0) \text{ are inverse points of membership value } \alpha\}.$$

Theorem 2.4.1. For two continuous S-type space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$,

(i) $\widetilde{D}\left(\widetilde{P_1}, \widetilde{P_2}\right)(\alpha) = \{d(u,v) :\ u \in \widetilde{P}_1(\alpha) \text{ and } v \in \widetilde{P}_2(\alpha) \text{ are inverse points of membership value } \alpha\}$ for all $\alpha \in (0, 1]$.

(ii) $\widetilde{D}$ is a fuzzy number in $\mathbb{R}$.

Proof. Similar to Theorem 4.1 in [1]. □

Example 2.4.1. (*Fuzzy distance*). Let $\widetilde{P}_1(1,0,1)$ and $\widetilde{P}_2(2,0,2)$ be two S-type space fuzzy points with the following membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_1(1,0,1)\right) = \begin{cases} 1 - 2\sqrt{\{(x-1)^2 + y^2 + (z-1)^2\}} & \text{if } (x-1)^2 + y^2 + (z-1)^2 \leq \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_2(2,0,2)\right) = \begin{cases} 1 - 2\sqrt{\{(x-2)^2 + y^2 + (z-2)^2\}} & \text{if } (x-2)^2 + y^2 + (z-2)^2 \leq \frac{1}{4} \\ 0 & \text{otherwise.} \end{cases}$$

For each $\alpha \in [0,1]$, the inverse points with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$ with the membership value α are

$$\left(u^1\right)^{\alpha}_{\theta\varphi} : \left(1 + \tfrac{(1-\alpha)\sin\varphi\cos\theta}{2}, \tfrac{(1-\alpha)\sin\varphi\sin\theta}{2}, 1 + \tfrac{(1-\alpha)\cos\varphi}{2}\right)$$

and

$$\left(v^2\right)^{\alpha}_{\theta\varphi} : \left(2 - \tfrac{(1-\alpha)\sin\varphi\cos\theta}{2}, -\tfrac{(1-\alpha)\sin\varphi\sin\theta}{2}, 2 - \tfrac{(1-\alpha)\cos\varphi}{2}\right),$$

respectively, where $0 \leq \varphi \leq \pi,\ 0 \leq \theta \leq 2\pi$.

The distance between $(u^1)^{\alpha}_{\theta\varphi}$ and $(v^2)^{\alpha}_{\theta\varphi}$ is

$$d\left(\left(u^1\right)^{\alpha}_{\theta\varphi}, \left(v^2\right)^{\alpha}_{\theta\varphi}\right) = \sqrt{2 + (1-\alpha)^2 - 2(1-\alpha)(\sin\varphi\cos\theta + \cos\varphi)}.$$

We observe that

$$\inf_{\varphi\in[0,\pi],\ \theta\in[0,2\pi]} d\left(\left(u^1\right)^{\alpha}_{\theta\varphi}, \left(v^2\right)^{\alpha}_{\theta\varphi}\right) = \sqrt{2 + (1-\alpha)^2 - 2.8284(1-\alpha)},$$

and

$$\sup_{\varphi\in[0,\pi],\ \theta\in[0,2\pi]} d\left(\left(u^1\right)^{\alpha}_{\theta\varphi}, \left(v^2\right)^{\alpha}_{\theta\varphi}\right) = \sqrt{2 + (1-\alpha)^2 + 2.8284(1-\alpha)}.$$

Let the fuzzy distance between $\widetilde{P}_1$ and $\widetilde{P}_2$ is $\widetilde{D}$. Then, the membership value of any

$$d \in \left[\sqrt{2 + (1-\alpha)^2 - 2.8284(1-\alpha)}, \sqrt{2 + (1-\alpha)^2 + 2.8284(1-\alpha)}\right]$$

in $\widetilde{D}$ is at least α, i.e., $\mu(d|\widetilde{D}) \geq \alpha$. Thus, $\widetilde{D}$ has the membership function:

$$\mu(d|\widetilde{D}) = \begin{cases} d - 0.4142 & \text{if } 0.4142 \leq d \leq 1.4142, \\ 2.4142 - d & \text{if } 1.4142 \leq d \leq 2.4142, \\ 0 & \text{elsewhere.} \end{cases}$$

In the below, we give an algorithm to execute the fuzzy distance $\widetilde{D}$ between two continuous S-type space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$.

Algorithm 2.4.1: To evaluate the fuzzy distance $\widetilde{D}$ between two S-type space fuzzy points

---

Input: Given two S-type space fuzzy points $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ with continuous membership functions which are strictly decreasing along the rays emanated from their respective core points. We denote

$$\phi_i(\lambda_i) = f_i\left(\lambda_i \sin\varphi \cos\theta, \lambda_i \sin\varphi \sin\theta, \lambda_i \cos\varphi\right) \text{ for } \lambda_i \geq 0, i = 1, 2.$$

Output: Fuzzy distance $\widetilde{D}\left(\widetilde{P_1}, \widetilde{P_2}\right) = \bigvee_{\alpha \in [0,1]} \widetilde{D}(\alpha)$.

For $\alpha = 0$ to 1; step size $\delta\alpha$

  $d_{\min}^{\alpha} = M$, a very large number

  $d_{\max}^{\alpha} = -M$

  For $\theta = 0$ to 2π; step size $\delta\theta$

    $d_{\min}^{\theta} = M$

    $d_{\max}^{\theta} = -M$

    For $\varphi = 0$ to π; step size $\delta\varphi$

      Compute

      $\lambda_1 = \phi_1^{-1}(\alpha)$

      $\lambda_2 = \phi_2^{-1}(\alpha)$

      Compute the inverse points

      $\left(u^1\right)_{\theta\varphi}^{\alpha} = \left(a_1 + (\sin\varphi\cos\theta)\phi_1^{-1}(\alpha), b_1 + (\sin\varphi\sin\theta)\phi_1^{-1}(\alpha), c_1 + (\cos\varphi)\phi_1^{-1}(\alpha)\right)$

      $\left(v^2\right)_{\theta\varphi}^{\alpha} = \left(a_2 - (\sin\varphi\cos\theta)\phi_2^{-1}(\alpha), b_2 - (\sin\varphi\sin\theta)\phi_2^{-1}(\alpha), c_2 - (\cos\varphi)\phi_2^{-1}(\alpha)\right)$

      Calculate the distance

      $d_{\varphi}^{\alpha} \leftarrow d\left(\left(u^1\right)_{\theta\varphi}^{\alpha}, \left(v^2\right)_{\theta\varphi}^{\alpha}\right)$

      if $d_{\varphi}^{\alpha} > d_{\max}^{\theta}$ **then**

        $d_{\max}^{\theta} \leftarrow d_{\varphi}^{\alpha}$

      end

      if $d_{\min}^{\theta} > d_{\varphi}^{\alpha}$ **then**

        $d_{\min}^{\theta} \leftarrow d_{\varphi}^{\alpha}$

      end

---

end
if $d^{\theta}_{\max} > d^{\alpha}_{\max}$ **then**
  $d^{\alpha}_{\max} \leftarrow d^{\theta}_{\max}$
end
if $d^{\alpha}_{\min} > d^{\theta}_{\min}$ **then**
  $d^{\alpha}_{\min} \leftarrow d^{\theta}_{\min}$
end
end
At the end of loop, $\widetilde{D}(\alpha) \leftarrow [d^{\alpha}_{\min}, d^{\alpha}_{\max}]$
end
return $\widetilde{D}(\widetilde{P_1}, \widetilde{P_2}) = \bigvee_{\alpha \in [0,1]} \widetilde{D}(\alpha)$

In the following example, we apply Algorithm 2.4.1 on three pairs of S-type space fuzzy points and calculate the fuzzy distances.

Example 2.4.2. Consider first pair of fuzzy points $\widetilde{P}_1(1, 2, 1)$ and $\widetilde{P}_2(-1, 0, 5)$ with membership functions

$$\mu\left((x, y, z)\middle|\widetilde{P}_1(1, 2, 1)\right) = \begin{cases} 1 - \sqrt{(x-1)^2 + \frac{1}{4}(y-2)^2 + (z-1)^2} & \text{if } (x-1)^2 + \frac{1}{4}(y-2)^2 + (z-1)^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x, y, z)\middle|\widetilde{P}_2(-1, 0, 5)\right) = \begin{cases} 1 - \sqrt{(x+1)^2 + (\frac{y}{2})^2 + (z-5)^2} & \text{if } (x+1)^2 + (\frac{y}{2})^2 + (z-5)^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here, the supports of $\widetilde{P}_1$ and $\widetilde{P}_2$ have empty intersection. The general expressions of the inverse points with respect to $\widetilde{P}_1$ and $\widetilde{P}_2$ are

$$\left(1+\tfrac{2(1-\alpha)\sin\varphi\cos\theta}{R^{\alpha}_{\theta\varphi}}, 2+\tfrac{2(1-\alpha)\sin\varphi\sin\theta}{R^{\alpha}_{\theta\varphi}}, 1+\tfrac{2(1-\alpha)\cos\varphi}{R^{\alpha}_{\theta\varphi}}\right)$$

and

$$\left(-1-\tfrac{2(1-\alpha)\sin\varphi\cos\theta}{R^{\alpha}_{\theta\varphi}}, -\tfrac{2(1-\alpha)\sin\varphi\sin\theta}{R^{\alpha}_{\theta\varphi}}, 5-\tfrac{2(1-\alpha)\cos\varphi}{R^{\alpha}_{\theta\varphi}}\right),$$

respectively, where

$$R^{\alpha}_{\theta\varphi} = \sqrt{4\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + 4\cos^2\varphi}.$$

Consider second pair of fuzzy points $\widetilde{Q}_1(1,2,1)$ and $\widetilde{Q}_2(-1,0,5)$ with membership functions

$$\begin{aligned}&\mu\left((x,y,z)\middle|\widetilde{Q}_1(1,2,1)\right)\\ =&\begin{cases}1-\sqrt{(x-1)^2+(y-2)^2+4(z-1)^2} & \text{if } (x-1)^2+(y-2)^2+4(z-1)^2\le 1\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{Q}_2(-1,0,5)\right) = \exp\left(-\tfrac{1}{4}(x+1)^2 - 4y^2 - (z-5)^2\right).$$

Here, the intersection of the supports of the fuzzy points $\widetilde{Q}_1$ and $\widetilde{Q}_2$ is nonempty. The general expressions of the inverse points with respect to $\widetilde{Q}_1$ and $\widetilde{Q}_2$ are

$$\left(1+\tfrac{0.5(1-\alpha)\sin\varphi\cos\theta}{R^{\alpha}_{\theta\varphi}}, 2+\tfrac{0.5(1-\alpha)\sin\varphi\sin\theta}{R^{\alpha}_{\theta\varphi}}, 1+\tfrac{0.5(1-\alpha)\cos\varphi}{R^{\alpha}_{\theta\varphi}}\right)$$

and

$$\left(-1-\frac{\left(\sqrt{\log\frac{1}{\alpha}}\right)\sin\varphi\cos\theta}{S^{\alpha}_{\theta\varphi}},-\frac{\left(\sqrt{\log\frac{1}{\alpha}}\right)\sin\varphi\sin\theta}{S^{\alpha}_{\theta\varphi}},5-\frac{\left(\sqrt{\log\frac{1}{\alpha}}\right)\cos\varphi}{S^{\alpha}_{\theta\varphi}}\right),$$

respectively, with membership value $\alpha \in (0, 1]$, where

$$R^{\alpha}_{\theta\varphi} = \sqrt{0.25\sin^2\varphi\cos^2\theta + 0.25\sin^2\varphi\sin^2\theta + \cos^2\varphi}$$

and

$$S^{\alpha}_{\theta\varphi} = \sqrt{0.25\sin^2\varphi\cos^2\theta + 4\sin^2\varphi\sin^2\theta + \cos^2\varphi}.$$

Consider third pair of fuzzy points $\widetilde{R}_1(1, 2, 1)$ and $\widetilde{R}_2(1, 2, 1)$ with membership functions

$$\begin{aligned}&\mu\left((x,y,z)\middle|\widetilde{R}_1(1,2,1)\right)\\ &=\begin{cases}1-\frac{1}{2}\sqrt{(x-1)^2+(y-2)^2+(z-1)^2} & \text{if } (x-1)^2+(y-2)^2+(z-1)^2\le 4\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{R}_2(1,2,1)\right) = \exp\left(-\tfrac{1}{9}((x-1)^2+(y-2)^2+(z-1)^2\right).$$

Here, the support of $\widetilde{R}_1$ is a subset of the support of $\widetilde{R}_2$. The general expressions of the inverse points with respect to $\widetilde{R}_1$ and $\widetilde{R}_2$ are

$$(1+2(1-\alpha)\sin\varphi\cos\theta, 2+2(1-\alpha)\sin\varphi\sin\theta, 1+2(1-\alpha)\cos\varphi)$$

and

$$\left(1-3\left(\sqrt{\log\tfrac{1}{\alpha}}\right)\sin\varphi\cos\theta, 2-3\left(\sqrt{\log\tfrac{1}{\alpha}}\right)\sin\varphi\sin\theta, 1-3\left(\sqrt{\log\tfrac{1}{\alpha}}\right)\cos\varphi\right),$$

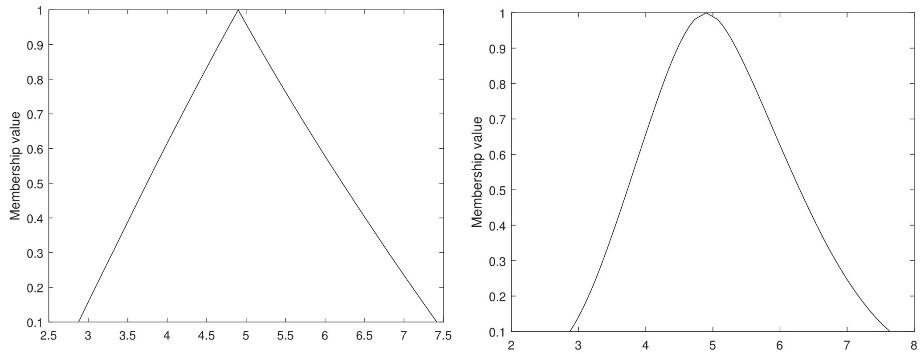
respectively, with membership value $\alpha \in (0,1]$.

The following Table 2.1 gives the α-cuts of the fuzzy distances $\widetilde{D}(\widetilde{P_1},\widetilde{P_2})$, $\widetilde{D}(\widetilde{Q_1},\widetilde{Q_2})$ and $\widetilde{D}(\widetilde{R_1},\widetilde{R_2})$ executed by the proposed Algorithm 2.4.1 with step sizes $\delta\alpha = 0.1$, $\delta\theta = 0.0706$ and $\delta\varphi = 0.0353$.

α	$\widetilde{D}(\widetilde{P_1},\widetilde{P_2})(\alpha)$	$\widetilde{D}(\widetilde{Q_1},\widetilde{Q_2})(\alpha)$	$\widetilde{D}(\widetilde{R_1},\widetilde{R_2})(\alpha)$
0.1	[2.8773, 7.4201]	[2.8633, 7.6378]	6.3523
0.2	[3.0906, 7.1077]	[3.1466, 7.1684]	5.4059
0.3	[3.3060, 6.8047]	[3.3615, 6.8343]	4.6918
0.4	[3.5227, 6.5088]	[3.5485, 6.5574]	4.0717
0.5	[3.7428, 6.2206]	[3.7247, 6.3042]	3.4977
0.6	[3.9650, 5.9396]	[3.8980, 6.0687]	2.9442
0.7	[4.1912, 5.6669]	[4.0716, 5.8403]	2.3917
0.8	[4.4219, 5.4030]	[4.2571, 5.6083]	1.8171
0.9	[4.6573, 5.1474]	[4.4730, 5.3545]	1.1738
1.0	4.8990	4.8990	0

TABLE 2.1: α-cuts of $\widetilde{D}(\widetilde{P_1},\widetilde{P_2})$, $\widetilde{D}(\widetilde{Q_1},\widetilde{Q_2})$ and $\widetilde{D}(\widetilde{R_1},\widetilde{R_2})$ by Algorithm 2.4.1 for Example 2.4.2

The following Figures 2.5 and 2.6 depict the membership functions of $\widetilde{D}(\widetilde{P_1},\widetilde{P_2})$, $\widetilde{D}(\widetilde{Q_1},\widetilde{Q_2})$ and $\widetilde{D}(\widetilde{R_1},\widetilde{R_2})$ obtained from Table 2.1. From Figures 2.5 and 2.6, we observe the following.



(a) Fuzzy distance $\widetilde{D}(\widetilde{P_1}, \widetilde{P_2})$ (b) Fuzzy distance $\widetilde{D}(\widetilde{Q_1}, \widetilde{Q_2})$

FIGURE 2.5: Fuzzy distance $\widetilde{D}(\widetilde{P_1}, \widetilde{P_2})$ and $\widetilde{D}(\widetilde{Q_1}, \widetilde{Q_2})$ by Algorithm 2.4.1 for Example 2.4.2

- For each different pairs of the S-type space fuzzy points in Example 2.4.2, the graphs of $\widetilde{D}(\widetilde{P_1}, \widetilde{P_2})$, $\widetilde{D}(\widetilde{Q_1}, \widetilde{Q_2})$ and $\widetilde{D}(\widetilde{R_1}, \widetilde{R_2})$ support the Theorem 2.4.1.

- For the fuzzy point $\widetilde{R}_1$ and $\widetilde{R}_2$, the graph $\widetilde{D}(\widetilde{R_1}, \widetilde{R_2})$ is the fuzzy number $\widetilde{0}$, i.e., $\mu\left(0 \middle| \widetilde{D}\right) = 1$.

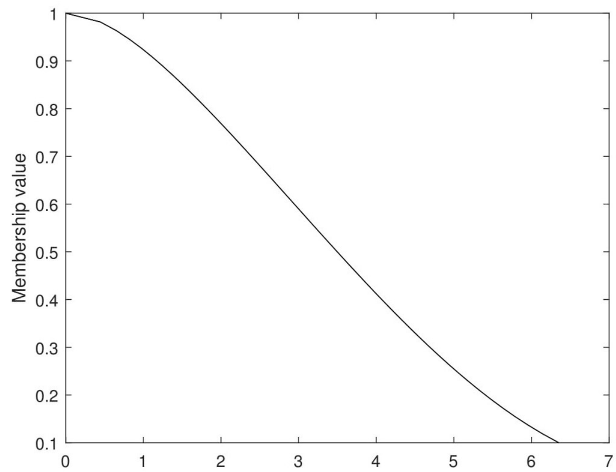


FIGURE 2.6: Fuzzy distance $\widetilde{D}(\widetilde{R_1}, \widetilde{R_2})$ by Algorithm 2.4.1 for Example 2.4.2

Next, we provide an Algorithm 2.4.2 to evaluate the membership value of a point in the fuzzy distance $\widetilde{D}$ between two S-type space fuzzy points.

Algorithm 2.4.2: To evaluate the membership value of a point in the fuzzy distance $\widetilde{D}$ between two S-type space fuzzy points

Input: Given two S-type space fuzzy points $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ with continuous membership functions which are strictly decreasing along the rays emanated from their respective core points. We denote

$$\phi_i(\lambda_i) = f_i\left(\lambda_i \sin\varphi \cos\theta, \lambda_i \sin\varphi \sin\theta, \lambda_i \cos\varphi\right) \text{ for } \lambda_i \geq 0, i = 1, 2.$$

We denote the fuzzy distance between $\widetilde{P}_1$ and $\widetilde{P}_2$ by $\widetilde{D}$.

Given $d \in \mathbb{R}$ for which the membership value $\mu\left(d \middle| \widetilde{D}\right)$ has to be calculated.

Output: The membership value $\mu\left(d \middle| \widetilde{D}\right)$.

Initialize $\alpha_{\text{sup}} \leftarrow 0$

loop:

For $\alpha = 0$ to 1; step size $\delta\alpha$

  For $\theta = 0$ to 2π; step size $\delta\theta$

    For $\varphi = 0$ to π; step size $\delta\varphi$

      Compute

      $\lambda_1 = \phi_1^{-1}(\alpha)$

      $\lambda_2 = \phi_2^{-1}(\alpha)$

      Compute the inverse points

      $(u^1)^{\alpha}_{\theta\varphi} =$

      $\left(a_1 + (\sin\varphi\cos\theta)\phi_1^{-1}(\alpha), b_1 + (\sin\varphi\sin\theta)\phi_1^{-1}(\alpha), c_1 + (\cos\varphi)\phi_1^{-1}(\alpha)\right)$

      $(v^2)^{\alpha}_{\theta\varphi} =$

      $\left(a_2 - (\sin\varphi\cos\theta)\phi_2^{-1}(\alpha), b_2 - (\sin\varphi\sin\theta)\phi_2^{-1}(\alpha), c_2 - (\cos\varphi)\phi_2^{-1}(\alpha)\right)$

      Calculate the distance

      $d^{\alpha} \leftarrow d\left((u^1)^{\alpha}_{\theta\varphi}, (v^2)^{\alpha}_{\theta\varphi}\right)$

```
            if d = d^α then
                if α_sup < α then
                    α_sup ← α
                else
                    goto loop
                end
            end
        end
    end
end
return μ(d | D̃) = α_sup
```

Example 2.4.3. (*Evaluation of the membership values of some points in the fuzzy distance* $\widetilde{D}\left(\widetilde{P_1}, \widetilde{P_2}\right)$). Consider the fuzzy points $\widetilde{P}_1(1,2,1)$ and $\widetilde{P}_2(-1,0,5)$ of Example 2.4.2. Table 2.2 shows the membership value of some number in the fuzzy distance $\widetilde{D}\left(\widetilde{P_1}, \widetilde{P_2}\right)$, obtained by executing Algorithm 2.4.2.

d	Membership value	Step sizes
6.023	0.5615	$\delta\alpha = 0.0231$, $\delta\theta = 0.1611$ and $\delta\varphi = 0.0806$
4.5231	0.8405	$\delta\alpha = 0.0114$, $\delta\theta = 0.0849$ and $\delta\varphi = 0.0425$
5.0000	0.9571	$\delta\alpha = 0.0107$, $\delta\theta = 0.0748$ and $\delta\varphi = 0.0374$

TABLE 2.2: Membership values of some points of $\widetilde{D}\left(\widetilde{P_1}, \widetilde{P_2}\right)(0)$ produced by Algorithm 2.4.2 for Example 2.4.3

Next, we define the degree of fuzzy coincidence, say ζ, of two S-type space fuzzy points.

Definition 2.4.2. (*Coincidence of two S-type space fuzzy points*). The degree of fuzzy coincidence (ζ) of two S-type space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ is defined by:

$$\zeta = \begin{cases} 0 & \text{if } \widetilde{P}_1(1) \neq \widetilde{P}_2(1), \\ 1 & \text{if } \widetilde{P}_1 = \widetilde{P}_2, \\ 1 - \sup\limits_{(x,y,z)\in\mathbb{R}^3} \left|\mu\left((x,y,z)\middle|\widetilde{P}_1\right) - \mu\left((x,y,z)\middle|\widetilde{P}_2\right)\right| & \text{if } \widetilde{P}_1(1) = \widetilde{P}_2(1) \text{ but } \widetilde{P}_1 \neq \widetilde{P}_2. \end{cases}$$

Example 2.4.4. Let $\widetilde{P}_1(0,0,0)$ and $\widetilde{P}_2(0,0,0)$ be two S-type space fuzzy points with the membership functions $\mu_1((x,y,z)|\widetilde{P}_1) = \max\{0, 1 - \sqrt{x^2+y^2+z^2}\}$ and $\mu_2((x,y,z)|\widetilde{P}_2) = \max\{0, 1 - \frac{1}{3}\sqrt{x^2+y^2+z^2}\}$. The degree of coincidence of $\widetilde{P}_1$ and $\widetilde{P}_2$ is

$$1 - \sup_{(x,y,z)\in\mathbb{R}^3} \left|\mu\left((x,y,z)\middle|\widetilde{P}_1\right) - \mu\left((x,y,z)\middle|\widetilde{P}_2\right)\right| = \tfrac{1}{\sqrt{3}}.$$

Note 6. If the degree of coincidence of two space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ is positive, then according to Theorem 2.4.1 the fuzzy distance $\widetilde{D}(\widetilde{P_1}, \widetilde{P_2})$ between the pair of fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ is the fuzzy number $\widetilde{0}$.

2.5 Space fuzzy line segments

A fuzzy line segment joining two fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ is the union of all possible convex combinations of $\widetilde{P}_1$ and $\widetilde{P}_2$, i.e.,

$$\bigcup_{\lambda\in[0,1]} \left(\lambda\widetilde{P}_1 + (1-\lambda)\widetilde{P}_2\right).$$

Note that Theorem 2.3.2 implies that only the combinations of the same points are sufficient to evaluate $\widetilde{P}_1 + \widetilde{P}_2$. These combinations are also sufficient to evaluate the

$$\lambda\widetilde{P}_1 + (1-\lambda)\widetilde{P}_2$$

for any $\lambda \in [0,1]$; the reason has been described in [1]. Therefore, we only need the same points of S-type space fuzzy points to construct the space fuzzy line segment. Thus, a space fuzzy line segment is formulated as follows.

Definition 2.5.1. (*Space fuzzy line segment joining two S-type space fuzzy points*). Let $\widetilde{P}_1$ and $\widetilde{P}_2$ be two S-type space fuzzy points. The space fuzzy line segment joining the fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$, $\widetilde{\widetilde{L}}_{P_1P_2}$ say, can be formulated by the membership function:

$$
\begin{aligned}
&\mu\left((x,y,z)\middle|\widetilde{\widetilde{L}}_{P_1P_2}\right)\\
= \;& \sup\{\alpha : \text{where } \;(x,y,z) \text{ lies on the line joining the same points } u \in \widetilde{P}_1(0) \text{ and}\\
& v \in \widetilde{P}_2(0) \;\text{ of membership value } \alpha\}.
\end{aligned}
$$

More explicitly,

$$
\begin{aligned}
&\mu\left((x,y,z)\middle|\widetilde{\widetilde{L}}_{P_1P_2}\right)\\
= \;& \sup\{\alpha : (x,y,z) = tu + (1-t)v, u \in \widetilde{P}_1(0) \text{ and } v \in \widetilde{P}_2(0) \text{ are same points of}\\
& \text{membership value } \alpha \text{ and } t \in [0,1]\}. \qquad (2.6)
\end{aligned}
$$

Theorem 2.5.1. *(α-cut of $\widetilde{\widetilde{L}}_{P_1P_2}$).* For any $\alpha \in [0,1]$, the α-cut of the fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2}$ is given by

$$\widetilde{\widetilde{L}}_{P_1P_2}(\alpha) = \bigvee\{l : l \text{ is the line segment joining same points in } \widetilde{P}_1(\alpha) \text{ and } \widetilde{P}_2(\alpha)\}.$$

Proof. The proof is directly followed from (2.6). □

Example 2.5.1. (*Space fuzzy line segment*). Let $\widetilde{P}_1(1,2,1)$ and $\widetilde{P}_2(-1,0,5)$ be two S-type space fuzzy points with the membership functions

$$\mu\left((x,y,z)\middle|\widetilde{P}_1(1,2,1)\right)$$
$$= \begin{cases} 1-\sqrt{(x-1)^2+(y-2)^2+(z-1)^2\}} & \text{if } (x-1)^2+(y-2)^2+(z-1)^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y,z)\middle|\widetilde{P}_2(-1,0,5)\right) = \begin{cases} 1-\sqrt{(x+1)^2+y^2+(z-5)^2\}} & \text{if } (x+1)^2+y^2+(z-5)^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For a particular $\alpha \in [0,1]$, the same points with membership value $\alpha \in [0,1]$ on $\widetilde{P}_1(1,2,1)$ and $\widetilde{P}_2(-1,0,5)$ are:

$$\left(u^1\right)^{\alpha}_{\theta\varphi} : (1+(1-\alpha)\sin\varphi\cos\theta, 2+(1-\alpha)\sin\varphi\sin\theta, 1+(1-\alpha)\cos\varphi)$$

and

$$\left(u^2\right)^{\alpha}_{\theta\varphi} : (-1+(1-\alpha)\sin\varphi\cos\theta, (1-\alpha)\sin\varphi\sin\theta, 5+(1-\alpha)\cos\varphi)\,,$$

respectively. The space fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2}$ is the collection of the line segments whose extremities are the same points of $\widetilde{P}_1$ and $\widetilde{P}_2$ for different value of α, θ and φ, i.e.,

$$\widetilde{\widetilde{L}}_{P_1P_2}(0)$$
$$= \bigcup_{\alpha\in[0,1]}\bigcup_{\varphi\in[0,\pi]}\bigcup_{\theta\in[0,2\pi]}\left\{(x,y,z) : \tfrac{x-(1+(1-\alpha)\sin\varphi\cos\theta)}{2} = \tfrac{y-(2+(1-\alpha)\sin\varphi\sin\theta)}{2} = \tfrac{z-(1+(1-\alpha)\cos\varphi)}{-4}\right\}.$$

The core line is

$$\widetilde{\widetilde{L}}_{P_1P_2}(1) : \tfrac{x-1}{2} = \tfrac{y-2}{2} = \tfrac{z-1}{-4}.$$

In the following Algorithm 2.5.1, we give the process to obtain a space fuzzy line segment.

Algorithm 2.5.1: To evaluate the space fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2}$ joining two S-type space fuzzy points

---

Input: Given two S-type space fuzzy points $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ with continuous membership functions which are strictly decreasing along the rays emanated from their respective core points. We denote

$$\phi_i(\lambda_i) = f_i\left(\lambda_i \sin\varphi\cos\theta, \lambda_i \sin\varphi\sin\theta, \lambda_i\cos\varphi\right) \text{ for } \lambda_i \geq 0, i = 1, 2.$$

Output: Space fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2} = \bigvee_{\alpha\in[0,1]} \widetilde{\widetilde{L}}_{P_1P_2}(\alpha)$.

for $\alpha = 0$ to 1; with step size $\delta\alpha$ **do**
  for $\theta = 0$ to 2π with step size $\delta\theta$ **do**
    for $\varphi = 0$ to π with step size $\delta\varphi$ **do**
      Compute
      $\lambda_1 = \phi_1^{-1}(\alpha)$
      $\lambda_2 = \phi_2^{-1}(\alpha)$
      Compute the same points
      $\left(u^1\right)^\alpha_{\theta\varphi} = \left(a_1 + (\sin\varphi\cos\theta)\phi_1^{-1}(\alpha), b_1 + (\sin\varphi\sin\theta)\phi_1^{-1}(\alpha), c_1 + (\cos\varphi)\phi_1^{-1}(\alpha)\right)$
      $\left(u^2\right)^\alpha_{\theta\varphi} = \left(a_2 + (\sin\varphi\cos\theta)\phi_2^{-1}(\alpha), b_2 + (\sin\varphi\sin\theta)\phi_2^{-1}(\alpha), c_2 + (\cos\varphi)\phi_2^{-1}(\alpha)\right)$
      for $t = 0$ to 1 with step size δt **do**
        Compute the convex combinations
        $c^\alpha_{\theta\varphi} = t\left(u^1\right)^\alpha_{\theta\varphi} + (1-t)\left(u^2\right)^\alpha_{\theta\varphi}$
      end
    end
  end
  At the end of loop, $\widetilde{\widetilde{L}}_{P_1P_2}(\alpha) \leftarrow c^\alpha_{\theta\varphi}$
end
return $\widetilde{\widetilde{L}}_{P_1P_2} = \bigvee_{\alpha\in[0,1]} \widetilde{\widetilde{L}}_{P_1P_2}(\alpha)$

The following Algorithm 2.5.2 shows how to find the membership value of a point in space fuzzy line segment.

Algorithm 2.5.2: To evaluate the membership value of a point in the space fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2}$ joining two continuous S-type space fuzzy points

Input: Given two S-type space fuzzy points $\widetilde{P}_1(a_1, b_1, c_1)$ and $\widetilde{P}_2(a_2, b_2, c_2)$ with continuous membership functions which are strictly decreasing along the rays emanated from their respective core points. We denote

$$\phi_i(\lambda_i) = f_i\left(\lambda_i \sin\varphi\cos\theta, \lambda_i \sin\varphi\sin\theta, \lambda_i\cos\varphi\right) \text{ for } \lambda_i \geq 0, i = 1, 2.$$

Given a point (x, y, z) whose membership value in $\widetilde{\widetilde{L}}_{P_1P_2}$ is to be calculated.

Output: The membership value $\mu\left((x, y, z)\middle|\widetilde{\widetilde{L}}_{P_1P_2}\right)$.

initialize $\alpha_{\text{sup}} \leftarrow 0$

loop:

for $\alpha = 0$ to 1 with step size $\delta\alpha$ **do**

  for $\theta = 0$ to 2π with step size $\delta\theta$ **do**

    for $\varphi = 0$ to π with step size $\delta\varphi$ **do**

      Compute

      $\lambda_1 = \phi_1^{-1}(\alpha)$

      $\lambda_2 = \phi_2^{-1}(\alpha)$

      Compute the same points

      $\left(u^1\right)_{\theta\varphi}^{\alpha} = \left(a_1 + (\sin\varphi\cos\theta)\phi_1^{-1}(\alpha), b_1 + (\sin\varphi\sin\theta)\phi_1^{-1}(\alpha), c_1 + (\cos\varphi)\phi_1^{-1}(\alpha)\right)$

      $\left(u^2\right)_{\theta\varphi}^{\alpha} = \left(a_2 + (\sin\varphi\cos\theta)\phi_2^{-1}(\alpha), b_2 + (\sin\varphi\sin\theta)\phi_2^{-1}(\alpha), c_2 + (\cos\varphi)\phi_2^{-1}(\alpha)\right)$

      for $t = 0$ to 1 with step size δt **do**

        if $(x, y, z) = t\left(u^1\right)_{\theta\varphi}^{\alpha} + (1-t)\left(u^2\right)_{\theta\varphi}^{\alpha}$ **then**

          if $\alpha_{\text{sup}} < \alpha$ **then**

            $\alpha_{\text{sup}} \leftarrow \alpha$

          else

            goto loop

          end

        end

      end

    end

  end

end

return $\mu\left((x, y, z)\middle|\widetilde{\widetilde{L}}_{P_1P_2}\right) = \alpha_{\text{sup}}$

Example 2.5.2. (*Evaluation of the membership values of some points in the space fuzzy line segment* $\widetilde{\widetilde{L}}_{P_1P_2}(0)$). Consider the fuzzy points $\widetilde{P}_1(1, 2, 1)$ and $\widetilde{P}_2(-1, 0, 5)$ in Example 2.5.1.

The general expressions of the same points with the membership value $\alpha \in [0, 1]$ on $\widetilde{P}_1(1, 2, 1)$ and $\widetilde{P}_2(-1, 0, 5)$ are

$$\left(u^1\right)_{\theta\varphi}^{\alpha} : (1 + (1-\alpha)\sin\varphi\cos\theta, 2 + (1-\alpha)\sin\varphi\sin\theta, 1 + (1-\alpha)\cos\varphi)$$

and

$$\left(u^2\right)^{\alpha}_{\theta\varphi} : \left(-1 + (1-\alpha)\sin\varphi\cos\theta, (1-\alpha)\sin\varphi\sin\theta, 5 + (1-\alpha)\cos\varphi\right),$$

respectively. The following Table 2.3 shows the membership value of some points in the fuzzy line segment $\widetilde{\widetilde{L}}_{P_1P_2}(0)$ by execution of Algorithm 2.5.2.

(x, y, z)	Membership value	Step size
(1.1078, 1.8, 2.2457)	0.1000	$\delta\alpha = 0.2250$, $\delta\theta = 1.5708$, $\delta\varphi = 0.7854$ and $\delta t = 0.1$
(1, 2.6750, 1)	0.3250	$\delta\alpha = 0.2250$, $\delta\theta = 1.5708$, $\delta\varphi = 0.7854$ and $\delta t = 0.1$
(1.45, 2, 1)	0.5500	$\delta\alpha = 0.2250$, $\delta\theta = 1.5708$, $\delta\varphi = 0.7854$ and $\delta t = 0.1$
(1, 2, 1.4)	0.6000	$\delta\alpha = 0.1$, $\delta\theta = 0.6981$, $\delta\varphi = 0.3491$ and $\delta t = 0.1$

TABLE 2.3: Membership value of some points of $\widetilde{\widetilde{L}}_{P_1P_2}(0)$ produced by Algorithm 2.5.2 for Example 2.5.2

2.6 Comparison

In this section, we compare the proposed formulations with the corresponding ideas in [7, 21, 30, 31, 37, 38, 40, 41, 39, 43, 44, 46].

- *S-type space fuzzy point*

  According to [21], a singleton set in $\mathbb{R}^3$ with a nonzero membership value is a space fuzzy point. Thus, a space fuzzy point of [21] may not be a normal fuzzy set in $\mathbb{R}^3$; hence, the core of space fuzzy point may not be a crisp point. On the other hand, the proposed space fuzzy point (see Definition 2.3.2) is a *normal*, convex, and connected fuzzy set in $\mathbb{R}^3$.

A fuzzy disk in [30] and a fuzzy ring in [31] are fuzzy points in [1, 5]. A three-dimensional extension of the two-dimensional fuzzy disk or fuzzy ring in [30, 31] is a space fuzzy point of [7] with a continuous membership function and having a circular base. Although space fuzzy points of Qui and Zhang [7] is very general and it neither considers only continuous membership functions nor has the possibility of having an empty-core, the expressions of same and inverse point for a pair of space fuzzy points of [7] is not easy to compute. It is not easy due to the variety of expressions for the possible membership functions. In order to ease the computation of the same and inverse points, we have proposed here the idea of three variable reference functions and have presented a space fuzzy point by a reference function (Definition 2.3.2). The idea of expressing a space fuzzy point by three-variable reference functions essentially unifies the general and wide variety of membership functions. In fact, this unification has made the computation of the same and inverse points easier and general (see Subsections 2.3.3.4 and 2.3.3.5). In addition, these general expressions of same and inverse points facilitated the analysis throughout the paper.

- *Fuzzy distance*

An extensive list of fuzzy distances and a review of them has been reported by Bloch [47], where fuzzy distances are classified into two categories. In the first category, fuzzy distances between a pair of fuzzy sets are measured by comparing the membership functions. The second category of the distances calculates the distance by combining both spatial and membership functions. The second category of distances is more appropriate as they produce a fuzzy value, unlike crisp values by the first category. In the second category of distances, there are four different approaches—geometrical, morphological, tolerance-based, and graph-theoretic. As the proposed definition in this study (Definition 2.4.1) falls in the second category with the geometrical approach, we compare the proposed definition with the existing geometrical approaches.

Distance (side length) between two vertices of a fuzzy triangle is crisp in [38]. The diameter, according to [39], of a fuzzy line segment can be considered as a fuzzy distance between the extremities, but this measurement of distance is a crisp number. In [44, 46], Hausdorff distances between a pair of fuzzy sets have been defined, but these measurements of distances are also non-fuzzy. However, the distance between fuzzy (imprecise) sets is expected to be imprecise [47].

Qiu and Zhang [7] defined the fuzzy distance between two fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ by

$$\widetilde{D} = \bigvee_{\alpha \in [0,1]} \{d : d = d(u, v),\ \text{where } u \in \widetilde{P}_1(\alpha) \text{ and } v \in \widetilde{P}_2(\alpha)\}. \tag{2.7}$$

By extension principle [15], this formulation of fuzzy distance is identical (see [7]) to that in [46] which is defined by

$$\mu\left(r \middle| \widetilde{D}\right) = \sup_{d(u,v)=r} \left[\inf\left\{\mu\left(u \middle| \widetilde{P}_1\right),\ \mu\left(v \middle| \widetilde{P}_2\right)\right\}\right]. \tag{2.8}$$

As per the approach of [43], the distance between $\widetilde{P}_1$ and $\widetilde{P}_2$ is defined by

$$\mu\left(r \middle| \widetilde{D}\right) = \sup_{d(u,v)\leq r} \left[\inf\left\{\mu\left(u \middle| \widetilde{P}_1\right),\ \mu\left(v \middle| \widetilde{P}_2\right)\right\}\right]. \tag{2.9}$$

All of (2.7), (2.8) and (2.9) produce fuzzy values for distances. Although the constraint set of (2.9) is a superset of that in (2.8), the distances (2.9) and (2.8) between two continuous space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ are identical (see Theorem 2.1 in [1]) since the membership functions of $\widetilde{P}_1$ and $\widetilde{P}_2$ are monotonically decreasing along any ray emanated from their respective core points.

In fact, the fuzzy distance for a pair of space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ by each of (2.7), (2.8) and (2.9) is given by (see [7] for details)

$$\bigvee_{\alpha\in[0,1]} \left[\bar{d}^{\alpha}_{\text{min}}, \bar{d}^{\alpha}_{\text{max}}\right], \tag{2.10}$$

where

$$\bar{d}^{\alpha}_{\text{min}} = \min\left\{d(u,v) : u \in \widetilde{P}_1(\alpha), v \in \widetilde{P}_2(\alpha)\right\}$$

and

$$\bar{d}^{\alpha}_{\text{max}} = \max\left\{d(u,v) : u \in \widetilde{P}_1(\alpha), v \in \widetilde{P}_2(\alpha)\right\}.$$

By contrast, the proposed distance evaluates the fuzzy distance (Definition 2.4.1 and Algorithm 2.4.1) by

$$\bigvee_{\alpha\in[0,1]} \left[d^{\alpha}_{\text{min}}, d^{\alpha}_{\text{max}}\right], \tag{2.11}$$

where

$$d^{\alpha}_{\text{min}} = \min\left\{d(u,v) :\ u \in \widetilde{P}_1(\alpha) \text{ and } v \in \widetilde{P}_2(\alpha) \text{ are inverse points}\right\}$$

and

$$d^{\alpha}_{\text{max}} = \max\left\{d(u,v) :\ u \in \widetilde{P}_1(\alpha) \text{ and } v \in \widetilde{P}_2(\alpha) \text{ are inverse points}\right\}.$$

Evidently,

$$\bar{d}^{\alpha}_{\min} \leq d^{\alpha}_{\min} \leq d^{\alpha}_{\max} \leq \bar{d}^{\alpha}_{\max},$$

and hence the support of the proposed fuzzy distance (2.11) is a subset of the support of the fuzzy distance (2.10). Therefore, the fuzzy distance evaluated by Algorithm 2.4.1 is less imprecise than that in [7].

In addition, since the explicit expression of inverse points for a pair of S-type space fuzzy points is known (see Subsection 2.3.3.5), the expressions $d^{\alpha}_{\min}$ and $d^{\alpha}_{\max}$ of (2.11) can be computed in terms of α. On the other hand, the arbitrariness of $u \in \widetilde{P}_1(\alpha)$ and $v \in \widetilde{P}_2(\alpha)$ in (2.10) makes the computation of $\bar{d}^{\alpha}_{\min}$ and $\bar{d}^{\alpha}_{\max}$ difficult in terms of α.

It is noteworthy that even for a particular $\alpha = \alpha_0$, the computation of $\bar{d}^{\alpha_0}_{\min}$ and $\bar{d}^{\alpha_0}_{\max}$ are difficult since the explicit expression of pertaining objective function $d(u, v)$ is not tractable. However, the objective function $d(u, v)$ for the computation of $d^{\alpha_0}_{\min}$ and $d^{\alpha_0}_{\max}$ are tractable due to Theorem 2.4.1 and the availability of the explicit expression of the inverse points of membership value α_0 (Subsection 2.3.3.5).

- *Space fuzzy line segment*

  Sides of a space fuzzy triangle in [38] can be considered as space fuzzy line segments. A fuzzy set in the $\mathbb{R}^3$-space whose support is a crisp line segment and that has a constant membership function is a fuzzy line segment in [21]. The space fuzzy line segment, according to [39], is the shortest path between two space points with a constant membership function. The core of the fuzzy line-segments of [38, 21, 39] can possibly be empty, and hence may not be a crisp line-segment. Moreover, even if the core of the fuzzy line segment in [38] is nonempty, the membership function of one side of the core is zero. In comparison, the proposed space fuzzy line-segment (Definition 2.5.1) neither has an empty core nor on one side of the core the membership function is zero. Qui and Zhang [7] defined the space fuzzy line segment joining two space fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ by

$$\widetilde{\widetilde{L}}_{P_1P_2} = \bigvee \{l : l \text{ is a line segment joining a point in } \widetilde{P}_1(0) \text{ to a point in } \widetilde{P}_2(0)\}.$$

On the other hand, we have evaluated the fuzzy line segment by

$$\widetilde{\widetilde{L}}_{P_1P_2} = \bigvee \{l : l \text{ is a line segment joining same points in } \widetilde{P}_1(0) \text{ and } \widetilde{P}_2(0)\}.$$

According to the proposed Algorithm 2.5.1, the space fuzzy line segment is formulated by

$$\widetilde{\widetilde{L}}_{P_1P_2} = \bigvee_{\alpha \in [0,1]} \widetilde{\widetilde{L}}_{P_1P_2}(\alpha),$$

where

$$\widetilde{\widetilde{L}}_{P_1P_2}(\alpha) = \bigcup_{\lambda \in [0,1]} \{\lambda u + (1-\lambda)v : \text{ where } u \in \widetilde{P}_1(\alpha) \text{ and } v \in \widetilde{P}_2(\alpha) \text{ are same points}\}.$$

Whereas the fuzzy line segment in [7] collects all line segments joining points in the supports of the fuzzy points. Our proposed method considers only the combinations of the same points. Therefore, the support of the proposed fuzzy line segment is a subset (less imprecise) of that in [7]. In addition, the explicit expressions (Subsection 2.3.3.4) of the same points for a pair of S-type space fuzzy point make the evaluation (Algorithm 2.5.1) of the membership function for $\widetilde{\widetilde{L}}_{P_1P_2}$ easier when compared to the formulations in [7].

2.7 Conclusion

In this paper, we have formulated a basic idea of space fuzzy point with the help of a three-variable reference function, namely, S-type space fuzzy point. We also have proposed the idea of the same and inverse points with respect to continuous S-type space fuzzy points. With the help of these concepts, analysis on the fuzzy distance between two S-type space fuzzy points, convex combinations of S-type space fuzzy points, the coincidence of two S-type space fuzzy points and space fuzzy line segment have been performed. Importantly, since the the proposed ideas depend on an unified representation of space fuzzy points by three-variable reference functions, just by extending the number of variables, all the three-dimensional fuzzy geometrical concepts can find the corresponding concepts in n-dimensional Euclidean space.

With the help of S-type representation of fuzzy points, the general expressions of the same and inverse points have been given that are used throughout the paper. Future studies can find the general explicit expressions of the same and inverse points for two fuzzy points in $\mathbb{R}^n$. Towards this, one needs an S-type representation of a fuzzy point in $\mathbb{R}^n$, which can be found by replacing $\mathbb{S}^2$ by $\mathbb{S}^{n-1}$ in the definition of reference function given in the Section 2.3.1. One can also try to develop the idea of the same and inverse points for discontinuous fuzzy points.

Our next research on fuzzy space geometry will be continued on the detailed analysis of fuzzy triangles, fuzzy planes, fuzzy spheres, fuzzy ellipsoids and their properties.

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