Chapter 1

# Introduction

Since its inception in 1965, the theory of fuzzy sets has advanced in various ways and in many disciplines. The fuzzy set theory facilitates the precise and rigorous study of vague conceptual phenomena by providing a strict mathematical framework (fuzzy set theory is not fuzzy!). The fuzzy set theory describes how to deal with uncertainty in real world. According to fuzzy set theory, the universe is non-fuzzy, and the way we look any object is fuzzy or vague. To deal with the vague/imprecise shapes in the universe, we extend the classical geometry using the fuzzy technique.

In research on fuzzy space geometry, we aim to mathematically characterize the fuzzy space geometrical elements using the membership functions. Like the main objective of the study on conventional geometry, we are trying to identify fuzzy space geometrical objects algebraically as a collection of three-dimensional points with varied membership values. The interest in analyzing fuzzy space geometry is that the theory of fuzzy geometrical objects provides a useful framework for dealing with vague/imprecise information, which many of us encounter on a daily basis. A general goal has been to establish fuzzy shape analysis that can be implemented to fuzzy object detection in an image. This thesis focuses on developing the fuzzy space geometrical notions (fuzzy distance, lines, planes, spheres, cones, etc.) and their applications in fuzzy image processing. At first, we shall attempt to develop fuzzy space geometry. Next, we shall turn up to implement the theory of fuzzy space geometry in fuzzy image processing.

## 1.1 Fuzzy space geometry

A profound study of fuzzy plane geometry has been seen in [1, 2, 3, 4, 5, 6]. In 1997, Buckley and Eslami [5, 6] investigated the fuzzy plane geometry along with the concept of fuzzy point in $\mathbb{R}^2$-plane. Afterward, Ghosh and Chakraborty redefined the fuzzy plane geometry more rigorously in [1, 2, 3, 4]. In spite of the great interest seen in fuzzy plane geometry, very few studies focus on its extension, such as fuzzy space geometrical notions in $\mathbb{R}^3$. It is also notable that fuzzy plane geometrical shapes have been far more studied than fuzzy shapes in $\mathbb{R}^3$. Only Qiu and Zhang [7] introduced some concepts on the representation and formulation of the fuzzy space geometrical objects in a fuzzy mathematical framework. After that, there is no study focusing on fuzzy space geometrical objects in $\mathbb{R}^3$. Additionally, we intend to study fuzzy space objects because they exist all around us. In addition, the ideas can also be extended to higher dimensions.

Fuzzy space geometry studies the size, shape, and nature of a surface enclosed by vague curves in $\mathbb{R}^3$. In this study, shape analysis has been performed in order to obtain the mathematical equations for vague/imprecise space objects. The fuzzy space entities have been perceived geometrically, and the formulation procedure has been suggested to seen them in the three-dimensional Euclidean space. Three-dimensional analytical fuzzy geometry is the main focus of this study.

A conventional analytical geometry represents the mathematical or algebraical concepts on the basis of a standard coordinate system with a graphical representation. Apart from analyzing, analytical geometry helps to build a structural analogy. In the Cartesian coordinate system, the axes $OX$, $OY$, and $OZ$ are just three real lines being placed perpendicular to each other. These perpendicular axes form a reference frame to describe an equation $f(x, y, z) = 0$ geometrically. In this system, every triplet described a unique point that is crisp. Although conventional analytical geometrical concepts may be potentially very vigorous in studying the topology of image subsets, but they cannot describe imprecise locations, imprecise objects, or imprecise shapes. For example, the point $(2, 4, 1)$ can be located precisely by a point in the three-dimensional Euclidean space. However, conventional geometry cannot answer the questions-how to locate 'nearly $(2, 4, 1)$' or 'around $(2, 4, 1)$'? How to locate an imprecise object? How to get the distance between two imprecise objects? At this point, we may require to explore the notions of fuzzy space geometry. The fuzzy space geometry adopted the frame of references as a crisp line, i.e., like classical geometry frame of references is crisp. This choice of a crisp frame of reference is based on the idea that though there are imprecise or fuzzy objects in practice, the mathematical universe of discourse is chosen to be crisp.

A space fuzzy point is the basic defining element in fuzzy space geometry. The delineation on any fuzzy space geometric objects (such as a line, plane, sphere, cone, etc.) generally depends on the entity space fuzzy point. A space fuzzy point can be envisioned as a set with a cluster or bunch or a chunk of points (in $\mathbb{R}^3$-space) with the membership function. The membership function is a mathematical function that describes the belongingness of a point to the set. The membership functions of space fuzzy points on the space $\mathbb{R}^3$ are realized as surfaces in space $\mathbb{R}^4$. Let us consider certain examples to perceive a space fuzzy point.

1. Many clusters, nebulae and galaxies can be seen as a space fuzzy point in $\mathbb{R}^3$-space.
2. We try to locate any point that appear randomly in a small region of $\mathbb{R}^3$-space.
3. We try to see a black boll by a hazy glass.

Geometrically, a space fuzzy point refers to a vague/imprecise location in $\mathbb{R}^3$-space.

If a fuzzy point is moving in space, the locus of the fuzzy point can be perceived as a fuzzy curve [8]. A fuzzy curve can be defined geometrically as a cluster of mathematical curves that are contained at different levels. The mathematical representation of the fuzzy curves and geometrical construction of fuzzy objects can be dealt with by studying fuzzy space geometry. Applications of the fuzzy space geometrical properties can be seen in medical image analysis to analyze and interpret a fuzzy image. Next, we show how the fuzzy geometrical notions will be helpful to feature extraction in fuzzy image processing.

## 1.2 Application of fuzzy geometry on fuzzy image processing

The fuzzy geometry describes how to deal with uncertainty in real life, so it has many practical applications. Fuzzy plane geometry is of great interest in fuzzy image processing for dealing with imprecise shapes and features in $2D$ images (see Subsection 1.4.3). The growing interest in dealing with blurred $2D$ images in the application field such as medical imaging motivates addressing the further tasks following the study of fuzzy space geometry. A successful application of fuzzy plane geometry [1, 2, 3, 9] in fuzzy image processing has been articulated in this thesis. In the proposed application of the fuzzy geometry, the concept of fuzzy lines and fuzzy circles introduced by Ghosh and Chakraborty [1, 2, 3, 9] is being used.

### 1.2.1 Fuzzy image processing

The term 'image processing' refers to the process of performing some operations on an image in order to enhance it or obtain some useful information from it. Prior to further processing, an image is preprocessed to improve its data at the lowest level. In preprocessing, important features are enhanced or highlighted and unwanted information is suppressed that is irrelevant to image processing. It has been found that using conventional mathematical tools to extract features from images was inefficient due to the inherent ambiguity and vagueness present in the images. Image boundaries and edges are not always precisely defined, and sometimes they are viewed as fuzzy subsets of a image due to the vague nature of their boundaries and edges. During the imaging process, noise can introduce fuzziness into images. This demands further exploring efficient methods (fuzzy image processing) to analyze digital images better. Fuzzy image processing, as its name implies, deals with how various fuzzy mathematical procedures can be used to process images.

Fuzzy set-theoretic techniques provide a variety of rich methodologies for processing diverse images in contrast to classical image-analysis techniques that use crisp mathematics. Fuzzy set theory is widely used for enhancement, segmentation, and feature extraction in image processing, with an explosion in interest and significant growth (see Subsection 1.4.3). A fuzzy image processing is a cluster of varieties of fuzzy approaches to image processing that enables the images to be understood, represented, and processed. Fuzzy geometries are important frameworks for building the foundations of fuzzy image processing. In computer vision and image processing, fuzzy geometry can be used to handle a variety of uncertainties. The application areas of geometrical fuzziness are image representation, feature extraction, and image segmentation.

Real images, such as medical images, images gathered by remote sensing, and others, often have edges that are not distinguishable because of unequal illumination and poor contrast. In such cases, it is very difficult to determine whether an edge exists in an image. In this case, a fuzzy edge detection approach has been adopted that considers the image as a fuzzy set. In fuzzy images, the edges are poor contrast, and edges are vague/imprecise, blurred, etc. A critical component of medical imaging is edge detection, particularly when the images are poorly contrasted. It is necessary to address that problem as the improper selection of edges can lead to disease misdiagnoses. Fuzzy geometry is helpful to alleviate such a type of problem. Fuzzy geometry plays a significant role to takes into account the ambiguity and vagueness exist in the image. The ideas of fuzzy geometrical elements such as fuzzy points, the distance between two fuzzy points, fuzzy lines, and fuzzy circles in the image play a key role in extracting the feature and analyzing a fuzzy image.

In order to better understand why fuzzy notions are useful in image processing, let us examine Prewitt's rationale [10]:

1. An image is a two-dimensional projection of a three-dimensional world, which is mapped by a camera. Cameras can operate in optical, near-infrared, or even thermal bands of the electromagnetic spectrum. Some information is lost when the space is transformed into images, leading to an increase in uncertainty.

2. Due to the variability of gray values, there is uncertainty in multi-valued gray-level images. A digital image's pixel values need not be considered precise constants, but rather fuzzy values.

3. The boundaries of objects or the homogeneity of segments in many images are vague/imprecise in nature that require fuzzy notions for characterization.

4. In addition, the poor contrast between the objects and the background can also be perceived as fuzzy.

5. Images are often interpreted in an ambiguous manner. Hence, it is important to incorporate soft decision-making since a hard decision can often prove to be costly as human understanding cannot be crisp or precise.

### 1.2.2 Fuzzy Hough transform

The image processing technique that identifies brightness discontinuities is edge detection. It detects the boundaries of objects in the image. There are several edge detection methods, including the Hough transform, thresholding methods, active contour model and edge detectors. A feature extraction technique known as Hough transform is used in digital image processing, image analysis and computer vision. The Hough transform can be used to locate objects within an image that possess known shapes and sizes, such as lines, circles, squares, rectangles, etc. The basic idea of this technique is to find curves like lines, circles, ellipse, etc., in suitable parameter space. It was originally used for parameterizable shapes like lines, circles, etc., but was extended later to include arbitrary shapes.

It is common for edges to be vague and not prominent, which makes them difficult to detect. Fuzzy methods are useful in solving this type of problem because the edges are vague. Also, shapes are often vague/ill-defined in real-life images due to noise, digitization errors, and shape variations. Due to external illumination, the gray values in the object region vary rapidly. In other words, grayness and spatial ambiguity lead to ill-defined shapes. Thus, finding a single peak in the parameter space may be difficult after applying the classical Hough transform (see Subsection 1.3.2). Specifically, the parameter space of an image may have more than one local peak for a single shape. In turn, this will decrease the amplitude of the peaks in parameter space, making detection more difficult. These peaks will represent multiple shapes even if they can be identified, leading to a misinterpretation of the data. In the literature on fuzzy Hough transforms, the majority of contributions focus on solutions to these problems [11, 12, 13, 14].

The classical Hough transform (see Subsection 1.3.2) is not enough to deal with all practical problems because real-life images cannot be captured precisely. However, in the literature, no method yet exists for capturing fuzzy lines and circles. In order to deal with the imprecise nature of objects, fuzzy Hough transform (FHT) is extensively presented in this thesis. The FHT addresses the lines and circles in imprecise environments.

## 1.3 Preliminaries

In this section, we provide the notations and basic definitions that are used throughout the paper. We use the notations $\mathbb{R}$ and $\mathbb{R}_+$ to represent the set of all real numbers and the set of all non-negative real numbers, respectively. We place a *tilde bar* over capital or small letters, i.e., $\widetilde{A}, \widetilde{B}, \widetilde{C}, \ldots, \widetilde{a}, \widetilde{b}, \widetilde{c}, \ldots,$ to denote fuzzy sets in $\mathbb{R}^n, n = 1, 2, 3$. The membership function of a fuzzy set $\widetilde{A}$ in $\mathbb{R}^n$ is denoted by $\mu(x|\widetilde{A})$, $x \in \mathbb{R}^n$. If $\mu(x|\widetilde{A})$ is continuous, we call $\widetilde{A}$ a continuous fuzzy set.

### 1.3.1 Fuzzy sets

*Definition* 1.3.1. ($\alpha$*-cut of a fuzzy set [15]*). For a fuzzy set $\widetilde{A}$, its $\alpha$-cut is denoted by $\widetilde{A}(\alpha)$ and is defined by

$$\widetilde{A}(\alpha) = \begin{cases} \{x : \mu(x|\widetilde{A}) \geq \alpha\} & \text{if } 0 < \alpha \leq 1 \\ \text{closure}\{x : \mu(x|\widetilde{A}) > 0\} & \text{if } \alpha = 0. \end{cases}$$

The set $\{x \in \mathbb{R}^n : \mu(x|\widetilde{A}) > 0\}$ is called *support* of the fuzzy set $\widetilde{A}$.

The set $\widetilde{A}(1) = \{x \in \mathbb{R}^n : \mu(x|\widetilde{A}) = 1\}$ is said to be *core* of the fuzzy set $\widetilde{A}$.

We adopt a notation $\bigvee\{x : x \in \widetilde{A}(0)\}$ to represent the construction of membership function of a fuzzy set $\widetilde{A}$, which is frequently used in place of $\mu(x|\widetilde{A}) = \sup\{\alpha : x \in \widetilde{A}(\alpha)\}$.

*Definition* 1.3.2. (*Fuzzy numbers [5]*). A fuzzy set $\widetilde{A}$ in $\mathbb{R}$ is called a fuzzy number if its membership function has the following properties:

(i) $\mu\left(x\middle|\widetilde{A}\right)$ is upper semi-continuous,

(ii) there exists a real number $m$ such that $\mu\left(m\middle|\widetilde{A}\right) = 1$, and

(iii) there exist two non-negative real numbers $\ell$ and $r$ such that $\mu\left(x\middle|\widetilde{A}\right) = 0$ outside the interval $[m-\ell, m+r]$, and $\mu\left(x\middle|\widetilde{A}\right)$ is non-decreasing on $[m-\ell, m]$ and $\mu\left(x\middle|\widetilde{A}\right)$ is non-increasing on $[m, m+r]$.

The $\alpha$-cut of a fuzzy number $\widetilde{A}$ is a closed and bounded interval for all $\alpha$ in $[0, 1]$ since $\mu\left(x\middle|\widetilde{A}\right)$ is upper semi-continuous.

*Definition* 1.3.3. (*LR-type fuzzy number [15]*). A function $L : \mathbb{R} \to [0, 1]$ which is symmetric about '0', non-increasing on $[0, \infty)$ and satisfying either of the following two is called a reference function of a fuzzy number:

(i) $L(0) = 1$ and $L(1) = 0$, or

(ii) $L(0) = 1$, $L(x) > 0$ for all $x$, and $\lim_{x\to+\infty} L(x) = 0$.

A fuzzy number $\widetilde{A}$ is called an $LR$-type fuzzy number if there exist two reference functions $L$ and $R$, a real number $m$, and two numbers $\ell > 0$ and $r > 0$ such that $\mu(x|\widetilde{A})$ can be written as:

$$\mu\left(x\middle|\widetilde{A}\right) = \begin{cases} L\left(\frac{m-x}{\ell}\right) & \text{if } x \leq m \\ R\left(\frac{x-m}{r}\right) & \text{if } x \geq m. \end{cases}$$

The numbers $\ell$ and $r$ are called the left spread and right spread, respectively, of the fuzzy number $\widetilde{A}$.

It is noteworthy that any fuzzy number can be represented by an $LR$-type fuzzy number (see Theorem 4.1 of [16]). We use the notation $(m-\ell/m/m+r)_{LR}$ to represent an $LR$-type fuzzy number. In particular, if $L(x)=R(x)=\max\{0,1-|x|\}$, the fuzzy number $\widetilde{A}$ is called a triangular fuzzy number. A triangular fuzzy number is denoted by $(m-\ell/m/m+r)$.

*Definition* 1.3.4. (*Same and inverse points for fuzzy numbers [1]*). Let $\widetilde{A}_1$ and $\widetilde{A}_2$ be two continuous fuzzy numbers whose cores are $m_1$ and $m_2$, respectively. Two numbers $x_1$ and $x_2$ in the supports of $\widetilde{A}_1$ and $\widetilde{A}_2$, respectively, are called same points with respect to $\widetilde{A}_1$ and $\widetilde{A}_2$ if

(i) $\mu\left(x_1\middle|\widetilde{A}_1\right)=\mu\left(x_2\middle|\widetilde{A}_2\right)$, and

(ii) $x_1\le m_1$ and $x_2\le m_2$, or $x_1\ge m_1$ and $x_2\ge m_2$.

The numbers $x_1$ and $x_2$ are said to be inverse points with respect to $\widetilde{A}_1$ and $\widetilde{A}_2$ if $x_1$ and $-x_2$ are same points with respect to $\widetilde{A}_1$ and $-\widetilde{A}_2$ (see [1]).

*Example* 1.3.1. (*General expressions of same and inverse points for fuzzy numbers*). Consider two fuzzy numbers $\widetilde{A}_1=(m_1-\ell_1/m_1/m_1+r_1)_{L_1R_1}$ and $\widetilde{A}_2=(m_2-\ell_2/m_2/m_2+r_2)_{L_2R_2}$, where the reference functions $L_1$, $L_2$, $R_1$ and $R_2$ are continuous and strictly monotone. With respect to $\widetilde{A}_1$ and $\widetilde{A}_2$, for any $\alpha\in(0,1]$,

(i) the same points of membership value $\alpha$ are $m_1-\ell_1L_1^{-1}(\alpha)$ and $m_2-\ell_2L_2^{-1}(\alpha)$, or $m_1+r_1R_1^{-1}(\alpha)$ and $m_2+r_2R_2^{-1}(\alpha)$, and

(ii) the inverse points of membership value $\alpha$ are $m_1-\ell_1L_1^{-1}(\alpha)$ and $m_2+r_2R_2^{-1}(\alpha)$, or $m_1+r_1R_1^{-1}(\alpha)$ and $m_2-\ell_2L_2^{-1}(\alpha)$.

*Definition* 1.3.5. (*Plane fuzzy points [5]*). A fuzzy point at $(a,b)$ in $\mathbb{R}^2$, denoted $\widetilde{P}(a,b)$, is a fuzzy set in $\mathbb{R}^2$ whose membership function has the following properties:

(i) $\mu\left((x,y)\middle|\widetilde{P}(a,b)\right) = 1$ only at $(x,y) = (a,b)$, and

(ii) $\widetilde{P}(a,b)(\alpha)$ is a compact convex subset of $\mathbb{R}^2$ for all $\alpha$ in $[0,1]$.

*Definition* 1.3.6. (*Same and inverse points for plane fuzzy points [1]*). Let $(x_1,y_1)$ and $(x_2,y_2)$ be two points in the supports of the continuous fuzzy points $\widetilde{P}_1(a_1,b_1)$ and $\widetilde{P}_2(a_2,b_2)$, respectively.

Let $\widetilde{r}_1$ be the fuzzy number on the support of $\widetilde{P}_1(a_1,b_1)$ along the line $L_1$ joining $(x_1,y_1)$ and $(a_1,b_1)$ (see Definition 3.3 in [1] for details).

Let $\widetilde{r}_2$ be the fuzzy number on the support of $\widetilde{P}_2(a_2,b_2)$ along the line $L_2$ joining $(x_2,y_2)$ and $(a_2,b_2)$.

The points $(x_1,y_1)$ and $(x_2,y_2)$ are said to be same points with respect to $\widetilde{P}_1(a_1,b_1)$ and $\widetilde{P}_2(a_2,b_2)$ if:

(i) $(x_1,y_1)$ and $(x_2,y_2)$ are same points with respect to $\widetilde{r}_1$ and $\widetilde{r}_2$, and

(ii) the lines $L_1$ and $L_2$ make the same angle with the line joining $(a_1,b_1)$ and $(a_2,b_2)$.

The points $(x_1,y_1)$ and $(x_2,y_2)$ are said to be inverse points if $(x_1,y_1)$ and $(-x_2,-y_2)$ are same points with respect to $\widetilde{P}_1(a_1,b_1)$ and $-\widetilde{P}_2(a_2,b_2)$.

*Example* 1.3.2. (*Same and inverse points for plane fuzzy points*). Consider two continuous fuzzy points $\widetilde{P}_1(a_1,b_1)$ and $\widetilde{P}_2(a_2,b_2)$ whose membership functions are

$$\mu\left((x,y)\middle|\widetilde{P}_1(a_1,b_1)\right) = \begin{cases} 1 - \sqrt{\left(\frac{x-a_1}{p_1}\right)^2 + \left(\frac{y-b_1}{q_1}\right)^2} & \text{if } \left(\frac{x-a_1}{p_1}\right)^2 + \left(\frac{y-b_1}{q_1}\right)^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu\left((x,y)\middle|\widetilde{P}_2(a_2,b_2)\right)=\begin{cases}1-\sqrt{\left(\frac{x-a_2}{p_2}\right)^2+\left(\frac{y-b_2}{q_2}\right)^2} & \text{if } \left(\frac{x-a_2}{p_2}\right)^2+\left(\frac{y-b_2}{q_2}\right)^2\leq 1\\ 0 & \text{otherwise.}\end{cases}$$

With respect to $\widetilde{P}_1(a_1,b_1)$ and $\widetilde{P}_2(a_2,b_2)$, for any $\alpha\in(0,1]$,

(i) the same points of membership value $\alpha$ are

$$(a_1+t^\alpha_{1\theta}\cos\theta, b_1+t^\alpha_{1\theta}\sin\theta) \quad \text{and} \quad (a_2+t^\alpha_{2\theta}\cos\theta, b_2+t^\alpha_{2\theta}\sin\theta) \text{ for all } \theta\in[0,2\pi],$$

and

(ii) the inverse points of membership value $\alpha$ are

$$(a_1+t^\alpha_{1\theta}\cos\theta, b_1+t^\alpha_{1\theta}\sin\theta) \quad \text{and} \quad (a_2-t^\alpha_{2\theta}\cos\theta, b_2-t^\alpha_{2\theta}\sin\theta) \text{ for all } \theta\in[0,2\pi],$$

where

$$t^\alpha_{1\theta}=\frac{1-\alpha}{\sqrt{\frac{\cos^2\theta}{p_1^2}+\frac{\sin^2\theta}{q_1^2}}} \text{ and } t^\alpha_{2\theta}=\frac{1-\alpha}{\sqrt{\frac{\cos^2\theta}{p_2^2}+\frac{\sin^2\theta}{q_2^2}}}.$$

*Definition* 1.3.7. (*Slope-intercept form of a fuzzy line [2]*). Let $\widetilde{m}$ and $\widetilde{c}$ be two fuzzy numbers. The fuzzy line $\widetilde{L}_{SI}$ with slope $\widetilde{m}$ and $y$-intercept $\widetilde{c}$ may be defined by its membership function as

$$\mu\left((x,y)\middle|\widetilde{L}_{SI}\right)=\sup\{\alpha: \text{where } (x,y) \text{ lies on the line with slope } m\in\widetilde{m}(0) \text{ and } y\text{-intercept } c\in\widetilde{c}(0) \text{ with } \mu(m|\widetilde{m})=\mu(c|\widetilde{c})=\alpha\}.$$

More precisely,

$$\mu\left((x,y)\middle|\widetilde{L}_{SI}\right)=\sup\{\alpha: y=mx+c \text{ where } m\in\widetilde{m}(0),\ c\in\widetilde{c}(0) \text{ with } \mu(m|\widetilde{m})=\mu(c|\widetilde{c})=\alpha\}.$$

Note that, for each $m$ in $\widetilde{m}(0)$, there exist two values of $c$ in $\widetilde{c}(0)$ with the same membership grades. In addition, for each $c$, there exist two values of $m$ with the same membership values as that of $c$. From the definition of $\widetilde{L}_{SI}$, we observe that $\widetilde{L}_{SI}$ is constructed by taking the union of the lines with slope $m$ that have exactly the same membership grade as the $y$-intercept. By Theorem 3.2 in [9], there always exists a fuzzy number on $\widetilde{L}$ along a line perpendicular to $\widetilde{L}(1)$.

*Definition* 1.3.8. (*Symmetric and non-symmetric fuzzy lines [9]*). If all the fuzzy numbers, which are situated on a fuzzy line $\widetilde{L}$ and along the perpendicular lines of $\widetilde{L}(1)$, are having same spread, then $\widetilde{L}$ is said to be a symmetric fuzzy line and otherwise non-symmetric.

*Definition* 1.3.9. (*Fuzzy circle [3]*). Let $\widetilde{P}_1(a_1, b_1)$, $\widetilde{P}_2(a_2, b_2)$, and $\widetilde{P}_3(a_3, b_3)$ be three fuzzy points. The fuzzy circle, $\widetilde{C}$, that passes through these three fuzzy points can be defined by its membership function as

$$\mu\left((x, y)\middle|\widetilde{C}\right) = \sup\{\alpha : (x, y) \text{ lies on the circle that passes through the three same point on } \widetilde{P}_1,\ \widetilde{P}_2 \text{ and } \widetilde{P}_3 \text{ with membership value } \alpha\}.$$

According to this definition, a fuzzy circle is constructed as a set of crisp points with various membership values. The definition of $\mu((x, y)|\widetilde{C})$ shows that a fuzzy circle is the union of all the crisp circles that pass through the same three points on the supports of $\widetilde{P}_1$, $\widetilde{P}_2$, and $\widetilde{P}_3$.

### 1.3.2 Classical Hough transform

The definitions concerning Hough transform are adopted from [17, 12, 13].

Hough [17] has proposed a procedure for detecting lines in a given image. This procedure also can be used for more general curve fitting [18]. Let us suppose that we are looking for straight lines in an image. If we consider a point $(x', y')$ in the image, all lines which pass through that point have the form $y' = mx' + c$ for varying values of $m$ and $c$ (see Figure 1.1). The equation $y' = mx' + c$, i.e., $c = -x'm + y'$, where $m$ and $c$ being variables is drawn in Figure 1.2. Note that, this is a straight line on a graph of $c$ against $m$ ($(m, c)$-space). Each different line through the point $(x', y')$ corresponds to one of the points on the line in $(m, c)$-space (termed as Hough space).

Explicitly, suppose a two dimensional image is given and consider two pixels $p$ and $q$ in $(x, y)$-space which lie on the same line. For each pixel, all of the possible lines effectively represent the collection of points on a single line in $(m, c)$-space. Thus the single line in $(x, y)$-space which goes through both pixels lies on the intersection of the two lines representing $p$ and $q$ in $(m, c)$-space (see Figure 1.3). All pixels which lie on the same line in the $(x, y)$-space are represented by lines that all pass through a single point in $(m, c)$-space. The single point through which they all pass gives the values of $m$ and $c$ in the line equation $y = mx + c$.

The algorithmic representation of the classical Hough transform is as follows.

**Algorithm 1.3.1:** Classical Hough transform

**Step 1:** Quantize $(m, c)$-space into a two-dimensional array $A$ for appropriate steps of $m$ and $c$.

**Step 2:** Initialize all elements of $A(m, c)$ to zero.

**Step 3:** For each point $(x', y')$ which lies on some line in the image, we add 1 to all elements of $A(m, c)$ whose indices $m$ and $c$ satisfy $y' = mx' + c$.

**Step 4:** Search for elements of $A(m, c)$ which have local peaks. Each one found corresponds to a line in the image.

*Note* 1. One practical difficulty is that the $y = mx+c$ form for representing a straight line breaks down for lines whose slopes $m$ have a large value. It is better to avoid this problem by using the alternative formulation of a line as $x\cos\theta + y\sin\theta = \rho$, where $\rho$ is the perpendicular distance of the line from the origin, and $\theta$ is the angle subtended by the normal with respect to the $x$-axis (see Figure 1.4). However, a point in $(x, y)$-space is now represented by a curve in $(\rho, \theta)$-space rather than a straight line. Otherwise, the method remains unchanged.

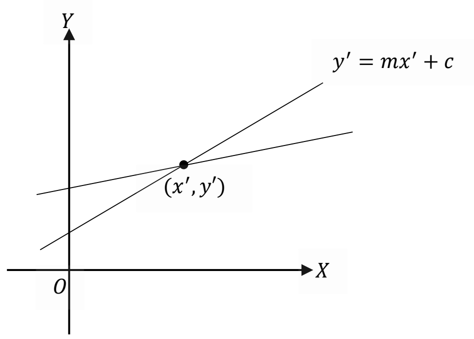


FIGURE 1.1: Lines passing through the point $(x', y')$

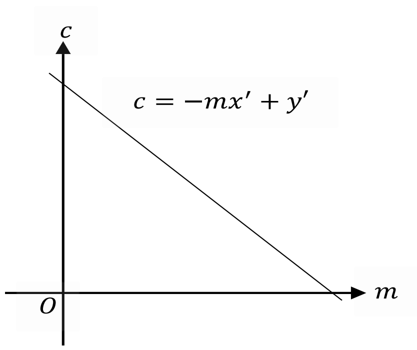


FIGURE 1.2: A line in $(m, c)$-space with slope $-x'$ and intercept $y'$

The terminology of Ballard [19], is based on the detection of generalized shapes. Consider analytic shapes of the form $f(\boldsymbol{x}, \boldsymbol{a}) = 0$, where $\boldsymbol{x}$ is an image point and $\boldsymbol{a}$ is a parameter vector. Consider the following example to gain a general idea of how the Hough transform works.

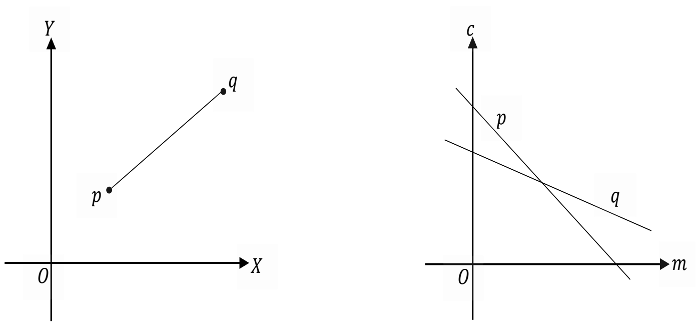


FIGURE 1.3: Classical Hough transform

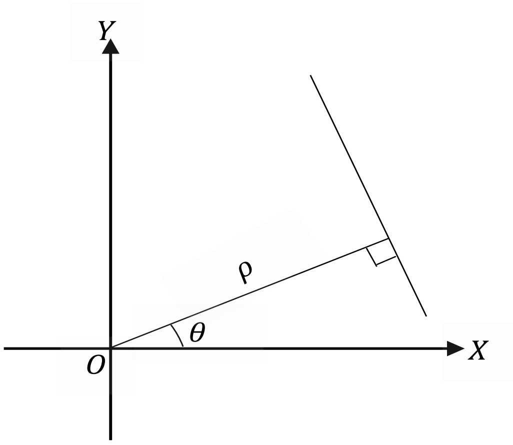


FIGURE 1.4: $(\rho, \theta)$-representation of a line

Suppose a circle of known radius is to be detected in an image. The equation of a circle is

$$(x - a)^2 + (y - b)^2 = r^2$$

in $(x, y)$-space. If $x$ and $y$ are fixed, and the parameters $a$ and $b$ are variable, then a circle in $(a, b)$-space is defined. The loci of the center for each point $(x, y)$ that truly lies on the circle will intersect at a single point, say $(a, b)$, in the parameter space. The coordinates $(a, b)$ represent the center of the circle in the image. By maintaining an initial zero accumulator array $A(a, b)$, the Hough transform can be implemented. For each $(x, y)$, all values of the parameters $a$ and $b$ are computed such that

$$(x-a)^2 + (y-b)^2 - r^2 = 0.$$

The accumulator cells $A(a, b)$ are incremented as follows:

$$A(a, b) = A(a, b) + 1.$$

In the accumulator array, when all border pixels are considered in the image, the maximum value is obtained. Essentially, it refers to the point at which the loci intersect or the center of a circle. It is possible to extend this phenomenon to circles of any radius by making the radius $r$ variable. In this case, $A$ will be a three-dimensional array.

A generalized version of the Hough transform for arbitrary shape is described in [13] as follows.

In the Hough transform, pixels in an image lying on a curve are transformed into the parameter space representing the curve's analytical (mathematical) form. Let a curve be analytically defined as $f(x, y, \boldsymbol{\theta}) = 0$, where $(x, y)$ are the coordinates of a pixel on the curve concerning some standard reference frame. The vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ is the set of parameters representing the shape of the curve. To illustrate, consider the equation $x \cos\theta + y \sin\theta = \rho$ for a straight line, where $(\rho, \theta)$ are the parameters determining the straight line, $\rho$ is the perpendicular distance from the origin, and $\theta$ is the angle made by the normal to the $x$-axis. Similarly, a circle can be characterized as

$$(x-a)^2 + (y-b)^2 = r^2,$$

where $(r, a, b)$ are the parameters illustrating the circle, $(a, b)$ are the coordinates of the center, and $r$ is the radius.

Given the parametric form of a curve in an image, in the classical Hough transform, all possible parameter values are determined for each object pixel lying along a curve. The parameter space is quantized, and an accumulator array $A$ is defined. The algorithm for detecting curves using the classical Hough transform can be illustrated as follows [13].

**Algorithm 1.3.2:** To detect the curves using the classical Hough transform

**Step 1:** Quantize the $\boldsymbol{\theta}$-space within the limits
$\theta_{1\,\min} \leq \theta_1 \leq \theta_{1\,\max}$, $\theta_{2\,\min} \leq \theta_2 \leq \theta_{2\,\max}$, . . ., $\theta_{n\,\min} \leq \theta_n \leq \theta_{n\,\max}$.

**Step 2:** Form an array $A(\boldsymbol{\theta})$ and initialize all elements to zero.

**Step 3:** Add 1 to all elements of $A(\boldsymbol{\theta})$ whose indices $\theta_1, \theta_2, \ldots, \theta_n$ satisfy $f(x, y, \boldsymbol{\theta}) = 0$, for every point $(x, y)$ on a curve in the image.

**Step 4:** Identify elements of $A(\boldsymbol{\theta})$ that have a local peak. These correspond to curves in the image.

In classical Hough transformation, images are only transformed into parameter spaces or accumulators. The points in the image space correspond to the curves in the parametric space. Specifically, a curve in the image is transformed into a family of curves in the parameter space. Ideally, this family of curves should pass through a single point in the parameter space representing the curve's parameter value. Consequently, the accumulator value will be maximum in the quantization slot of parameter space containing the point through which the family of curves passes. It is necessary to detect the local maxima in the accumulator space in order to detect a curve in the image.

## 1.4 Literature survey

### 1.4.1 Literature on fuzzy geometry

Although the ideas of fuzzy geometrical notions have been proposed by many researchers but fuzzy geometry has been of interest only when Buckley and Eslami [5, 6] gave some ideas on the construction and representation of the basic fuzzy geometrical entities in a mathematical framework. After that, Ghosh and Chakraborty [1, 2, 3, 4] gave a prominent contribution to construct and represent the basic fuzzy geometrical notions in a mathematical framework. Chakraborty and Ghosh [1] were the first to simplify the extension principle with the introduction of same and inverse points for two fuzzy numbers and fuzzy points. Along with the idea of same and inverse points, Chakraborty and Ghosh redefined the fundamental concepts of fuzzy geometry, e.g., fuzzy distance, line, circle, ellipse and parabola [1, 2, 3, 4].

In [1], for the fuzzy geometry on the $\mathbb{R}^2$ plane, the basic ideas of fuzzy reference frames, fuzzy points, fuzzy distances, fuzzy angles, and linear combinations of fuzzy points are studied. A detailed account of the fuzzy reference frame can be found in [4].

A rigorous formulation of *fuzzy lines* in four different forms—two-point form, point-slope form, slope-intercept form and intercept-form—has been demonstrated in [2]. A fuzzy line has been perceived as a locus of fuzzy points in a particular direction in [2]. Prior to the study of Ghosh and Chakraborty [2], a fuzzy line has been constructed by Buckley and Eslami [5]. The spread of the support of a fuzzy line in [2] cannot suddenly widen as described by Buckley and Eslami [5]. Also, the proposed methodologies and definitions in [2] are less imprecise than existing methods [5]. Obradović et al. [20] proposed a fuzzy linear combination on the linear fuzzy space

$\mathbb{R}^2$. The fuzzy line obtained by Obradović et al. [20] is similar to the fuzzy line segment constructed in [5]. Prior to the work of Buckley and Eslami [5], fuzzy lines were introduced in [21], but their cores can be empty sets. Pham [22] presented a mathematical representation of fuzzy shapes such as fuzzy points and fuzzy lines, etc. Fuzzy lines in [22] is a collection of line segments with varied membership values. Rosenfeld [23] defined geometric properties of a set of lines and a fuzzy set of lines by associating a set of points and fuzzy points with each line. To model imprecise lines: a set of lines in plane, a right way was investigated by Löffler and Kreveld [24]. A fuzzy line obtained by Gupta and Ray [25] is a fuzzy set when its support set is a straight line. The fuzzy lines obtained in Chaudhuri [21] and Gupta and Ray [25] can have an empty core. Zadeh [8] indicated that, in Euclidean geometry, the counterpart of a crisp line $C$ is a fuzzy line. The fuzzy transform of $C$, which is a one-to-many function, is a fuzzy line. Here, $C$ enjoys the role of the prototype of the fuzzy line. It is helpful to visualize a fuzzy line as a fuzzy transform of $C$ drawn by a spray pen [8]. In [26], a normal to a fuzzy line segment is formulated using some spatial transformations. A general form of a fuzzy line and symmetric fuzzy lines are defined in [27]. Recently, Das and Chakraborty conceptualized a fuzzy line as a union of fuzzy points in the geometrical plane [28].

*Fuzzy circles* in two different forms have been formulated in [3]. In the first form, a fuzzy circle is defined for a given center and radius. In the second form, a fuzzy circle passing through three fuzzy points has been described. Prior to work in [3], Buckley and Eslami [6], and its explication [29] used sup-min compositions to construct the fuzzy circles. Buckley and Eslami [6] defined fuzzy circle through the fuzzy algebraic extension of the well-known classical equations of a circle. The defined fuzzy circle in [6] may be violating the customary classical definition of a circle. Some deficiency in the construction of the fuzzy circles in [6] is reported in [3]. In [8], Zadeh mentioned that the fuzzy transform of a crisp circle $C$ in Euclidean geometry is a fuzzy circle, where the fuzzy transformation is a one-to-many function. Here, $C$ plays the role of the prototype of the fuzzy circle. A fuzzy transformation of $C$ drawn by a spray pen can be perceived as a fuzzy circle [8]. The formulations of a fuzzy disk and fuzzy perimeter were studied by Rosenfeld and Haber [30]. The fuzzy disk in [30] is a fuzzy point in [5]. An idea of a fuzzy ring is found in [31]. In [20], Obradović et al. defined a fuzzy circle whose center is a crisp point. The core of the fuzzy circle used in [32, 33] is not a crisp circle. Chaudhuri defined some fuzzy geometric shapes like points, lines, circles, ellipses and polygons in [21]. The level sets of the fuzzy circle defined in [21] are concentric circles. The center or foci of the fuzzy conics used in [34] as prototypes for fuzzy criterion clustering of a finite number of crisp or fuzzy data are crisp points.

In [4], construction of a symmetric fuzzy parabola and two different forms of *Fuzzy parabola* is given (see Chapter 6). The ideas of fuzzy trigonometry and *fuzzy triangle* are demonstrated in [35].

It is easy to observe that the theories of same and inverse points play a fundamental role in developing all the research given in [1, 2, 3, 4, 35, 27, 26]. Before introducing the concepts of *the same and inverse points*, Buckely and Eslami [5, 6] developed a few ideas of fuzzy plane geometry by directly applying the *extension principle* [36]. Further, in the lines of Buckley and Eslami [5, 6], the work on fuzzy plane geometry is extended to the fuzzy space geometry by Qiu and Zhang [7].

The concept of fuzzy half-plane in first introduced by Rosenfeld [37]. In [37], author defined fuzzy rectangle and fuzzy polygon as the intersection of fuzzy half-planes. In [38], Rosenfeld proposed the idea of discrete fuzzy triangle and defined it as an intersection of three discrete fuzzy half-planes. A fuzzy plane is a thin planar shell with variable thickness [22]. The planar shell includes a family of crisp planes which is an extension of the family of crisp lines representing the fuzzy line in [22]. In the sequel, definition of fuzzy half-plane proposed by Ghosh and Chakraborty [9], which can be explicitly written in a mathematical form.

By using real integrals, the theory of height, width, and diameter of fuzzy sets was developed by Rosenfeld [39]. Thereafter, a brief study on the fuzzy topology and geometry of image subsets containing adjacency, separation, connectedness, distance, and relative position was presented by Rosenfeld [40]. In [41], they provided a comprehensive analysis of their work to date on the fuzzy topology and geometry of image subsets, as well as some applications of these techniques in image processing and analysis. Bogomolny made the observation that the definitions introduced in [39, 30] lack inner conformity when reduced to the corresponding customary definitions for crisp sets. Thus, modified definitions of the height, width, and diameter of a fuzzy set were given by Bogomolny [31] using a projection of the fuzzy sets onto two mutually perpendicular directions. The measurements of the defined height, width, perimeter, etc. in [39, 30, 31] are considered as crisp numbers, which is not appreciable. This is due to the fact that if the region is ill-defined, it is difficult to verify the measurement as precise one [42].

The shortest distance between two given fuzzy subsets of a metric space (e.g., Euclidean plane) was established by Rosenfeld [43]. Chaudhuri and Rosenfeld [44] defined a metric distance between two given fuzzy subsets on the support set $S$ in a metric (e.g., Euclidean) space. In [45], Gadjiev and Rustanov provided the aspects of fuzzy geometry based on the notions of length, width, and height, together with methods for determining the distance between fuzzy sets. Two methods for defining the fuzzy distance between fuzzy points and their applications to plausible reasoning were explored by Dubois and Prade [46]. One method is based on the extension principle, while the other generalises the Hausdorff distance. In [47], fuzzy distances are thoroughly reviewed. The mathematical morphology-based fuzzy distances are explored, as well, in [47]. Some suggestions for evaluating the geometric characteristics of fuzzy regions of a fuzzy image, such as connectedness, adjacency and surroundedness, starshapcdness, convexity, adjacency, and so on, were investigated in [48, 49].

In [50], the concept of fuzzy spheres and the relation between the folding of fuzzy graphs and fuzzy spheres are introduced. A sphere can be viewed as a fuzzy sphere that is homeomorphic to a fuzzy graph [50]. A fuzzy three-sphere is constructed as a subspace of fuzzy complex projective space of complex dimension three in [51]. Fuzzy geodesics of a fuzzy hyperboloid can also provide fuzzy hyperboloid and fuzzy spheres [52].

The idea of convex fuzzy sets was investigated by Zadeh [53] and Lowen [54]. Ammar presented some properties of convex fuzzy sets and convex fuzzy cones [55]. In [56], it is shown that a graph of convex fuzzy process is a convex fuzzy cone. In [57], Li and Guo investigated and developed techniques for modelling fuzzy objects based on fuzzy set theory. Fuzzy conics are defined by blurring their boundaries using a smooth unit step function and implicit functions [57]. The core of the fuzzy conics in [57] contains all the points which lie outside the conic instead of the points on the boundary of the desired conic. After that, Esogbue and Liu [34] used fuzzy conics as a prototype in fuzzy criterion clustering of a finite number of crisp or fuzzy data. The center or foci of the fuzzy conics in [34] are crisp points.

The idea of fuzzy spheres [50, 51, 52] and fuzzy cones [53, 54, 55, 56] are not yet extensively studied from the analytic geometrical viewpoint.

### 1.4.2 Literature on fuzzy Hough transform

Classical Hough transform was first invented by Hough to detect geometric features like straight lines in digital images [58, 17], and later modified by Duda and Hart [18]. This transform is a feature extraction technique, especially used in computer vision, image analysis and digital image processing. It uses a voting procedure to find imperfect instances of objects within a certain class of shapes, and the voting procedure is performed in parameter space. In a so-called accumulator space, which is explicitly constructed by the Hough transform algorithm, the object candidates obtained are considered as local maxima in accumulator space. Later, to identify positions of arbitrary shapes, like circles or ellipses, etc., from the binary image, the Hough transform was extended in [19, 59, 60, 61, 62], and further refined in [63, 64, 65, 66]. A state-of-the-art survey describes the variations and the implementation techniques of the classical Hough transform [67, 68, 69, 70]. Real-life applications of the Hough transform can be seen in the works of [71, 72, 73, 74, 75].

For finding shapes, Han et al. [11] developed a fuzzy Hough transform technique by fitting the data points to some given parametric shapes. In [11], every data point close to the perfect shape contributes to the accumulator space. In this way, peaks can be clearly identified even for distorted image shapes since the accumulator values exhibit smooth transitions between higher and lower values. Also, to obtain a faster computation for the same, one can take the convolution of the accumulator space with a Gaussian window [11]. Philip et al. [12] presented a fuzzy Hough transform technique for extracting features based on a similar concept as in [11], where accumulator values were computed from a thick shaded zone around the distorted shape. The fuzzification of the distorted input image shapes was done in [11, 12] before identifying shapes by fuzzy Hough transform. However, the amount of thickness of the shaded zone was not specified. Basak and Pal [13] took care of these problems. Theoretical quantification of shape distortion in fuzzy Hough transform is described in [13]. The voting procedure in the Hough transform is fuzzified in [76] to derive the generalized fuzzy Hough transform, which is used to detect arbitrary shapes in a vague and noisy image. A fuzzy-probabilistic model of the generalized Hough transform based on qualitative labeling of scene features has been presented in [14]. Much successful application of fuzzy Hough transform has been seen in the papers [77, 78, 79, 80, 81]. In [77], fuzzy Hough transform is used in face detection in color images. In [79, 78], Soodamani used fuzzy Hough transform for shape description and a novel fuzzy Hough transform for shape representation. Also, to reduce the uncertainty/precision duality, Strauss [80] used the fuzzy Hough transform technique. The fuzzy Hough Transform has been used for crack detection in [81].

### 1.4.3 Literature on applications of fuzzy geometry

A number of papers reported the applications of fuzzy geometry. Safi et al. [82] used the fuzzy geometrical concepts of Buckley and Eslami to solve fuzzy linear programming problems. Ghosh and Chakraborty [27] performed a study on the general form of fuzzy lines and their use in fuzzy line fitting. A fuzzy regression technique for fuzzy points through the same-points in fuzzy geometry has been developed in [83]. Chakraborty et al. [84] used fuzzy geometry to solve multi-objective fuzzy geometric programming problems. In [85], fuzzy geometry has been used to solve separable fuzzy nonlinear programming problems. Recently, Ghosh and Chakraborty [86, 87, 88] focused on fuzzy optimization problems by using a fuzzy geometrical viewpoint.

Raut and Pal [89] embarked on a comprehensive study of fuzzy intersection graphs by defining fuzzy graphs and fuzzy intersection regions with the help of fuzzy points and fuzzy line segments. Aliev [90] developed a decision-making process based on fuzzy geometry. Pap et al. [91] investigated mathematical models of the imprecise basic plane geometric objects (fuzzy line, fuzzy triangle and fuzzy circle) and their properties. The applications of the fuzzy mathematical models [91] can be visualized in [71, 92]. Li and Guo [57] developed the concept of fuzzy geometric object modeling. In [47], Bloch reported the use of fuzzy distance image processing. Further, fuzzy sets for image processing are demonstrated in [93]. In [94], new definitions concerning the relative positions between two objects in the fuzzy set framework were investigated.

Location discovery based on fuzzy geometry is used in passive sensor networks by Rui et al. [95]. In [96], Ibánez et al. used fuzzy geometrical ideas to define a fuzzy landmark to deal with the skull-face overlay uncertainty in forensic identification. Li et al. [97] gave a novel visual codebook model based on fuzzy geometry for large-scale image classification. Obradović et al. [71] applied the concepts of linear fuzzy space in the road lane model and detection. In [98], the idea of fuzzy points in [5] has been revisited to formulate the fuzzy measure for a class of two-dimensional fuzzy numbers. Mayburav [99] developed commutative fuzzy geometry and geometric quantum mechanics with the help of the fuzzy geometry in [5]. Lee et al. [100] gave an application of half-circle fuzzy numbers and triangular fuzzy numbers in fuzzy control. In [101], Wang et al. studied fuzzy geometric localization for triangular grid deployment in passive sensor networks. Asasian [102] studied the analysis and critique of artworks by providing definitions of meaningful geometrical forms (point and line) in visual arts with the help of fuzzy geometry.

The concepts of fuzzy circles (mentioned in Subsection 1.4.1) are applied in fuzzy retraction and folding of fuzzy-orientable compact manifold [103], validity measure for fuzzy criterion clustering [34], solving and learning a tractable class of soft temporal constraints [104], fuzzy voronoi diagram [105], uncertain voronoi diagram [106], fuzziness vs. probability [107], and the probability monopoly [108]. In [109], a fuzzy circle is defined as a collection of several concentric circles with a continuous radius and applied in iris localization. The fuzzy circle defined in [30] has been used to measure the geometric properties of fuzzy subsets of the plane in [41, 39].

In the literature, a detailed study has been found to construct the similarity measure between fuzzy numbers [110, 111, 112, 113, 114]. It is found that the similarity measure between fuzzy numbers in [110, 111, 112, 113, 114] is a crisp value. Thereafter the concept of similarity measure has been modified in [42].

A successful application of fuzzy geometry on fuzzy interpolation can be seen in [115, 116]. In [115], Das et al. has established a mathematical demonstration of the pattern of the existing fuzzy rule base using fuzzy geometry. A fuzzy geometry-based inverse fuzzy rule base interpolation has been suggested in [116]. The fuzzy geometrical tools are successfully applied in feature extraction techniques [11, 12, 13].

## 1.5 Objective of the Thesis

The objectives of the thesis are:

- To define three variable reference function, $S$-type space fuzzy points, same and inverse points with respect to two continuous $S$-type space fuzzy points,
- to construct fuzzy space geometrical elements, like

- fuzzy distance between two continuous $S$-type space fuzzy points,
- space fuzzy line segments,
- space fuzzy lines and skew fuzzy lines,
- shortest distance between two skew fuzzy lines,
- fuzzy planes,
- fuzzy spheres,
- fuzzy cones and fuzzy conics,

- to generalize Hough transform for detecting fuzzy lines and fuzzy circles in the image space.

## 1.6 Outline of the Thesis

The thesis consists of six chapters, including an introductory chapter and a chapter containing a conclusion and future scopes. In the introductory chapter, a concise but adequate literature review has been provided on these topics. It also defines the objective of this thesis. The outline of the thesis is as follows:

In **Chapter 2**, basic tools of fuzzy space geometry such that a three-variable reference function, $S$-type space fuzzy point, fuzzy number along a direction, the addition operation of two fuzzy points, a general expression of same and inverse points by a reference function, scalar multiplication of space fuzzy point, and a linear combination of two space fuzzy points are investigated. The fuzzy space geometrical object like fuzzy distance and space fuzzy line segment are also delineated in this chapter. The proposed results are compared with existing ones.

In **Chapter 3**, space fuzzy lines and a symmetric fuzzy line are described. The formulation of skew fuzzy lines and the shortest distance between them are also provided. Sequentially, we propose three forms of fuzzy planes— a three-point form, an intercept form, and a fuzzy plane passing through an $S$-type space fuzzy point and perpendicular to a given crisp direction. This chapter presents all the forms of fuzzy planes supported by step-wise algorithms. We discuss and compare our results to those produced by existing methods.

**Chapter 4** deals with the formulations of all the forms of fuzzy spheres and their intersection by a crisp plane. The formulation of a fuzzy cone and its intersection by a crisp plane is also explained. This chapter includes discussing and comparing the proposed fuzzy sphere and fuzzy cone with existing ideas.

**Chapter 5** demonstrates the generalization of Hough transform technique for detecting fuzzy lines and fuzzy circles in image space. A brief study on the generalized version of the fuzzy Hough transform is also described. Sequentially, an idea of similarity measure between two fuzzy shapes is investigated. Effective implementation of the proposed method to detect fuzzy lines and circles in authentic images is shown in this chapter.

Finally, **Chapter 6** summarizes the main conclusions and gives the potential directions for future research.

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