A STUDY ON ANALYTICAL FUZZY SPACE GEOMETRY AND SOME APPLICATIONS

The thesis submitted in partial fulfillment

for the Award of Degree

DOCTOR OF PHILOSOPHY

by

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To

My Beloved Parents

Mrs. Pramila Devi

&

Mr. Shyam Kishor

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Diksha Gupta

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Symbols

$\widetilde{A}, \widetilde{B}, \ldots, \widetilde{a}, \widetilde{b}, \ldots$	fuzzy sets
$\widetilde{A}(\alpha), \widetilde{P}(\alpha), \ldots$	α-cut of fuzzy sets
$\widetilde{P}(a, b, c)$	fuzzy point at (a, b, c)
$\mu(x \mid \widetilde{A})$	membership function of $\widetilde{A}$
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of nonnegative real numbers

PREFACE

Fuzzy space geometry is a study of imprecise locations, imprecise objects or imprecise shapes, and their topological properties. In this study, the fuzzy objects have been visualized geometrically, and the construction procedure has been suggested to visualize them in the Euclidean space $\mathbb{R}^3$. One of the applications of fuzzy objects can be seen in fuzzy image processing, such as fuzzy object detection. Fuzzy object detection is another approach to object detection that considers the image to be fuzzy. In most of the images, where the objects are not clearly defined, that is, vague/imprecise, etc., in that case, object detection becomes very difficult. As the objects are vague/imprecise, fuzzy Hough transform is useful in dealing with this type of problem. The basic idea of this technique is to find fuzzy curves like fuzzy lines, fuzzy circles, etc., in suitable parameter space.

This thesis is intended to explore fuzzy space geometry and its applications in fuzzy image processing. Sequentially three chapters (Chapter 2, 3, and 4) cover theoretical aspects of fuzzy space geometry in $\mathbb{R}^3$-space. Chapter 5 includes the application of fuzzy geometry in which we investigate a technique, namely fuzzy Hough transform, to detect fuzzy lines and fuzzy circles.

In this thesis, the author discusses fuzzy space geometrical objects (fuzzy distance, fuzzy lines, fuzzy planes, fuzzy spheres, fuzzy cones, etc.) in $\mathbb{R}^3$ and a fuzzy plane geometrical object detection technique. This thesis contains six chapters. The thesis is structured as follows.

Chapter 1 is an introduction that describes fuzzy space geometry and the literature survey of fuzzy geometry and its applications. This chapter provides notations and basic definitions of fuzzy set theory. Some basic definitions and concepts of fuzzy plane geometry are also included in this chapter. These concepts play an important role in the investigation of fuzzy space geometry and its applications.

Chapter 2 introduces basic tools such as S-type space fuzzy point, the concept of same and inverse points with respect to S-type space fuzzy point, and fuzzy geometrical elements (fuzzy distance and space fuzzy line segment). A three-variable reference function, representation of space fuzzy point by a reference function, fuzzy number along a direction, the addition operation of two space fuzzy points, a general expression of same and inverse points by a reference function, scalar multiplication of space fuzzy point, and a linear combination of two space fuzzy points are proposed in this chapter. Employing the concept of same and inverse points, we define the fuzzy distance between two space fuzzy points and space fuzzy line segments. All the provided ideas are supported with numerical examples and necessary pictorial illustrations.

Chapter 3 explores the construct of space fuzzy lines and three different forms of fuzzy planes in $\mathbb{R}^3$— a three-point form, an intercept form, and a fuzzy plane passing through an S-type space fuzzy point and perpendicular to a given crisp direction. We define a space fuzzy line as a bi-infinite extension of a space fuzzy line segment. Particularly, we also formulate symmetric fuzzy lines. Importantly, the concept of skew fuzzy lines and the shortest distance between two skew fuzzy lines are discussed in $\mathbb{R}^3$. We introduce the angle between two fuzzy planes and the fuzzy distance between a fuzzy point and a fuzzy plane. Geometric properties of the proposed space fuzzy lines and all the proposed forms of fuzzy planes are also explored. Numerical examples support all formulations and studies.

In chapter 4, the constructions of all the different forms of fuzzy spheres and their intersection by a crisp plane are explained. The formulation of a fuzzy cone and its intersection by a crisp plane is also delineated in this chapter. Three different methodologies to formulate fuzzy spheres depend on the information available for the fuzzy sphere, such as a fuzzy point and fuzzy distance or diameter of the fuzzy sphere or four fuzzy points. We establish the notions of translation and rotation of a fuzzy point. With the help of these notions, we construct the diameter form of a fuzzy sphere and a fuzzy cone. This chapter incorporates the concept of a great fuzzy circle and its rotation. We show that the rotation of a great fuzzy circle about its diameter is a fuzzy sphere. In this sequel, the notions of the fuzzy cone, convex fuzzy cone, and its intersection by a crisp plane are initiated here. An idea of degenerated and non-degenerated fuzzy conics is explored.

Chapter 5 includes the study of the application of fuzzy plane geometrical elements like fuzzy lines and fuzzy circles. We introduce a technique, say fuzzy Hough transform (FHT), for detecting fuzzy lines and fuzzy circles in the image space. This technique aims to find imprecise objects within a certain class of imprecise shapes by a voting procedure. A brief study on the generalized version of the fuzzy Hough transform is also described. Sequentially, a concept of similarity measure between two fuzzy shapes is delineated. Moreover, we implement the proposed technique in authentic images to detect fuzzy lines and fuzzy circles.

Proper care has been taken so that every concept coincides with the conventional definitions in classical geometry with zero uncertainty. It is also shown that the proposed concepts of same and inverse points are the basis for this study and facilitate the computations of membership functions of all the proposed fuzzy geometrical entities in $\mathbb{R}^3$. The ideas on fuzzy geometry are applied to the detection of fuzzy geometrical elements in fuzzy image processing.