Chapter 5

Nonlinear Polytopic Systems with Predefined-Time Convergence

5.1 Introduction

The problem of designing control for uncertain systems has been a topic of considerable interest in the control community. When uncertainty is present in the control affine model, one can represent the system as a polytopic form that has a versatile modeling structure. The polytopic model is an effective way to characterize the plant uncertainty due to its simpler design. In this approach, parametric uncertain systems are described by the parameters of a set of models that have a convex structure [80–84]. This chapter focuses on nonlinear polytopic systems with predefined-time convergence, where the convergence time is invariant with respect to the initial conditions of the system and can be chosen by the designer in advance.

The motivation for the work here is drawn from the limitation of fixed-time stability [10], where the settling time function depends on system parameters. This dependence requires adjusting the system parameters to achieve different convergence time. In many modern applications, the desired convergence time needs to be chosen at the outset. Therefore, the importance of settling time functions with minimal dependence on system parameters is increasing. A part of the work is also motivated by the design of controllers with arbitrary convergence time, where the settling time function is independent on the initial values and system parameters and can be chosen as per our own choice, as noted by [37].

The main contributions are:

1. Predefined-time convergence is enabled for a class of nonlinear polytopic systems.
2. The control Lyapunov function is employed, which yields less conservative results.
3. The proposed predefined-time controller demonstrates robustness to uncertain system parameters.

The rest of this chapter is organized as follows. The problem formulation of nonlinear polytopic systems is described in section 5.2. Section 5.3 presents the predefined time control of the nonlinear polytopic systems, followed by their respective proof. Section 5.4 demonstrates the application of the proposed method in the practical example with the simulation results. Finally, section 5.4 summarizes the chapter.

5.2 Problem Formulation

Consider a nonlinear system given by

$$\dot{z} = f(z);\ z(0) = z_0 \tag{5.1}$$

where, $z \in \mathbb{R}^n$ denotes the state vector and $f : \mathbb{R}^n \to \mathbb{R}^n$ represents a continuous function and the origin is a fixed point.

Definition 8 *(Predefined-Time Stable)*
The origin of the system (5.1) *is said to be predefined-time stable, if it is fixed time stable and any solution* $z(t, t_0, z_0)$ *of the system* (5.1) *converges to the origin within some predefined time and the settling time function is independent of the initial conditions of the system.*

Consider the following nonlinear polytopic system

$$\dot{z} = \sum_{p=1}^{K} \alpha_p f_p(z) + \sum_{q=1}^{L} \beta_q g_q(z) u \tag{5.2}$$

where, $z \in \mathbb{R}^n$ denotes the state vector, $u \in \mathbb{R}$ represents the control input vector, $f_p : \mathbb{R}^n \to \mathbb{R}^n$, $p = 1, 2, \cdots, K$ and $g_q : \mathbb{R}^n \to \mathbb{R}^n$, $q = 1, 2, \cdots, L$ are sufficiently smooth nonlinear functions, $\alpha = [\alpha_1, \cdots, \alpha_K]^T$ and $\beta = [\beta_1, \cdots, \beta_L]^T$ are uncertain parameters. Let's assume that the system given in (5.2) is controllable.

Assumption 6 *The uncertain parameters αs and βs satisfy: $\alpha_1 + \cdots + \alpha_K = \beta_1 + \cdots + \beta_L = 1$, $\alpha_p \geq 0$, $\beta_q \geq 0$; $p = 1, 2, \cdots, K$, $q = 1, 2, \cdots, L$ and can be completely independent.*

An unforced part and control part of the nonlinear polytopic system (5.2) are given by $\sum\limits_{p=1}^{K} \alpha_p f_p(z)$ and $\sum\limits_{q=1}^{L} \beta_q g_q(z)$, respectively. Suppose that the nonlinear polytopic system (5.2) is predefined-time stabilizable for all possible α and β. Furthermore, consider the function $f_p(0) = 0 \ \forall p \in \{1, \cdots, K\}$.

Next, the objective is to design a continuous state feedback control function $u = v(z) : \mathbb{R}^n \to \mathbb{R}$, such that the closed-loop system

$$\dot{z} = \sum_{p=1}^{K} \alpha_p f_p(z) + \sum_{q=1}^{L} \beta_q g_q(z) v(z) \tag{5.3}$$

is predefined time stabilizable for all possible α and β.

5.3 Predefined-Time Control

In this section, firstly, stability criteria of the unforced nonlinear polytopic system are discussed. For the existence of a continuous and predefined-time stable state feedback controller, a sufficient condition is defined. After that, it is established that the achieved sufficient condition is also necessary, such that the closed-loop nonlinear polytopic system has the RCLF for all possible uncertainties.

Consider the following nonlinear unforced polytopic system

$$\dot{z} = \sum_{p=1}^{K} \alpha_p f_p(z) \tag{5.4}$$

where, $z \in \mathbb{R}^n$ is the system state, $f_p : \mathbb{R}^n \to \mathbb{R}^n$, $p = 1, 2, \cdots, K$ is sufficiently smooth nonlinear function.

Lemma 6 *Consider the nonlinear unforced system* (5.4). *If there exists a positive-definite, smooth, and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that*

$$\frac{\partial V}{\partial z} f_p(z) \leq -\gamma \frac{(1 - e^{-V})}{(t_s - t)}, \ \forall z \neq 0, \ t_0 \leq t < t_s, \ \gamma > 1 \tag{5.5}$$

then, the system (5.4) is predefined-time stable for all possible α_p; $p = 1, 2, \cdots, K$, satisfying $\sum_{p=1}^{K} \alpha_p = 1$.

Proof: The time derivative of Lyapunov function along the system trajectory is given by

$$\dot{V} = \frac{\partial V}{\partial z}\dot{z} = \frac{\partial V}{\partial z}\sum_{p=1}^{K}\alpha_p f_p(z) \leq -\gamma\frac{(1-e^{-V})}{(t_s-t)}, \ \gamma > 1 \tag{5.6}$$

where, t_s is the predefined time within which the system trajectory converges to the origin.

From the aforementioned discussions, it is ensured that (5.6) shows predefined-time convergent dynamics. Thus, using (5.6) one obtains $V = 0 \ \forall \ t \geq t_s$, and since V is a Lyapunov function in z therefore it implies that z is zero for every $t \geq t_s$. Therefore, the nonlinear unforced polytopic system (5.4) leads to predefined-time stability. The proof is completed. ∎

Remark 9 *The above proof shows that the convex combination of predefined-time stable dynamics is also predefined-time stable.*

At this point, it is interesting to investigate whether for a predefined-time stable system there exist convex combinations such that the elements of each combination are predefined time stable. For predefined time stable system we have $\dot{V} \leq \frac{-\gamma(1-e^{-V})}{(t_s-t)}$, which leads to

$$\frac{\partial V}{\partial z}(\alpha_1 f_1(z) + \cdots + \alpha_K f_K(z)) \leq \frac{-\gamma(1-e^{-V})}{(t_s-t)}(\alpha_1 + \cdots + \alpha_K). \tag{5.7}$$

From (5.7), it follows that a straightforward converse results for lemma 6 cannot be drawn. However, it is possible to outline a generalization in connection to lemma 6. To that end, the following lemma is given:

Lemma 7 *Let there exists a positive-definite, smooth and radially unbounded function* $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *for the nonlinear unforced system* (5.4). *Assume that S is an index set of $r(< K)$ elements of a particular convex combination satisfying* $\frac{\partial V}{\partial z} f_i \leq -\phi_i, i \in S$, *for some* $\phi_i \geq 0$. *Let the remaining $K-r$ elements satisfy* $\frac{\partial V}{\partial z} f_j \leq \frac{-\gamma(1-e^{-V})}{(t_s-t)}$, $j \in T$,*where T is the index set of those $K-r$ elements. Suppose* $\phi_i > \frac{\gamma(1-e^{-V})}{(t_s-t)}$ *for all $i \in S$, then system* (5.4) *is predefined-time stable.*

Proof: Let us consider an arbitrary r less than K. Without loss of generality, we assume the index set $S = \{1, 2, \cdots, r\}$, over which $\frac{\partial V}{\partial z} f_i \leq -\phi_i$, is satisfied for some $\phi_i \geq 0$. Then we can write

$$\dot{V} \leq -\alpha_1\phi_1 - \alpha_2\phi_2 - \cdots - \alpha_r\phi_r - \frac{\gamma(1-e^{-V})}{(t_s-t)}(\alpha_{K-r} + \alpha_{K-r+1} + \alpha_{K-r+2}\cdots + \alpha_K)$$
$$= -\alpha_1\left(\phi_1 - \frac{\gamma(1-e^{-V})}{(t_s-t)}\right) - \alpha_2\left(\phi_2 - \frac{\gamma(1-e^{-V})}{(t_s-t)}\right) - \cdots$$
$$- \alpha_r\left(\phi_r - \frac{\gamma(1-e^{-V})}{(t_s-t)}\right) - \frac{\gamma(1-e^{-V})}{(t_s-t)}(\alpha_1 + \alpha_2 + \cdots + \alpha_K).$$

Now if $\phi_i > \frac{\gamma(1-e^{-V})}{(t_s-t)}$, $i \in S$, then $\dot{V} \leq \frac{-\gamma(1-e^{-V})}{(t_s-t)}$, which implies predefined-time stability. ■

Further, having some knowledge about the gains of the polytopic system (5.4) allows us to make the following conclusions:

Lemma 8 *The required condition $\phi_i > \frac{\gamma(1-e^{-V})}{(t_s-t)}$ for predefined-time stability in lemma 7 can be relaxed to any $\phi_i \geq 0$, if $\sum_{j\in T}\gamma\alpha_j > 1$.*

A positive-definite, smooth, and radially unbounded function $V(z)$ that satisfies (5.5) is called as robust Lyapunov function (RLF) for the unforced nonlinear polytopic system (5.4).

In the above discussions, if a continuous control $v(z)$ is established such that there exists a Lyapunov function $V(z)$, which holds the following small control property [89], for all possible α and β,

$$\frac{\partial V(z)}{\partial z}\left(\sum_{p=1}^{K}\alpha_p f_p(z) + \sum_{q=1}^{L}\beta_q g_q(z)v(z)\right) \leq -\gamma\frac{(1-e^{-V})}{(t_s-t)},\ \forall z \neq 0,\ t_0 \leq t < t_s \tag{5.8}$$

then, $v(z)$ is predefined time stable controller for the nonlinear polytopic system (5.2). It is noticed that if (5.8) is satisfied, then

$$\frac{\partial V(z)}{\partial z}\sum_{q=1}^{L}\beta_q g_q(z) = 0 \Rightarrow \frac{\partial V(z)}{\partial z}\sum_{p=1}^{K}\alpha_p f_p(z) \leq -\gamma\frac{(1-e^{-V})}{(t_s-t)}. \tag{5.9}$$

In the given RCLF $V(z)$ of the nonlinear polytopic system (5.2), let

$$\Phi_p(z) = \frac{\partial V(z)}{\partial z}f_p(z) \text{ and } \Pi_q(z) = \frac{\partial V(z)}{\partial z}g_q(z).$$

Next, define

$$S_p(z) \equiv \{q \in \{1, 2, \cdots, L\} | \Pi_q(z) > 0\}$$
$$S_n(z) \equiv \{q \in \{1, 2, \cdots, L\} | \Pi_q(z) < 0\}$$
$$S_z(z) \equiv \{q \in \{1, 2, \cdots, L\} | \Pi_q(z) = 0\}.$$

Moreover, consider

$$D_z \equiv \{z \in \mathbb{R}^n | S_z(z) \neq \emptyset\}$$
$$D_p \equiv \{z \in \mathbb{R}^n | S_z(z) = \emptyset, S_p(z) \neq \emptyset, S_n(z) = \emptyset\}$$
$$D_n \equiv \{z \in \mathbb{R}^n | S_z(z) = \emptyset, S_p(z) = \emptyset, S_n(z) \neq \emptyset\}$$
$$D_m \equiv \{z \in \mathbb{R}^n | S_z(z) = \emptyset, S_p(z) \neq \emptyset, S_n(z) \neq \emptyset\}$$

where $\emptyset$ is an empty set. One can obtain $S_p(z) \cup S_n(z) \cup S_z(z) = \{1, 2, \cdots, L\}$ and $D_z \cup D_p \cup D_n \cup D_m \subset \mathbb{R}^n$.

The following assumptions are considered for proofing the theorems.

Assumption 7 *There exists a Lyapunov function $V(z)$, such that for all $z \in D_z(z) \backslash \{0\}$ and for all $z \in D_m$, $\Phi_p(z) \leq -\gamma \frac{(1-e^{-V})}{(t_s-t)}$, $\gamma > 1$, $t_0 \leq t < t_s$, $\forall p \in \{1, 2, \cdots, K\}$.*

Assumption 8 *RCLF $V(z)$ of the nonlinear polytopic system (5.2) satisfies the small control property:*

$\forall \epsilon > 0$, $\exists\ \delta > 0$ such that if $z \neq 0$ satisfies $||z|| < \delta$, then there exists some $u = v(z)$ with $||u|| < \epsilon$ such that $\forall p \in \{1, 2, \cdots, K\}$ and $q \in \{1, 2, \cdots, L\}$,

$$\frac{\partial V(z)}{\partial z} f_p(z) + \frac{\partial V(z)}{\partial z} g_q(z) v(z) \leq -\gamma \frac{(1 - e^{-V})}{(t_s - t)},\ \gamma > 1,\ t_0 \leq t < t_s.$$

Lemma 9 *[128] If a state feedback control function $v(z)$ is continuous at the origin and there exists a radially unbounded, positive definite and smooth Lyapunov function $V(z)$, for all possible α and β, i.e.*

$$\frac{\partial V(z)}{\partial z} \left(\sum_{p=1}^{K} \alpha_p f_p(z) + \sum_{q=1}^{L} \beta_q g_q(z) v(z) \right) \leq -\gamma \frac{(1 - e^{-V})}{(t_s - t)},\ \forall z \neq 0$$

then $v(z)$ is a finite-time stabilizing controller for the system (5.2).

Necessary and Sufficient Condition

Theorem 3 *For a given nonlinear polytopic system (5.2), which is predefined-time stabilizable for all possible α and β, if the assumptions 7 and 8 holds, then there exists a continuous control $u = v(z)$ and a positive-definite, smooth, and radially unbounded RCLF $V(z)$ that satisfies*

$$\sqrt{\Phi_p^2(z) + \Pi_q^4(z)\rho_1^2(z)} \geq \gamma \frac{(1-e^{-V})}{(t_s - t)}, \forall z \neq 0,\; z \in D_p \subset \mathbb{R}^n \tag{5.10}$$

$$\sqrt{\Phi_p^2(z) + \Pi_q^4(z)\rho_2^2(z)} \geq \gamma \frac{(1-e^{-V})}{(t_s - t)}, \forall z \neq 0,\; z \in D_n \subset \mathbb{R}^n \tag{5.11}$$

where, $\gamma > 1$ and $t_0 \leq t < t_s$. Moreover, the control is structured as

$$u = v(z) = \begin{cases} v_z(z), \text{ if } z \in D_z \subset \mathbb{R}^n \\ v_p(z), \text{ if } z \in D_p \subset \mathbb{R}^n \\ v_n(z), \text{ if } z \in D_n \subset \mathbb{R}^n \\ v_m(z), \text{ if } z \in D_m \subset \mathbb{R}^n \end{cases} \tag{5.12}$$

with

$$\begin{aligned} v_z(z) =& v_m(z) = 0 \\ v_p(z) =& -\frac{\max_p\{\Phi_p(z)\} + \sqrt{\max_p\{\Phi_p(z)\}^2 + \min_q\{\Pi_q(z)\}^4\left(1+\rho_1^2(z)\right)}}{\min_q\{\Pi_q(z)\}} \\ v_n(z) =& -\frac{\max_p\{\Phi_p(z)\} + \sqrt{\max_p\{\Phi_p(z)\}^2 + \max_q\{\Pi_q(z)\}^4\left(1+\rho_2^2(z)\right)}}{\max_q\{\Pi_q(z)\}} \end{aligned}$$

where

$$\rho_1^2(z) = \max\left\{0, \frac{\Theta^2(V(z)) - \overline{\Phi}^2(z)}{\underline{\Pi}^4(z)}\right\}, \rho_2^2(z) = \max\left\{0, \frac{\Theta^2(V(z)) - \overline{\Phi}^2(z)}{\overline{\Pi}^4(z)}\right\}$$

$$\begin{aligned} \Theta(z) :=& \gamma \frac{(1-e^{-z})}{(t_s - t)},\; \gamma > 1,\; \overline{\Phi}(z) := \max\{\Phi_1(z),\; \Phi_2(z)\}, \\ \underline{\Pi}(z) :=& \min\{\Pi_1(z)\sqrt[4]{1+\rho_1^2(z)},\; \Pi_2(z)\sqrt[4]{1+\rho_1^2(z)}\}, \\ \overline{\Pi}(z) :=& \max\{\Pi_1(z)\sqrt[4]{1+\rho_2^2(z)},\; \Pi_2(z)\sqrt[4]{1+\rho_2^2(z)}\}. \end{aligned}$$

Proof: Firstly, the proof of continuity of state feedback control law $v(z)$ is presented. Then, the predefined-time stabilization of closed-loop nonlinear polytopic system (5.3) is proven by considering that,

$$\dot{V}(z) = \sum_{p=1}^{K} \alpha_p \Phi_p(z) + \sum_{q=1}^{L} \beta_q \Pi_q(z) v(z) \leq -\gamma \frac{(1-e^{-V})}{(t_s - t)} \tag{5.13}$$

for each possible α and β, where $\gamma > 1$ and $t_0 \leq t < t_s$.

A. Continuous Control

The $\Phi_p(z)$ and $\Pi_q(z)$ are smooth functions, on the respective domain of the definition of $v(z)$. Employing the small control property, it can be shown that $v(z)$ is continuous in the interior regions of D_z, D_p, D_n, and D_m. The only feasible point of discontinuity of $v(z)$ occurs on the boundary between the following sets D_z and D_p or the sets D_z and D_n or the sets D_z and D_m.

Consider $\{z^r\} \in D_p$ is a series of vectors. It converges to a point $\tilde{z} \in D_z$ on the boundary of D_z and D_p, $\min_q\{\Pi_q(\tilde{z})\} = 0$. If $\tilde{z} \neq 0$, $\lim\limits_{r\to\infty} \Phi_p(z^r) \leq -\gamma\frac{(1-e^{-V})}{(t_s-t)}, \forall p = 1, 2, \cdots, K$ from the assumption 7. Moreover,

$$\lim_{r\to\infty} v(z^r) = \lim_{r\to\infty} v_p(z^r) = \lim_{r\to\infty}\left(-\frac{\max_p\{\Phi_p(z^r)\} + \Xi(z^r)}{\min_q\{\Pi_q(z^r)\}}\right) = 0 = v_z(\tilde{z}) \tag{5.14}$$

where $\Xi(z^r) := \sqrt{\max_p\{\Phi_p(z^r)\}^2 + \min_q\{\Pi_q(z^r)\}^4\,(1+\rho_1^2(z^r))}$.
If $\tilde{z} = 0$, then $\lim\limits_{r\to\infty} \Phi_p\{z^r\} = 0, \forall p = 1, 2, \cdots, K$. Thus, (5.14) also holds the small control property. This indicates that on the boundary between the sets D_z and D_p the controller $v(z)$ is continuous.

Similarly, for the boundary between the sets D_z and D_n, the controller $v(z)$ is continuous. Given that the controller $v(z) = 0$ in both the sets D_z and D_m, it is also continuous on the boundary of the sets D_z and D_m.

B. Predefined-Time Stability

i) For $z \in D_z\backslash\{0\}$

From the assumption 7, if $z \in D_z\backslash\{0\}$, then $\Phi_p(z) \leq -\gamma\frac{(1-e^{-V})}{(t_s-t)}, \forall p = 1, 2, \cdots, K$. For this region, $u = v(z) = v_z(z) = 0$, such that for all possible α and β,

$$\dot{V}(z) = \sum_{p=1}^{K}\alpha_p\Phi_p(z) + \sum_{q=1}^{L}\beta_q\Pi_q(z)v(z) = \sum_{p=1}^{K}\alpha_p\Phi_p(z) \leq -\gamma\frac{(1-e^{-V})}{(t_s-t)} \tag{5.15}$$

thus, (5.13) is guaranteed in this region (see the *Lemma 6*).

ii) $z \in D_p$

For this region, $u = v(z) \equiv v_p(z) < 0$. So, for all possible α and β,

$$\begin{aligned}
\dot{V}(z) &= \sum_{p=1}^{K}\alpha_p\Phi_p(z) + \sum_{q=1}^{L}\beta_q\Pi_q(z)v_p(z) \leq \sum_{p=1}^{K}\alpha_p\max_i\{\Phi_i(z)\} + \sum_{q=1}^{L}\beta_q\min_j\{\Pi_j(z)\}v_p(z)\\
&= \max_i\{\Phi_i(z)\} + \sum_{q=1}^{L}\beta_q\left\{-\max_i\{\Phi_i(z)\} - \sqrt{\max_i\{\Phi_i(z)\}^2 + \min_j\{\Pi_j(z)\}^4\left(1+\rho_1^2(z)\right)}\right\}\\
&= -\sum_{q=1}^{L}\beta_q\sqrt{\max_i\{\Phi_i(z)\}^2 + \min_j\{\Pi_j(z)\}^4\left(1+\rho_1^2(z)\right)}\\
&= -\sqrt{\max_i\{\Phi_i(z)\}^2 + \min_j\{\Pi_j(z)\}^4\left(1+\rho_1^2(z)\right)} \leq -\gamma\frac{(1-e^{-V})}{(t_s-t)},\ \gamma > 1,\ t_0 \leq t < t_s.
\end{aligned} \tag{5.16}$$

Thus, one can observe that (5.13) is satisfied in this case.

$iii)$ Let $z \in D_n$

From the case (ii), it is easy to show that (5.13) holds in this region with the particular choice of $u = v(z) \equiv v_n(z)$.

$iv)$ Let $z \in D_m$

From the case (i), it follows that (5.13) is satisfied in this region with the particular choice of $u = v(z) \equiv v_m(z) = 0$.

According to the aforementioned discussions, if $v(z)$ and $V(z)$ satisfies continuity and small control property, respectively, then the origin of the system (5.2) is predefined-time stabilizable for all possible α and β. Here, the proof is completed. ■

Remark 10 *The system states $z \in D_z, D_p, D_n, D_m \subset \mathbb{R}^n$ where D_z, D_p, D_n and D_m are all compact sets which can be chosen by the designer based on the concerned system.*

Remark 11 *There exists a trade-off between the control input and settling time, i.e., large control input is required for obtaining small settling time.*

Remark 12 *In the aforementioned Theorem, a sufficient condition is presented for the existence of continuous stable control of the nonlinear polytopic system* (5.2)*. Moreover, for designing stable controllers, a universal formula is also provided.*

Theorem 4 *For the given nonlinear polytopic system* (5.2)*, if there exists a continuous control $u = v(z)$ then the closed-loop nonlinear polytopic system* (5.3) *is predefined time stable and has the RCLF $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and satisfies assumptions 7 and 8 for all possible α and β.*

Proof: The RCLF $V(z)$ of the closed-loop nonlinear polytopic system (5.3) for all possible α and β yields

$$\sum_{p=1}^{K} \alpha_p \Phi_p(z) + \sum_{q=1}^{L} \beta_q \Pi_q(z) v(z) \leq -\gamma \frac{(1 - e^{-V})}{(t_s - t)}, \ \forall z \neq 0.$$

For the regions $D_z \backslash \{0\}$ and D_m, it is always possible to find some parameters $\hat{\beta}_q$, $q = 1, 2, \cdots, L$ such that $\sum_{q=1}^{L} \hat{\beta}_q \Pi_q(z) = 0$. Thus, in such regions, for all possible α_p, $p = 1, 2, \cdots, K$, following is necessary,

$$\sum_{p=1}^{K} \alpha_p \Phi_p(z) + \sum_{q=1}^{L} \beta_q \Pi_q(z) v(z) = \sum_{p=1}^{K} \alpha_p \Phi_p(z) \leq -\gamma \frac{(1 - e^{-V})}{(t_s - t)}.$$

For each $z \in D_z \backslash \{0\}$ and for each $z \in D_m$, $\Phi_p(z) \leq -\gamma \frac{(1-e^{-V})}{(t_s - t)}$ $\forall\ p = 1, 2, \cdots, K$ are necessary.

From the continuity of $v(z)$, the small control property holds the assumption 7 in $V(z)$. Therefore, this completes the proof. ■

5.4 Simulation results

To illustrate the efficacy of the proposed control design. Let's take, a continuous stirred tank reactor [93]. In the reactor, an isothermal parallel/series Van de Vusse reaction is considered.

The continuous stirred reactor dynamics is given follows as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -r_1 z_1 - r_3 z_1^2 \\ r_1 z_1 - r_2 z_2 \end{bmatrix} + \begin{bmatrix} c_{A_0} - z_1 \\ -z_2 \end{bmatrix} u \tag{5.17}$$

where, $c_{A_0} = 1.2$ gmol.l^{-1} represents the concentration of species A in feed stream. Reaction rate constants are $r_1 = 40$ h^{-1}, $r_2 = 80$ h^{-1} and $r_3 = 100$ (gmol.h)$^{-1}$. From the system (5.17), one can easily write

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{z_1}} r_1 z_1^{\frac{3}{2}} - \frac{1}{\sqrt{z_1}} r_3 z_1^{\frac{5}{2}} \\ \frac{1}{\sqrt{z_1}} r_1 z_1^{\frac{3}{2}} - r_2 z_2 \end{bmatrix} + \begin{bmatrix} \frac{c_{A_0}}{\sqrt{z_1}} \left(\sqrt{z_1} - \frac{z_1^{\frac{3}{2}}}{c_{A_0}} \right) \\ -z_2 \end{bmatrix} u. \tag{5.18}$$

Suppose that the uncertainty is present in the state z_1 which is unknown but bounded i.e. $z_1 \in [z_{1\min}, z_{1\max}] = [1.5, 70]$.

Next, we can write the system (5.18) into the form as mentioned in the nonlinear polytopic system (5.2)

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} =& \alpha_1 \begin{bmatrix} -45.6 z_1^{\frac{3}{2}} - 114 z_1^{\frac{5}{2}} \\ 45.6 z_1^{\frac{3}{2}} - 113.6 z_2 \end{bmatrix} + \alpha_2 \begin{bmatrix} -13.2 z_1^{\frac{3}{2}} - 33 z_1^{\frac{5}{2}} \\ 13.2 z_1^{\frac{3}{2}} - 266.4 z_2 \end{bmatrix} \\ &+ \beta_1 \begin{bmatrix} 4.9\sqrt{z_1} - 4.08 z_1^{\frac{3}{2}} \\ -5 z_2 \end{bmatrix} u + \beta_2 \begin{bmatrix} 0.18\sqrt{z_1} - 0.15 z_1^{\frac{3}{2}} \\ -1.25 z_2 \end{bmatrix} u. \end{aligned} \tag{5.19}$$

Now, consider the RCLF $V(z) = \frac{1}{2}z_1^2 + z_2^2$. Using RCLF, it can be easily calculated

$$\begin{aligned}
\Phi_1(z) =& 45.6z_1^{\frac{3}{2}}\left(2z_2 - z_1\right) - 114z_1^{\frac{7}{2}} - 227.2z_2^2 \\
\Phi_2(z) =& 13.2z_1^{\frac{3}{2}}\left(2z_2 - z_1\right) - 33z_1^{\frac{7}{2}} - 532.8z_2^2 \\
\Pi_1(z) =& 4.9z_1^{\frac{3}{2}} - 4.08z_1^{\frac{5}{2}} - 10z_2^2 \\
\Pi_2(z) =& 0.18z_1^{\frac{3}{2}} - 0.15z_1^{\frac{5}{2}} - 2.5z_2^2.
\end{aligned}$$

From the conditions of the sets D_z, D_p, D_n and D_m

$$\begin{aligned}
D_z &\equiv \{z_1 \in \mathbb{R}, z_2 \in \mathbb{R} | \Pi_1(z) = 0 \text{ or } \Pi_2(z) = 0\} \\
D_p &\equiv \{z_1 \in \mathbb{R}, z_2 \in \mathbb{R} | \Pi_1(z) > 0 \text{ and } \Pi_2(z) > 0\} \\
D_n &\equiv \{z_1 \in \mathbb{R}, z_2 \in \mathbb{R} | \Pi_1(z) < 0 \text{ and } \Pi_2(z) < 0\} \\
D_m &\equiv \{z_1 \in \mathbb{R}, z_2 \in \mathbb{R} | \Pi_1(z)\Pi_2(z) < 0\}.
\end{aligned}$$

Furthermore, the controller u is a continuous predefined-time stable for the nonlinear polytopic system (5.19) as follows

$$u = v(z) = \begin{cases} v_z(z) & = 0 \\ v_p(z) & = -\sqrt[4]{1+\rho_1^2(z)}\frac{\overline{\Phi}(z)+\sqrt{\overline{\Phi}^2(z)+\underline{\Pi}^4(z)}}{\underline{\Pi}(z)} \\ v_n(z) & = -\sqrt[4]{1+\rho_2^2(z)}\frac{\overline{\Phi}(z)+\sqrt{\overline{\Phi}^2(z)+\overline{\Pi}^4(z)}}{\overline{\Pi}(z)} \\ v_m(z) & = 0. \end{cases} \tag{5.20}$$

In the simulation, the uncertain parameters $\alpha_1 = 0.7$, $\alpha_2 = 0.3$, $\beta_1 = 0.2$ and $\beta_2 = 0.8$ are taken for Figure 5.1 and Figure 5.2. The constant parameters $\gamma = 4$, $t_s = 0.7$ and $z(0) = \begin{bmatrix} 3 & 4 \end{bmatrix}$ are chosen for the Figure 5.1. In the same way, for the Figure 5.2, $\gamma = 8.5$, $t_s = 0.7$ and $z(0) = \begin{bmatrix} 7 & -5 \end{bmatrix}$ are selected. The simulation results are depicted in Figure 5.1 and Figure 5.2. With the controller (5.20), the system trajectory converges to the origin at the desired time t_s without bearing to the initial conditions (see Figure 5.1 (a), (b) and Figure 5.2 (a), (b)). Moreover, the approach invokes a very nominal control effort (see Figure 5.1 (c) and Figure 5.2 (c)). More importantly observe that irrespective of the different initial conditions assumed, the convergence happens within the chosen predefined time. This is the key property of predefined-time control.

The proposed results are compared with the existing approaches in [10] and [11]. It is found that the proposed approach provides the exact convergence at the predefined time ($t_s = 0.7$ sec.), which is independent of initial conditions. While, the existing results on fixed-time [10] and finite-time [11], provides larger convergence time as compared to the proposed approach. Moreover, in finite-time approach [11], the convergence time depends on initial conditions and system parameters whereas in fixed-time approach [10], the convergence time depends on system parameters. Therefore, one can easily see that the origin of the nonlinear polytopic system (5.19) is stabilized in the desired time t_s irrespective of the values of initial conditions.

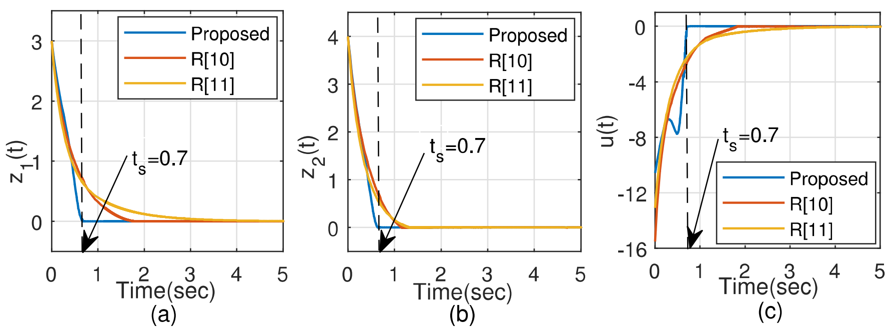

Figure 5.1: (a) state trajectory $z_1(t)$, (b) state trajectory $z_2(t)$, and (c) control input $u(t)$.

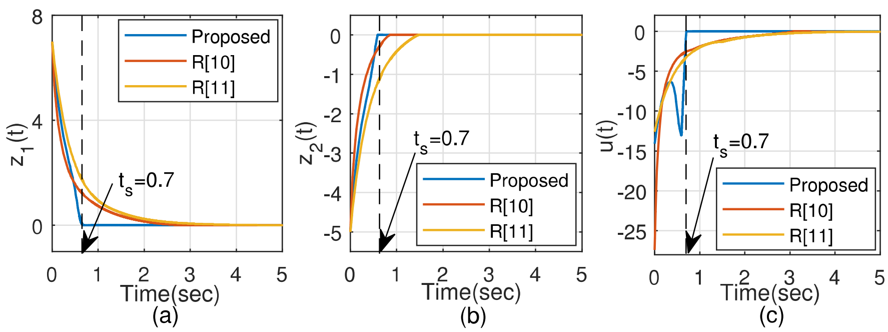

Figure 5.2: (a) state trajectory $z_1(t)$, (b) state trajectory $z_2(t)$, and (c) control input $u(t)$.

5.5 Summary

This chapter suggested predefined-time controller for nonlinear polytopic systems. With this type of controller, the settling time function (t_s) is invariant with respect to initial conditions and can be chosen as per own choice. Such kind of systems is said to be predefined time stable. Stability analysis of the nonlinear polytopic systems is discussed by using the control Lyapunov function. The effectiveness of the proposed results is studied through the simulations.