# Chapter 2

# Preliminaries

## 2.1 Notations

Symbol $\mathbb{R}$ denotes set of real numbers, $\mathbb{R}_{\geq c} = \{z \in \mathbb{R} : z \geq c\}$ and $\mathbb{R}^n$ represents set of real numbers having n-components. $\lceil \epsilon_1 \rfloor^\theta = |\epsilon_1|^\theta \text{sign}(\epsilon_1)$, with $\theta \in (0\ 1)$ and $\epsilon_1 \in \mathbb{R}$. The vector $z = [z_1,\ z_2, \cdots, z_n]^T \in \mathbb{R}^n$, $||z||_p = (\sum_{i=1}^{n} |z_i|^p)^{1/p}$ refers to the Euclidean p-norm respectively on $\mathbb{R}^n$. For a matrix $A \in \mathbb{R}^{n\times n}$, $A^T$ represents transpose matrix.

The signum function is defined as follows:

$$\text{sign}(z) = \begin{cases} -1 & \text{if } z < 0 \\ [-1, 1] & \text{if } z = 0 \\ 1 & \text{if } z > 0 \end{cases}$$

where $\text{sign}(z)$ is a multivalued function.

The saturation function $sat_\omega(\cdot)$ is defined as follows

$$sat_\omega(z) = \begin{cases} 1, & z > \omega \\ \frac{z}{\omega}, & |z| \leq \omega \\ -1, & z < -\omega \end{cases}$$

where $\omega$ is a small positive constant.

## 2.2 Stability Notions and Definitions

We here present some standard definitions in context of stability. Consider the nonautonomous system

$$\dot{x} = f(t, x),\ x(t_0) = x_0 \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the system state and $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function such that $f(t, 0) = 0$, i.e., origin $x = 0$ is an equilibrium point of (2.1), $t_0 \geq 0$ is the initial time and $x_0$ is the initial state.

Now, let's delve into stability notions. Stability theory holds a central role in systems theory and engineering, where various kinds of stability notions emerge in the study of dynamical systems. These stability notions can be classified as follows:

**Definition 1** *[21](Lyapunov stability) The origin of the system* (2.1) *is said to be Lyapunov stable if for* $\forall \epsilon \in \mathbb{R}_+$ *and* $\forall t_0 \in \mathbb{R}$ *there exists* $\delta = \delta(\epsilon, t) \in \mathbb{R}_+$ *such that for* $\forall x_0 \in B(\delta)$

1. *any solution* $x(t, t, x_0)$ *of Cauchy problem* (2.1), $x_0$ *exists for* $t > t_0$
2. $x(t, t, x_0) \in B(\epsilon)$ *for* $t > t_0$.

**Definition 2** *[21] (Asymptotic stability). The origin of the system* (2.1) *is said to be asymptotically stable if it is Lyapunov stable and asymptotically attractive.*

*If* $U(t_0) = \mathbb{R}^n$ *then the asymptotically stable (attractive) origin of the system* (2.1) *is called globally asymptotically stable (attractive). The set* $U(t_0)$ *is called attraction domain.*

**Remark 1** *It is important to note that Definition 1 and 2 highlight the notion of unrated stability [21].*

**Definition 3** *[21] (Exponential stability). The origin of the system* (2.1) *is said to be exponentially stable if there exist an attraction domain* $U \subseteq \mathbb{R}^n : 0 \in int(U)$ *and number* $C, r \in \mathbb{R}_+$ *such that*

$$||x(t, t_0, x_0)|| \leq C||x_0||e^{-r(t-t_0)}; t > t_0$$

*for* $t_0 \in \mathbb{R}$ *and* $x_0 \in U$.

**Definition 4** [20] (Global finite-time stability).
*The origin of the system* (2.1) *is said to be globally finite-time stable if it is globally asymptotically stable and any solution* $x(t, t_0, x_0)$ *of* (2.1) *converges to the origin at some finite time, i.e.,* $\forall\, t \geq t_0 + T(t_0, x_0),\ x(t, t_0, x_0) = 0,$ *where* $T\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0},$ *is the settling time function.*

**Definition 5** [20] (Fixed-time stability).
*The origin of the system* (2.1) *is said to be fixed-time stable if it is globally finite-time stable and the settling time function is bounded, i.e.,* $\exists\, T_{max} > 0 : \forall\, x_0 \in \mathbb{R}^n$ *and* $\forall\, t_0 \in \mathbb{R}_{\geq 0}, \quad T(t_0, x_0) \leq T_{max}$.

**Definition 6** *[37] (Predefined Time Stability)*
*The origin of the system* (2.1) *is said to be predefined time stable, if it is fixed time stable and any solution* $z(t, t_0, z_0)$ *of the system* (2.1) *converges to the origin within some predefined time and the settling time function is independent of the initial conditions of the system.*

**Remark 2** *Please note that Definition 3, 4, 5, and 6 emphasize the concept of rated stability [21].*

Now, let us define the next definition, which is related to Persistent Excitation (PE).

**Definition 7** *[14] (Persistent Excitation)*
*A signal* $u(t)$ *is persistently exciting of order* $r$ *if there exists a constant* $\gamma > 0$ *and a time* $T > 0$ *such that for all* $t \geq 0$ *and all sets of coefficients* $c_0, \ldots, c_r$*, the inequality*

$$\gamma \int_t^{t+T} \left\| \Phi(s,t) \begin{bmatrix} u(s) & \ldots & u^{(r)}(s) \end{bmatrix}^\top \right\|^2 ds \geq \sum_{i=0}^{r} c_i^2$$

*holds, where* $\Phi(s,t)$ *is the state transition matrix of the system at time* $s$*, starting from time* $t$ *and the parameters* $c_i$ *represent a set of arbitrary real numbers, and the condition must hold for all possible choices of these coefficients.*

## 2.3 Important Lemmas

This section presents essential lemmas, which are used in the upcoming chapters in order to get the desired results. These lemmas are common to chapters. For this reason, we have accumulated the lemmas here.

**Lemma 1** *[37]: Consider the nonlinear system* (2.1)*, if a positive-definite, smooth, radially unbounded function $V(x)$ exists such that*

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(t,x) \leq -\eta \frac{(1-e^{-V})}{(t_s - t)}, \ \forall x \neq 0, \quad t_0 \leq t < t_s \tag{2.2}$$

*where $\eta \in \mathbb{R}_{>1}$ and $t_s$ is the desired settling time. Then, the state of the system* (2.1) *converges to zero within the desired settling time.*

**Lemma 2** *[38]: For every $z \in \mathbb{R}$, the following condition holds*

$$-z\left(1-e^{-z}\right) \leq -|z|\left(1-e^{-|z|}\right).$$

## 2.4 High-Gain Observer

Let us consider the nonlinear dynamical system:

$$\begin{aligned} \dot{x} =& F(x) + \sum_{j=1}^{r} G_j(x) u_j; \qquad x(t_0) = x_0 \\ y =& h(x) \end{aligned} \tag{2.3}$$

where the functions $F$, $h$ and $G_j (j = 1, \cdots, r)$ are assumed to be sufficiently continuous differentiable in a domain $\mathbb{D} \subset \mathbb{R}^n$. The mappings $F : \mathbb{D} \to \mathbb{R}^n$ and $G_j : \mathbb{D} \to \mathbb{R}^n$ are vector fields on $\mathbb{D}$ and $h : \mathbb{D} \to \mathbb{R}$ is the scalar function on $\mathbb{D}$.
Now, the system (2.3) can be written as the following observable canonical form (see [12]):

$$\begin{aligned} \dot{z}_1 =& z_2 + \sum_{j=1}^{r} g_{1,j}(z_1) u_j \\ \dot{z}_2 =& z_3 + \sum_{j=1}^{r} g_{2,j}(z_1, z_2) u_j \\ &\vdots \\ \dot{z}_n =& \phi(z_1, \cdots, z_n) + \sum_{j=1}^{r} g_{n,j}(z_1, \cdots, z_n) u_j \\ y =& z_1 = Cz \end{aligned} \tag{2.4}$$

where $C = [1, \ 0, \ \cdots, \ 0]_{1\times n}$, $z = [z_1, \ z_2, \ \cdots, \ z_n]^T$, $\phi$ and $g_{i,j}$ $(i = 1, \ \cdots, \ n, \ j = 1, \ \cdots, \ r)$ are smooth enough functions.

**Lemma 3** *[31]: For the system (2.4), a global finite-time observer is designed for nonlinear systems that are uniformly observable and globally Lipschitz. This approach ensures that the observer's state converges to the actual states in a finite amount of time. The observer is formulated as follows:*

$$\begin{aligned}
\dot{\hat{z}}_1 =& \hat{z}_2 + l_1(\lceil \epsilon_1 \rfloor^{\beta_1} + \sigma\epsilon_1) + \sum_{j=1}^{r} g_{1,j}(\hat{z}_1)u_j \\
\dot{\hat{z}}_2 =& \hat{z}_3 + l_2(\lceil \epsilon_1 \rfloor^{\beta_2} + \sigma\epsilon_1) + \sum_{j=1}^{r} g_{2,j}(\hat{z}_1, \hat{z}_2)u_j \\
&\vdots \\
\dot{\hat{z}}_n =& \phi(\hat{z}_1, \cdots, \hat{z}_n) + l_n(\lceil \epsilon_1 \rfloor^{\beta_n} + \sigma\epsilon_1) + \sum_{j=1}^{r} g_{n,j}(\hat{z}_1, \cdots, \hat{z}_n)u_j
\end{aligned} \tag{2.5}$$

where $\epsilon_1 = z_1 - \hat{z}_1$, $\sigma$ be the positive constant and $\beta_i$ is defined as:

$$\beta_i = i\beta - (i-1); \; i = 1, \; \cdots, \; n, \; \beta \in \left(1 - \frac{1}{n}, 1\right).$$

The gain values $l_1, \; l_2, \; \cdots, \; l_n$ are given as:

$$[l_1, \; l_2, \; \cdots, \; l_n]^T = S^{-1}(\alpha)C^T; \; \alpha > 0$$

where $S(\alpha)$ is positive-definite symmetric $n \times n$ matrix.

## 2.5 Planar Missile-Target System

The planar missile-target interaction geometry is depicted in Figure 2.1. The missile and the target's relative motion is represented by $M$ and $T$, respectively. The distance between the missile and the target is $R$. The missile and the target velocities are represented by $V_M$ and $V_T$, respectively. The missile's and the target's flight paths are denoted by $\gamma_M$ and $\gamma_T$, respectively. The LOS angle is defined by $\sigma$. The kinematic equations of the planar motion shown in Figure 2.1 can be written as [96]:

$$\begin{aligned}
\dot{R} =& V_T \cos(\sigma - \gamma_T) - V_M \cos(\sigma - \gamma_M) \\
R\dot{\sigma} =& -V_T \sin(\sigma - \gamma_T) + V_M \sin(\sigma - \gamma_M)
\end{aligned}$$

where $\dot{\sigma}$ represents the LOS rate.

The target acceleration can be defined into three cases.

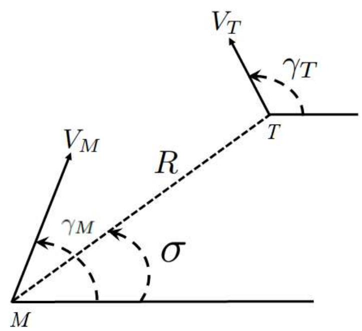


Figure 2.1: Planar Missile-Target Engagement Geometry

### 2.5.1 Non-Maneuvering Target

A non-maneuvering target is a type of target that does not change its trajectory or speed in response to external stimuli, such as a tracking system. In other words, a non-maneuvering target moves along a predictable path at a constant speed, which makes it relatively easy to track and predict its future location.

### 2.5.2 Constant Maneuvering Target

A constant maneuvering target is a type of maneuvering target that changes its motion characteristics at a constant rate over time. The target's acceleration or velocity changes at a fixed rate, which makes it more difficult to track using simple kinematic models.

### 2.5.3 Time-varying Maneuvering Target

Time-varying maneuvering targets combine the characteristics of both maneuvering and non-maneuvering targets. They may exhibit different acceleration profiles or follow specific patterns in their maneuvers. Therefore, their motion characteristics are more complex and unpredictable over time.

The term time-varying maneuvering target refers to a target whose motion or behavior changes over time due to intentional actions, external influences, or dynamic factors. The signal that drives the time-varying maneuver can originate from various sources and depends on the specific application or scenario. It can come from sources such as control inputs, environmental factors, interactions with other targets, mission objectives, or information gathered from onboard sensors. The specific driving signal depends on the target's nature, its operating environment, and the objectives it aims to achieve.