Chapter 3
Mathematical Formulation

3.1. Introduction

In this chapter, the mathematical formulations required for the analytical and FE modelling for structural analysis of the carbon nanotube reinforced composite plates and sandwich structures resting on an elastic foundation in the framework of the different non-polynomial shear deformation theories based on "secant function" and "inverse hyperbolic sine function" are discussed in detail. The non-polynomial high order shear deformation theory based upon different trigonometric function is used here for the modelling of the in-plane and transverse displacements of any point inside the plate. The trigonometric shear deformation theories used here contains non-polynomial shear strain functions of "secant function and inverse hyperbolic sine function" to introduce the non-linearity of transverse shear stresses through thickness at the cost of less number of field variables with respect to the high order shear deformation theories available in the literature which are generally of polynomial in nature. First order shear deformation theory does not have the required deformation modes to model thick carbon nanotube reinforced composite and sandwich plates and it is usually preferred to study the thin ones where shear deformation is not dominant. While the higher-order deformation modes (membrane and bending) are present in the polynomial based higher order shear deformation theories, yet their inclusion is only possible with a large number of higher-order terms which increases computational costs. In non-polynomial shear deformation theories, the non-linearity of shear deformation is accommodated with the aid of different non-polynomial functions i.e., “secant function” and “inverse hyperbolic sine function” in the kinematic field. Hence, efficient results are obtained at the cost of lesser computational efforts. Next, the non-polynomial shear deformation theory is inherently satisfying the traction free conditions of transverse shear stresses at the top and bottom surfaces of the plate while in most of the polynomial based higher order shear deformation theory’s, these conditions are generally not taken into consideration and in some cases, these conditions are artificially enforced.

Mechanical properties of composites depend on the volume fraction of reinforcement and matrix. The mechanical property of the composites varies as the orientation of the reinforcement changes. The mechanical property such as the elastic modulus, mass density, ultimate tensile strength, thermal conductivity, and electrical conductivity of the carbon nanotube reinforced composite plate can be found using rule of mixture. Rule of mixture uses weighted mean method to find out the various properties associated with the composite materials. The displacement field of trigonometric shear deformation theory is used here for model the carbon nanotube reinforced composite and sandwich plate. The trigonometric shear deformation theory fulfills the traction free boundary conditions and considers non-linear distribution of transverse shear stresses. The governing equilibrium equations are derived using Hamilton’s principle which generates five partial differential equations (PDEs) corresponding to the five primary variables in association with stress-resultants and inertia. The obtained five PDEs are associated with the corresponding stress-resultants which make the analysis indeterminate. This problem is solved using the plate-constitutive equations that make the analysis determinate. The solution for the governing differential equation is proposes using the analytical and numerical approach. Navier's solution technique is used for the analytical solutions. The Navier's solution technique uses the separation of the primary variables and the primary variables expressed in terms of double trigonometric series in the spatial domain by which the PDEs are transformed into the ODEs. The finite element method (FEM) is employed to solve the governing differential equations for the numerical solutions. The physical domain of the plate is discretized using the eight-noded isoparametric serendipity element. The primary variables for an element are expressed in the term of the shape functions and the generalized nodal coordinates.

The present chapter deals with the derivation of basic equations associated with the modelling of the carbon nanotube reinforced composite plate resting on an elastic foundation. The assumptions involve in the analysis followed by derivation of basic equations associated with the modelling of the carbon nanotube reinforced composite plate resting on an elastic foundation and the formulation of the governing equations and the method of solutions are discussed in detail.

3.2. Basic Assumptions

The CNTRC plate with the Cartesian coordinate system is shown in the Figure 3.1. The following assumptions are made for deriving the required mathematical formulations.

- The layers are so well connected such that there is no slip and separation.
- The reference plane is at the middle of the plate.
- The materials selected for the analysis obey Hooke's law.

- The lateral deflection in comparison to the in-plane dimensions of the plate structures is very small.
- The transverse normal stress in comparison to the other stresses is very small and therefore neglected.
- The thickness-stretching effect is neglected; since the transverse displacement does not varies across the thickness of the carbon nanotube reinforced composite plate.

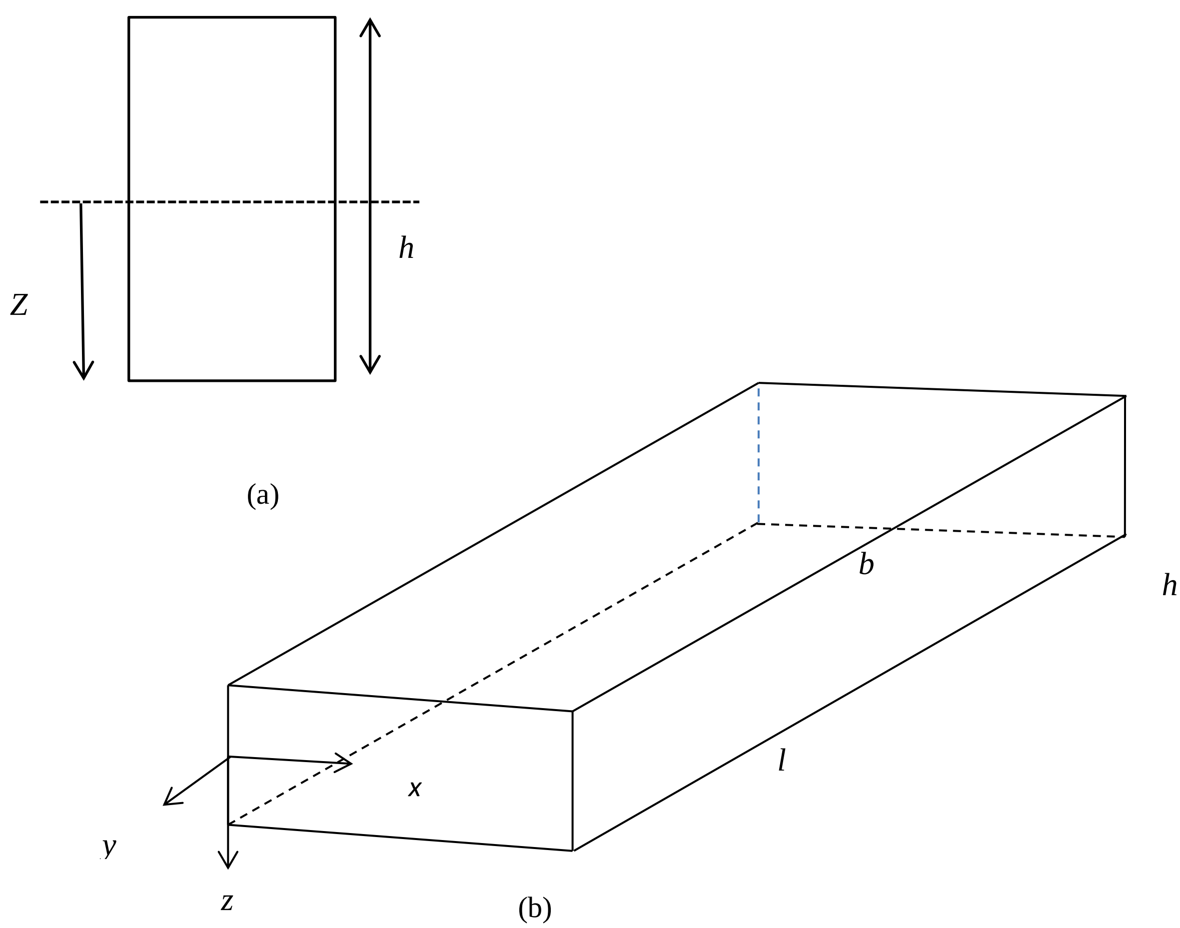

Figure 3.1: Co-ordinate system (a) CNTRC plate cross section (b) ‘*x*’ and ‘*y*’ are in plane and ‘*z*’ is along the thickness direction.

3.3. Properties of carbon nanotube reinforced composite plate

The Rule of mixture (ROM) is a method of calculating the mechanical properties of a composite structure, such as a composite plate. This method is based on the assumption that the mechanical properties of the composite plate are the weighted-average of the mechanical properties of the individual components. The ROM is used to predict the strength, stiffness, and other mechanical properties of a composite material based on the properties of its constituent materials. The material properties of the carbon nanotube reinforced composite plate can be easily found out using rule of mixture which is given as follows:

$$E_{11} = \eta_1 V_{CNT} E_{11}^{CNT} + V_m E^m \tag{3.1}$$

$$\frac{\eta_2}{E_{22}} = \frac{V_{CNT}}{E_{22}^{CNT}} + \frac{V_m}{E^m} \tag{3.2}$$

$$\frac{\eta_3}{G_{12}} = \frac{V_{CNT}}{G_{22}^{CNT}} + \frac{V_m}{G^m} \tag{3.3}$$

$$v_{12} = V_{CNT}^{*} v_{12}^{CNT} + V_m v^m \tag{3.4}$$

$$\rho = V_{CNT} \rho^{CNT} + V_m \rho^m \tag{3.5}$$

$$V_{CNT}^{*} = \frac{w_{CNT}}{w_{CNT} + (\rho^{CNT} / \rho^m) - (\rho^{CNT} / \rho^m) w_{CNT}} \tag{3.6}$$

where E_{11}^{CNT} and E_{22}^{CNT} are the Young's modulus of carbon nanotube reinforced composite plate in longitudinal and lateral direction respectively. G_{12}^{CNT} is the shear modulus of carbon nanotube reinforced composite plate. v_{12}^{CNT} and V_{CNT}^{*} are the Poisson's ratio and volume fraction of carbon nanotube reinforced composite respectively. E^m, G^m, v^m and V_m are the Young's modulus, shear modulus, Poisson's ratio and volume fraction of matrix respectively.

3.4. Stress-Strain Constitutive Relations

Stress-strain constitutive relations are the mathematical equations that describe the relationship between stresses and strains in a material. These equations can be used to predict the behavior of a material under various loading conditions. The equations are used to calculate the amount of strain a material can tolerate before it yields or fails. Stress-strain constitutive relations are useful in a wide range of engineering applications such as structural analysis, fatigue analysis, and stress analysis. They are used to predict the strength and stiffness of materials, to analyze the behavior of structures under load, and to model the behavior of materials when subjected to various types of loading. The stress-strain constitutive relationship is expressed in the following form for the orthotropic lamina in the carbon nanotube reinforced composite plate.

$$\{\sigma\}_n = [\mathrm{Q}_{ij}]_n \{\varepsilon\}_n \tag{3.7}$$

where, $\{\sigma\}_n$ and $\{\varepsilon\}_n$ the stress and strain vectors at any point in the n^{th} layer of carbon nanotube reinforced composite sandwich plate. $[\mathrm{Q}_{ij}]_n$ is the reduced stiffness coefficient matrix which establish the relation between the stress and strain vectors at any point in the n^{th} layer of carbon nanotube reinforced composite sandwich plate. The components of the stress and strain vectors are given below:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{bmatrix}_n = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0\\ Q_{21} & Q_{22} & Q_{26} & 0 & 0\\ Q_{16} & Q_{26} & Q_{66} & 0 & 0\\ 0 & 0 & 0 & Q_{44} & Q_{45}\\ 0 & 0 & 0 & Q_{45} & Q_{55}\end{bmatrix}_n \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{bmatrix}_n \qquad 3.8$$

where, $Q_{11} = \dfrac{E_{11}}{1-\nu_{12}\nu_{21}}; Q_{22} = \dfrac{E_{22}}{1-\nu_{12}\nu_{21}}; Q_{12} = \dfrac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}}; Q_{66} = G_{12}; Q_{44} = G_{23}$ and $Q_{55} = G_{13}$

The directions '1' and '2' refer for the directions along the fibers and perpendicular to the fibers, respectively.

3.5. Strain displacement relationships

Strain-displacement relationships are mathematical equations that describe the relationship between strains and displacements in a material. These equations are used to predict the behavior of a material under various loading conditions and to calculate the amount of strain a material can tolerate before it yields or fails. Strain-displacement relationships are useful in a wide range of engineering applications such as structural analysis, fatigue analysis, and stress analysis. They are used to predict the strength and stiffness of materials, to analyze the behavior of structures under load, and to model the behavior of materials when subjected to various types of loading. The strain displacement relationship given by using the following equations:

$$\varepsilon = \begin{bmatrix}\varepsilon_b\\ \varepsilon_s\end{bmatrix}, \quad \varepsilon_b = \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\end{bmatrix}, \quad \varepsilon_s = \begin{bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{bmatrix} \qquad 3.9$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x},\ \varepsilon_{yy} = \frac{\partial v}{\partial y},\ \varepsilon_{zz} = \frac{\partial w}{\partial z},\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad 3.10$$

3.6. Plates on elastic foundation

The "Winkler's hypothesis" is based on one parameter according to which in the case of the elastic soil the deflection at any point on a surface is proportional to the load being applied onto the surface and free from the load being applied on any other points on the surface. This one parameter model is referred as the Winkler model (Tanahashi, 2007). The mechanical model of the elastic soil by the "Winkler's hypothesis", is assumed by mutually independent vertical springs. The hypothesis leads to the limitation to this model that is the displacement is free from the load being applied on any other points on the surface. In this research, an improved model which eliminates the limitation of the Winkler model is selected for the analysis. The two parameter model is selected for the analysis considers the deflection of elastic soil due to the effect of the loads applied onto the surface and the load being applied on any other points on the surface due to which a continuity between the adjacent displacements point is established by considering shear interactions. This two parameter model is selected for the analysis is referred as the Pasternak's foundation model (Zenkour, 2010). The Pasternak's foundation model is as follow:

$$R_{EF} = \beta_w W - \beta_{s1} \frac{\partial^2 W}{\partial x^2} - \beta_{s2} \frac{\partial^2 W}{\partial y^2} \qquad 3.11$$

where R_{EF} is the Pasternak's elastic foundation reaction, β_{w} is the Winkler spring constants and, β_{s1} , $\beta_{\mathrm{s2}}$are shear layer spring constants respectively. In the case of homogenous and isotropic soil both the shear layer spring constants are same i.e. $\beta_{\mathrm{s1}} = \beta_{s2}= \beta_{\mathrm{s}}$. In the present analysis the soil is consider as the homogenous and isotropic. The modified Pasternak's foundation model is as follow:

$$R_{EF} = \beta_w W - \beta_s \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right) \tag{3.12}$$

If the β_s i.e. shear layer spring constants is neglected the Pasternak's foundation model is converted to the Winkler model. The Figure 3.2 shows, the CNTRC plate resting on the Pasternak's elastic foundation.

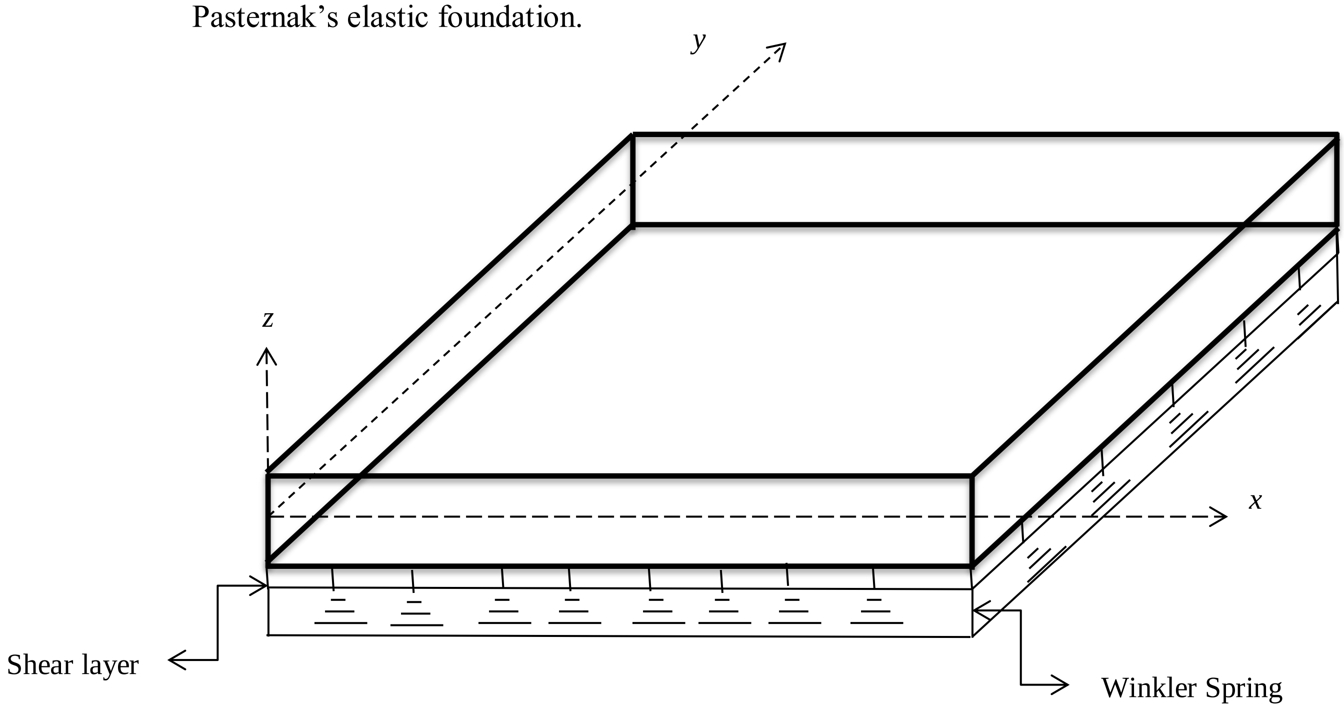

Figure 3.2 the CNTRC plate resting on the Pasternak's elastic foundation

3.7. Displacement field

The non-polynomial shear deformation theory based on different non-polynomial functions i.e., secant function and inverse hyperbolic sine function is used to model and analyze the behavior of thin-walled structures. This theory is based on the assumption that the cross-sectional deformation of the structure is governed by a combination of normal and shear deformations, with the latter being represented by trigonometric functions. The displacement field is a fundamental concept in the non-polynomial shear deformation theory, which describes how the structure deforms under external loads. In the non-polynomial shear deformation theory, the displacement field is expressed as a combination of polynomial and non-polynomial trigonometric functions, with the polynomial part accounting for the normal deformations and the non-polynomial trigonometric part representing the shear deformations. This approach allows for an accurate representation of the complex deformation behavior of thin-walled structures, which is often difficult to model using traditional methods. The non-polynomial shear deformation theory has been widely used in the analysis and design of various engineering structures, such as aircraft wings, bridges, and wind turbines. The accurate prediction of the displacement field is critical in determining the structural response to external loads, and therefore, it is essential to have a robust mathematical framework to model these behaviors. The non-polynomial shear deformation theory has been successfully applied in many engineering applications with minimum limitations. For instance, the accuracy of the method is highly dependent on the choice of trigonometric functions that is used to represent the shear deformation. Additionally, the non-polynomial shear deformation theory assumes that the deformation is uniform across the thickness of the structure, which may not always be the case in practice. The displacement field is a key concept in the trigonometric shear deformation theory, which provides a powerful mathematical framework for analyzing the behavior of thin-walled structures. While the non-polynomial shear deformation theory has been widely used in engineering applications, it is important to recognize its limitations and to carefully select the appropriate method for each specific application. The non-polynomial high order shear deformation theory based upon the different non-polynomial trigonometric function is used for the modelling the in-plane and transverse displacements of any point inside the plate. The non-polynomial trigonometric shear deformation theories used here contains non-polynomial shear strain functions of "inverse hyperbolic sine" and of "secant function" to introduce the non-linearity of transverse shear stresses through thickness at the cost of less number of field variables with respect to the high order shear deformation theories available in the literature which are generally of polynomial nature. While the higher-order deformation modes (membrane and bending) are present in the polynomial based high order shear deformation theories, yet their inclusion is only possible with a large number of higher-order terms which increases computational costs. In non-polynomial trigonometric shear deformation theories, the non-linearity of shear deformation is accommodated with the aid of a single non-polynomial functions "inverse hyperbolic sine function" and a "secant function" in the kinematic field. Hence, efficient results are obtained at the cost of lesser computational efforts. The displacement field for the non-polynomial shear deformation theory is given below:

$$\begin{bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{bmatrix} = \begin{bmatrix} u_0(x,y,t) \\ v_0(x,y,t) \\ w_0(x,y,t) \end{bmatrix} - z \begin{bmatrix} \dfrac{\partial w_0(x,y,t)}{\partial x} \\ \dfrac{\partial w_0(x,y,t)}{\partial y} \\ 0 \end{bmatrix} + f(z) \begin{bmatrix} \theta_x(x,y,t) \\ \theta_y(x,y,t) \\ 0 \end{bmatrix} \qquad 3.13$$

where u_0, v_0, w_0, and θ_x, θ_y are the displacement and shear deformations at the mid plane, respectively. In the case of secant function based non-polynomial shear deformation theory, trigonometric function ($f(z)$) is the non- polynomial function used to refine the bending profile of the system which is given below:

$$f(z) = g(z) + \Omega z \; ; g(z) = z\left(\sec\left(rz/h\right)\right) ; \; \Omega = -\sec\left(r/2\right)\left[1 + \left(r/2\right)\tan\left(r/2\right)\right] \tag{3.14}$$

where, *r* is the transverse shear stress constant parameter. The value of the *r* is determined by the post processing step using inverse method and the result for the post processing is compared with the 3D elasticity solutions. The value of the *r* is selected in such a way that it achieves the maximum efficiency in compression to the 3D elasticity solutions. In the case of trigonometric shear deformation theory based on the secant function the value of r = 0.1. Further, in the case of inverse hyperbolic sine function based non-polynomial shear deformation theory *f(z)* is defined as follows:

$$f(z) = g(z) + \Omega z$$

where, $g(z) = \left(\sinh^{-1}\left(\frac{rz}{h}\right)\right) ; \quad \Omega = -\frac{2r}{h\sqrt{r^2+4}}$ 3.15

In the case of non-polynomial shear deformation theory based on the inverse hyperbolic sine function the value of r = 3. Next, the trigonometric shear deformation theory is inherently satisfying the traction free conditions of transverse shear stresses at the top and bottom surfaces of the plate while in most of the polynomial based high order shear deformation theories, these conditions are generally not taken into consideration and in some cases, these conditions are artificially enforced.

3.8. Analytical Formulation

The strain displacement relations are established with the help of the following equations

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} + [g(z) + \Omega z]\frac{\partial \theta_x}{\partial x} \tag{3.16a}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} + [g(z) + \Omega z]\frac{\partial \theta_y}{\partial y} \qquad 3.16b$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \qquad 3.16c$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - z\frac{2\partial^2 w_0}{\partial x \partial y} + [g(z) + \Omega z]\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \qquad 3.16d$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\frac{\partial w_0}{\partial x} + f^{'}(z)\theta_x + \frac{\partial w_0}{\partial x} = f^{'}(z)\theta_x \qquad 3.16e$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\frac{\partial w_0}{\partial y} + f^{'}(z)\theta_y + \frac{\partial w_0}{\partial y} = f^{'}(z)\theta_y \qquad 3.16f$$

For the secant function based non-polynomial shear deformation theory,

$$f^{'}(z) = g^{'}(z) + \Omega\,;\;\; g^{'}(z) = r/h\sqrt{1 + \left(\frac{rz}{h}\right)^2} \qquad 3.17$$

whereas, for the inverse hyperbolic sine function based non-polynomial shear deformation theory, the *f(z)* is defined as given below:

$$f^{'}(z) = g^{'}(z) + \Omega\,;\;\; g^{'}(z) = \sec\left(\frac{rz}{h}\right)\left(1 + \frac{rz}{h}\tan\frac{rz}{h}\right) \qquad 3.18$$

The basic equations like the rule of mixture, displacement field, stain displacement relations and stress-strain constitutive relations required for the present investigation are presented above.

3.8.1. Equations of motion

Equations of motion are mathematical equations that describe the motion of a body or system. They are used to calculate the motion of a body under the influence of forces and torques. The equations are used to analyze the behavior of a system, to predict its response to an external force, and to design systems with desired characteristics. Equations of motion are useful in a wide range of engineering applications such as dynamics, vibration, and control. They are used to calculate the position, orientation, speed, and acceleration of a body, to analyze the behavior of a system under the influence of external forces, and to design systems with desired characteristics. Hamilton's principle is an important concept in classical mechanics which states that the motion of a system of particles or rigid bodies is such that the action integral of the system is minimized with respect to the motion of the particles and rigid bodies within the system. The principle is used to derive the equations of motion for a system and can be used to calculate the motion of a system with forces that are not known in advance. Hamilton's principle is widely used in a variety of engineering applications such as robotics, vibration analysis, and control. It is also used to analyze the behavior of dynamical systems, to predict the response of a system to external forces, and to design systems with desired characteristics. “The motion of a dynamical system in a given time interval is such as to maximize or minimize the action integral”. This statement is known as Hamilton's principle, and was first formulated in 1834 by the Irish mathematician William Hamilton. The Hamilton’s principle is used to derive the equation of motion of the carbon nanotubes reinforced composite plate with Pasternak elastic foundation.

$$\int_{-h/2}^{h/2} (\delta L)\,\mathrm{dt} = 0 \tag{3.19}$$

$$\delta L = \delta T - (\delta U_s + \delta U_f + \delta V) \tag{3.20}$$

where, δT, δU_s, δU_f ,and δV are the change in kinetic energy, change in strain energy, change in strain energy of elastic foundation and change in potential energy due to external applied load, respectively.

The change in the kinetic energy of the system is expressed as follows:

$$\delta T = \int_{\Omega_0} \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho[\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w]dz \right\} dxdy \qquad 3.21$$

The change in the strain energy of the system is expressed as follows:

$$\delta U = \int_{\Omega_0} \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} [\sigma_{xx}\delta \in_{xx} + \sigma_{yy}\delta \in_{yy} + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}]dz \right\} dxdy \qquad 3.22$$

The change in the potential energy of the system due to external applied load is expressed as follows:

$$\delta V = \int_{\Omega_0} q\delta w_0 dxdy \qquad 3.23$$

The change in the strain energy of the system due to Pasternak elastic foundation is expressed as follows:

$$\delta U_f = \int_{\Omega_0} k_w w_0 \delta w_0 dxdy + \int_{\Omega_0} k_s \left(\frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial x} + \frac{\partial w_0}{\partial y}\frac{\partial \delta w_0}{\partial y}\right) dxdy \qquad 3.24$$

where K_w and K_s are the Winkler and shear layer spring constants, respectively

$$K_w = \frac{\beta_w D_0}{a^4} \; ; \; K_s = \frac{\beta_s D_0}{a^2} \quad \text{and}\, D_0 = \frac{E^p h^3}{12[1-(\nu^p)^2]}$$

where, β_w and β_s are the Winkler and shear layer constant factor, respectively.

The carbon nanotubes reinforced composite plate with Pasternak elastic foundation is subjected to the transverse load q which is the bending load, in-plane compressive load of $\Psi_x N_{cr}$, and $\Psi_y N_{cr}$, respectively. The total potential energy of the system due to external applied load is expressed as follows:

$$\delta V_T = -\int_{\Omega_0} q\delta w_0 dxdy + \int_{\Omega_0} (\psi_x N_{cr} \frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial x} + \psi_y N_{cr} \frac{\partial w_0}{\partial y}\frac{\partial \delta w_0}{\partial y})dxdy \tag{3.25}$$

The variation of the energy of the system from Eq. (3.21-.3.25) is substituted in Eq. (3.20) and then by parts integration, the coefficients of variation of the mid plane displacements and rotations are separated which leads to develop the governing differential equations of the carbon nanotubes reinforced composite plate with Pasternak elastic foundation which are as follows:

$$\delta u_0 : \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial x} + I_3 \ddot{\theta}_x \tag{3.26}$$

$$\delta v_0 : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \ddot{v}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial y} + I_3 \ddot{\theta}_y \tag{3.27}$$

$$\begin{aligned}\delta w_0 : &\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q - k_w w_0 + k_s \frac{\partial^2 w_0}{\partial x^2} + k_s \frac{\partial^2 w_0}{\partial y^2} + \bar{N}_{xx} \frac{\partial^2 w_0}{\partial x^2} + 2\bar{N}_{xy} \frac{\partial^2 w_0}{\partial x \partial y} + \bar{N}_{yy} \frac{\partial^2 w_0}{\partial y^2} = \\ &I_1 \left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y} \right) - I_2 \left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2} \right) + I_4 \left(\frac{\partial \ddot{\theta}_x}{\partial x} + \frac{\partial \ddot{\theta}_y}{\partial y} \right) + I_0 \ddot{w}_0\end{aligned} \tag{3.28}$$

$$\delta \theta_x : \Omega \frac{\partial M_{xx}}{\partial x} + \frac{\partial P_{xx}}{\partial x} + \Omega \frac{\partial M_{xy}}{\partial y} + \frac{\partial P_{xy}}{\partial y} - \Omega Q_1 - K_1 = I_3 \ddot{u}_0 - I_4 \frac{\partial \ddot{w}_0}{\partial y} + I_5 \ddot{\theta}_x \tag{3.29}$$

$$\delta \theta_y : \Omega \frac{\partial M_{xy}}{\partial x} + \frac{\partial P_{xy}}{\partial x} + \Omega \frac{\partial M_{yy}}{\partial y} + \frac{\partial P_{yy}}{\partial y} - \Omega Q_2 - K_2 = I_3 \ddot{v}_0 - I_4 \frac{\partial \ddot{w}_0}{\partial y} + I_5 \ddot{\theta}_y \tag{3.30}$$

The above equation is indeterminate as there are fourteen stress-resultants in five equations. When solving a problem using elasticity formulations, the formulation starts with the equilibrium equations of elasticity in which 3 equations are associated with 6 unknown stresses. To make the problem determinate, the strain-displacement relations and stress-strain constitutive equations are utilized which results in 15 unknowns and 15 equations. Similarly, in the present problem, additional equations are defined with the help of Eq. (3.8) which are known as the plate constitutive relationships and then substituted for the stress-resultants in above equation. The plate constitutive relations for the functionally graded carbon nanotube reinforced composite plate resting on a Pasternak's elastic foundation are defined as follows:

$$\begin{bmatrix} N_{xx} & M_{xx} & P_{xx} \\ N_{yy} & M_{yy} & P_{yy} \\ N_{xy} & M_{xy} & P_{xy} \end{bmatrix} = \int_{h_k}^{h_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} [1 \;\; z \;\; g(z)]\, dz \tag{3.31a}$$

$$\begin{bmatrix} Q_2 & K_2 \\ Q_1 & K_1 \end{bmatrix} = \int_{h_k}^{h_{k+1}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} [1 \;\; g'(z)]\, dz \tag{3.31b}$$

$$[I_0 \;\; I_1 \;\; I_2 \;\; I_3 \;\; I_4 \;\; I_5] = \int_{h_k}^{h_{k+1}} \rho^{(k)} [1 \;\; z \;\; z^2 \;\; f(z) \;\; z.f(z) \;\; (f(z))^2]\, dz \tag{3.31c}$$

Partial differential equation terms of the primary variables are separated for different displacement modes derived at the mid plane are shown below:

$$A_{11}\left(\frac{\partial^2 u_0}{\partial x^2}\right) + B_{11}\left(\Omega\frac{\partial^2 \theta_x}{\partial x^2} - \frac{\partial^3 w_0}{\partial x^3}\right) + E_{11}\left(\frac{\partial^2 \theta_x}{\partial x^2}\right) + A_{12}\left(\frac{\partial^2 v_0}{\partial x \partial y}\right) + B_{12}\left(\Omega\frac{\partial^2 \theta_y}{\partial x \partial y} - \frac{\partial^3 w_0}{\partial x \partial y^2}\right) + E_{12}\left(\frac{\partial^2 \theta_y}{\partial x \partial y}\right)$$
$$+A_{66}\left(\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x \partial y}\right) - B_{66}\left(\Omega\left(\frac{\partial^2 \theta_x}{\partial y^2} + \frac{\partial^2 \theta_y}{\partial x \partial y}\right) - 2\frac{\partial^3 w_0}{\partial x \partial y^2}\right) + E_{66}\left(\frac{\partial^2 \theta_x}{\partial y^2} + \frac{\partial^2 \theta_y}{\partial x \partial y}\right) = I_0 \ddot{u} - +I_1 \frac{\partial \ddot{w}_0}{\partial x} + I_3 \ddot{\theta}_x \tag{3.32}$$

$$B_{11}\left(\frac{\partial^3 u_0}{\partial x^3}\right)+D_{11}\left(\Omega\frac{\partial^3\theta_x}{\partial x^3}-\frac{\partial^4 w_0}{\partial x^4}\right)+F_{11}\left(\frac{\partial^3\theta_x}{\partial x^3}\right)+B_{12}\left(\frac{\partial^3 u_0}{\partial x\partial y^2}+\frac{\partial^2 v_0}{\partial x^2\partial y}\right)+B_{22}\left(\frac{\partial^3 v_0}{\partial y^3}\right)$$
$$+D_{12}\left(\Omega\left(\frac{\partial^3\theta_x}{\partial y^2\partial x}+\frac{\partial^3\theta_y}{\partial x^2\partial y}\right)-2\frac{\partial^4 w_0}{\partial x^2\partial y^2}\right)+F_{12}\left(\frac{\partial^3\theta_y}{\partial x^2\partial y}+\frac{\partial^3\theta_x}{\partial x\partial y^2}\right)+D_{22}\left(\Omega\frac{\partial^3\theta_y}{\partial y^3}-\frac{\partial^4 w_0}{\partial y^4}\right)+F_{22}\left(\frac{\partial^3\theta_y}{\partial y^3}\right)$$
$$+2B_{66}\left(\frac{\partial^3 u_0}{\partial x\partial y^2}+\frac{\partial^3 v_0}{\partial x^2\partial y}\right)-2D_{66}\left(\Omega\left(\frac{\partial^3\theta_x}{\partial x\partial y^2}+\frac{\partial^3\theta_y}{\partial x^2\partial y}\right)-2\frac{\partial^4 w_0}{\partial x^2\partial y^2}\right)+2F_{66}\left(\frac{\partial^3\theta_x}{\partial x\partial y^2}+\frac{\partial^3\theta_y}{\partial x^2\partial y}\right)$$
$$=I_1\left(\frac{\partial\ddot{u}_0}{\partial x}+\frac{\partial\ddot{v}_0}{\partial y}\right)-I_2\left(\frac{\partial^2\ddot{w}_0}{\partial x^2}+\frac{\partial^2\ddot{w}_0}{\partial y^2}\right)+I_4\left(\frac{\partial\ddot{\theta}_x}{\partial x}+\frac{\partial\ddot{\theta}_y}{\partial y}\right)+I_0\ddot{w}_0$$

3.33

$$A_{12}\left(\frac{\partial^2 u_0}{\partial x\partial y}\right)+B_{12}\left(\Omega\frac{\partial^2\theta_x}{\partial x\partial y}-\frac{\partial^3 w_0}{\partial x^2\partial y}\right)+E_{12}\left(\frac{\partial^2\theta_x}{\partial x\partial y}\right)+A_{22}\left(\frac{\partial^2 v_0}{\partial y^2}\right)+B_{22}\left(\Omega\frac{\partial^2\theta_y}{\partial y^2}-\frac{\partial^3 w_0}{\partial y^3}\right)+E_{22}\left(\frac{\partial^2\theta_y}{\partial y^2}\right)$$
$$+A_{66}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}\right)-B_{66}\left(\Omega\left(\frac{\partial^2\theta_x}{\partial x\partial y}+\frac{\partial^2\theta_y}{\partial x^2}\right)-2\frac{\partial^3 w_0}{\partial x^2\partial y}\right)+E_{66}\left(\frac{\partial^2\theta_x}{\partial x\partial y}+\frac{\partial^2\theta_y}{\partial x^2}\right)+q-k_w w_0+k_s\frac{\partial^2 w_0}{\partial x^2}$$
$$+k_s\frac{\partial^2 w_0}{\partial y^2}+\bar{N}_{xx}\frac{\partial^2 w_0}{\partial x^2}+2\bar{N}_{xy}\frac{\partial^2 w_0}{\partial x\partial y}+\bar{N}_{yy}\frac{\partial^2 w_0}{\partial y^2}=I_0\ddot{v}_0+I_1\frac{\partial\ddot{w}_0}{\partial y}+I_3\ddot{\theta}_y$$

3.34

$$(\Omega B_{11}+E_{11})\left(\frac{\partial^2 u_0}{\partial x^2}\right)+(\Omega D_{11}+F_{11})\left(\Omega\frac{\partial^2\theta_x}{\partial x^2}-\frac{\partial^3 w_0}{\partial x^3}\right)+(\Omega F_{11}+H_{11})\left(\frac{\partial^2\theta_x}{\partial x^2}\right)$$
$$+(\Omega B_{66}+E_{66})\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}\right)+(\Omega B_{12}+E_{12})\left(\frac{\partial^2 v_0}{\partial x\partial y}\right)+(\Omega F_{12}+H_{12})\left(\frac{\partial^2\theta_y}{\partial x\partial y}\right)$$
$$+(\Omega D_{12}+F_{12})\left(\Omega\frac{\partial^2\theta_y}{\partial y\partial x}+\frac{\partial^3 w_0}{\partial x\partial y^2}\right)+(\Omega F_{66}+H_{66})\left(\frac{\partial^2\theta_x}{\partial y^2}+\frac{\partial^2\theta_y}{\partial x\partial y}\right)$$
$$+(\Omega D_{66}+F_{66})\left(\Omega\left(\frac{\partial^2\theta_x}{\partial y^2}+\frac{\partial^2\theta_y}{\partial x\partial y}\right)-2\frac{\partial^3 w_0}{\partial x\partial y^2}\right)-[\Omega^2\,\mathrm{A}_{55}+2\Omega K_{55}+\mathrm{L}_{55}]\theta_x=I_3\ddot{u}_0-I_4\frac{\partial w_0}{\partial x}+I_5\ddot{\theta}_x$$

3.35

$$(\Omega B_{12}+E_{12})\left(\frac{\partial^2 u_0}{\partial x\partial y}\right)+(\Omega D_{12}+F_{12})\left(\Omega\frac{\partial^2\theta_x}{\partial x\partial y}-\frac{\partial^3 w_0}{\partial x^2\partial y}\right)+(\Omega F_{12}+H_{12})\left(\frac{\partial^2\theta_x}{\partial x\partial y}\right)$$
$$+(\Omega B_{66}+E_{66})\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}\right)+(\Omega B_{22}+E_{22})\left(\frac{\partial^2 v_0}{\partial y^2}\right)+(\Omega F_{22}+H_{22})\left(\frac{\partial^2\theta_y}{\partial y^2}\right)$$
$$+(\Omega D_{22}+F_{22})\left(\Omega\frac{\partial^2\theta_y}{\partial y^2}+\frac{\partial^3 w_0}{\partial y^3}\right)+(\Omega F_{66}+H_{66})\left(\frac{\partial^2\theta_x}{\partial x\partial y}+\frac{\partial^2\theta_y}{\partial x^2}\right)$$
$$+(\Omega D_{66}+F_{66})\left(\Omega\left(\frac{\partial^2\theta_x}{\partial x\partial y}+\frac{\partial^2\theta_y}{\partial x^2}\right)-2\frac{\partial^3 w_0}{\partial x^2\partial y}\right)-[\Omega^2\,\mathrm{A}_{44}+2\Omega K_{44}+\mathrm{L}_{44}]\theta_y=I_3\ddot{v}_0-I_4\frac{\partial w_0}{\partial y}+I_5\ddot{\theta}_y$$

3.36

Where,

$$[A_{ij}\ \ B_{ij}\ \ D_{ij}\ \ E_{ij}\ \ F_{ij}\ \ H_{ij}] = \int_{h_k}^{h_{k+1}} \left[\overline{Q}_{ij}^{(k)}\right] [1\ \ z\ \ z^2\ \ g(z)\ \ z.g(z)\ \ (g(z))^2]\, dz$$

for i, j = 1, 2 and 3

$$[K_{ij}\ \ L_{ij}] = \int_{h_k}^{h_{k+1}} \left[\overline{Q}_{ij}^{(k)}\right]_{2\times 2} [g'(z)\ \ (g'(z))^2]\, dz \text{ for } i, j = 4 \text{ and } 5.$$

3.8.2. Navier's Solution Methodology

The system of PDEs in Eq. (3.32-3.36) consists of the spatial derivatives and time derivatives of the primary variables. To solve the equations, the boundary conditions and the initial conditions of the problem are required. The general boundary conditions of trigonometric shear deformation theory in terms of the primary variables and the stress-resultants are derived and presented in Eq. 3.37. Based on the boundary conditions of the problem, the displacements and stress-resultants are required to be specified at all the edges. At a particular edge, both the forces and displacements cannot be specified. For analytical solutions of Eq. (3.32-3.36), the solutions of the governing differential equations are obtained using Navier's closed form solution technique. The exact solution for the governing differential equation for carbon nanotube reinforced sandwich and composite plates are considered by assuming the simply supported boundary condition. The appropriate boundary condition is as follows:

At edge *x=0 and x=l*:

$$v_0 = 0,\quad w_0 = 0,\quad \theta_y = 0,$$
$$\theta_x = 0,\quad N_x = 0,\quad M_x = 0. \qquad 3.37a$$

At edge *y=0 and y=b*:

$$u_0 = 0, \quad w_0 = 0, \quad \theta_x = 0,$$
$$\theta_y = 0, \quad N_y = 0, \quad M_y = 0. \qquad 3.37b$$

The decision to restrict analytical solutions to simply supported square carbon nanotube reinforced composite plates raises pertinent questions about the rationale behind this choice and the challenges that may arise when dealing with different boundary conditions. This discussion delves into the motivations behind this decision and explores the complex challenges associated with varying boundary conditions in nanocomposite plate analysis. The decision to focus on simply supported square carbon nanotube reinforced composite plates in analytical solutions is likely rooted in a combination of factors, including analytical tractability, model complexity, and practical relevance. Analytical solutions offer valuable insights into the fundamental behavior of composite materials, aiding in the understanding of underlying mechanics and behaviors. By choosing simply supported square plates, researchers can simplify the problem, making it amenable to analytical treatment and mathematical formulation. This approach enables the derivation of closed-form solutions that provide essential information about stress distribution, deformation, and load-carrying capacity, all of which are crucial for designing and optimizing engineering structures. While the choice of focusing on simply supported square plates facilitates analytical treatment, it is essential to acknowledge the challenges that arise when dealing with other boundary conditions. Real-world engineering applications often involve a wide range of boundary conditions, each of which can significantly influence the behavior of composite plates. Some of the key challenges associated with different boundary conditions include:

Clamping and Fixed Boundary Conditions: When a nanocomposite plate is clamped or subjected to fixed boundary conditions, the stress distribution and deformation patterns can vary dramatically from those of simply supported plates. The presence of clamping constraints may induce complex stress concentrations and buckling phenomena that demand more sophisticated experimental validation.

Free Edge Boundary Conditions: Plates with free edges experience stress concentrations and edge effects that can affect load-carrying capacity and failure modes. Analyzing the behavior of nanocomposite plates with free edges requires accounting for these effects, potentially necessitating experimental investigations.

Mixed Boundary Conditions: In real-world engineering scenarios often involve mixed boundary conditions, where different edges of the plate may have distinct boundary conditions. Analyzing the response of nanocomposite plates under such conditions requires a comprehensive understanding of stress interactions and load transfer mechanisms, posing a significant challenge in terms of analytical formulation.

The decision to restrict analytical solutions to simply supported square carbon nanotube reinforced composite plates is likely driven by a balance between analytical tractability and practical relevance. While this choice facilitates the derivation of closed-form solutions and fundamental insights into plate behavior, it is crucial to recognize the challenges associated with different boundary conditions. Real-world engineering applications demand a holistic understanding of how nanocomposite plates respond under varying boundary conditions, each of which introduces unique complexities. Addressing these challenges requires a combination of analytical techniques, numerical simulations, and experimental validation, ultimately contributing to the advancement of nanocomposite materials and their diverse applications in engineering and technology.

3.8.2.1 Solution of differential equation for bending analysis

To reproduce the governing differential equation for the bending analysis of the carbon nanotube reinforced composite plates resting on Pasternak's elastic foundation subjected to different loading conditions from Eq. (3.32-3.36), the inertia components are neglected. The primary variables are expressed in terms of double trigonometric series by satisfying the boundary conditions in Eq. (3.37). The mathematical functions for the primary variables are assumed as follows:

$$u_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{0mn} \cos(ax)\sin(by) \tag{3.38a}$$

$$v_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{0mn} \sin(ax)\cos(by) \tag{3.38b}$$

$$w_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{0mn} \sin(ax)\sin(by) \tag{3.38c}$$

$$\theta_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{xmn} \cos(ax)\sin(by) \tag{3.38d}$$

$$\theta_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{ymn} \sin(ax)\cos(by) \tag{3.38e}$$

where, u_{0mn}, v_{0mn}, w_{0mn}, θ_{xmn} and θ_{ymn} are the arbitrary parameters.

The load is expressed as:

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn} \sin(ax)\sin(by) \tag{3.39}$$

where, $a = m\pi/l$, $b = n\pi/b$.

Sinusoidal load can be expressed as follows:

$$Q_{mn} = q_0, \ (m = n = 1)$$

Whereas uniformly distributed load (UDL) is expressed as follows:

$$Q_{mn} = \frac{16q_0}{mn\pi^2}, \ (m = n = 1,3,5.....)$$

By substituting the assumed solutions Eq. (3.38) in the governing differential equations Eq. (3.26-3.30), set of equations are obtained which are written in the generalized displacement as follows.

$$\{k\}\{\Delta_i\} = \{q_i\} \qquad 3.40$$

where, k is considered as the globe stiffness matrix in which operators of partial derivative with respect to x and y is involved. $\{\Delta_i\}$ vector containing the field variables, $\{q_i\}$ the external mechanical, respectively

$$\left\{[k]_{5\times5}\right\}\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_x \\ \theta_y \end{Bmatrix}_{5\times1} = \begin{Bmatrix} 0 \\ 0 \\ q(\mathrm{x},\mathrm{y}) \\ 0 \\ 0 \end{Bmatrix}_{5\times1} \qquad 3.41$$

3.8.2.2 Solution of differential equation for free vibration

For the free vibration response of a CNTRC plate, the following displacement variables are taken into consideration since they have met the boundary conditions stated in Eq. (3.37). The potential energy on the system caused by external loads and external loads' potential loads are ignored in the free vibration analysis.

$$u_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{0mn} e^{i\omega t} \sin(ax)\cos(by) \quad 3.42a$$

$$v_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{0mn} e^{i\omega t} \sin(ax)\cos(by) \quad 3.42b$$

$$w_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{0mn} e^{i\omega t} \sin(ax)\sin(by) \quad 3.42c$$

$$\theta_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{xmn} e^{i\omega t} \cos(ax)\sin(by) \quad 3.42d$$

$$\theta_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{ymn} e^{i\omega t} \sin(ax)\cos(by) \quad 3.42e$$

where, ω is the free vibration frequency and $i = \sqrt{-1}$.

By substituting the assumed displacement variables in Eq. (3.42) in the governing differential equations Eq. (3.26-3.30), set of homogenous equations are obtained which were written in the form of eigen-value problem:

$$\left\{K - \omega^2 M\right\}\left\{\Delta_i\right\} = \left\{0\right\} \quad 3.43$$

Where, [K] is stiffness matrix and [M]is the mass matrix.

$$\left\{\left[K\right]_{5\times5} - \omega^2\left[M\right]_{5\times5}\right\}\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_x \\ \theta_y \end{Bmatrix}_{5\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{5\times1} \quad 3.44$$

Here, the element of the stiffness matrix is referring as [K] and mass matrix is referring as [M].

3.8.2.3. Buckling analysis

For the buckling analysis of the carbon nanotube reinforced composite plates resting on Pasternak's elastic foundation the kinetic energy of the system is neglected from the governing differential equations Eq. (3.26-3.30).

$$u_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{0mn} \cos(ax)\sin(by) \tag{3.45a}$$

$$v_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{0mn} \sin(ax)\cos(by) \tag{3.45b}$$

$$w_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{0mn} \sin(ax)\sin(by) \tag{3.45c}$$

$$\theta_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{xmn} \cos(ax)\sin(by) \tag{3.45d}$$

$$\theta_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{ymn} \sin(ax)\cos(by) \tag{3.45d}$$

where, u_{0mn}, v_{0mn}, w_{0mn}, θ_{xmn} and θ_{ymn} are the arbitrary parameters.

By substituting the assumed displacement variables given in Eq. (3.45) to the governing differential equations of Eq. (3.26-3.30), set of homogenous equations are obtained which were written in the form of eigen-value problem:

$$\{[K]-\lambda[G]\}\{\Delta_i\}=\{0\} \tag{3.46}$$

Where, $[K]$ is stiffness matrix and $[G]$is the geometric stiffness matrix.

$$\left\{[K]_{5\times5}-\lambda[G]_{5\times5}\right\}\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_x \\ \theta_y \end{Bmatrix}_{5\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{5\times1} \tag{3.47}$$

The above equations are established for the bending, free vibration and buckling analysis for the carbon nanotube reinforced composite plates resting on Pasternak's elastic foundation. These equations can be reduced to the bending, free vibration and buckling analysis for the carbon nanotube reinforced composite plates by neglecting the β_w and β_s which are the Winkler and shear layer spring constants factors.

3.9. Finite Element (FE) Formulation

The finite element method is the numerical solution based approach. The displacement of the primary variables at the mid plane of the carbon nanotubes reinforced composite plate is expressed using Eq. (3.13) which is associated with the C^1 continuity which make the FE solution more complex. In order to make the problem mathematically economical the C^1 continuity is modified to the C^0 continuity by imposing the new degrees of freedom to the system as $\left(\frac{\partial w_0}{\partial x} - \phi_x\right) = 0$ and $\left(\frac{\partial w_0}{\partial y} - \phi_y\right) = 0$

To reduce the continuity conditions of the transverse displacement in the present FE formulation, modified displacement field introduced after enforcing the constraint which is expressed in the following form:

$$\begin{bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{bmatrix} = \begin{bmatrix} u_0(x,y,t) \\ v_0(x,y,t) \\ w_0(x,y,t) \end{bmatrix} - z\begin{bmatrix} \beta_x \\ \beta_y \\ 0 \end{bmatrix} + f(z)\begin{bmatrix} \theta_x(x,y,t) \\ \theta_y(x,y,t) \\ 0 \end{bmatrix}$$

In the present formulation, the constraint conditions are enforced with the help of penalty functions. The discretized strain-displacement relations are first expressed in the following form.

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \\ 0 \\ 0 \end{Bmatrix} + z \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} + \Omega \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} + \Omega \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \Omega \frac{\partial \theta_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + \Omega \frac{\partial \theta_y}{\partial x} \\ 0 \\ 0 \end{Bmatrix} + g(z) \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \phi_y + \Omega \theta_y + \frac{\partial w_0}{\partial y} \\ \phi_x + \Omega \theta_x + \frac{\partial w_0}{\partial x} \end{Bmatrix}$$

$$+ g^{'}(z) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_y \\ \theta_x \end{Bmatrix}$$

3.48

Now the Eq. 3.48 can be further discretized and expressed in a matrix-vector form in which the components of the matrix are the mathematical functions of the thickness-coordinate and the components of the vector are the mid-plane derivatives of the primary variables.

$$\{\in\}_{5\times 1} = [H]_{(5\times 13)} \{\overline{\in}\}_{(13\times 1)} \qquad 3.49$$

The equation can be express in its full-scale form as follows:

The [H] matrix given in Eq. (3.49) can be expressed as follows:

$$[\mathrm{H}] = \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 & g(z) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & z & 0 & 0 & g(z) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & z & 0 & 0 & g(z) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & g^{'}(z) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & g^{'}(z) \end{bmatrix} \qquad 3.50$$

$$
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 & g(\mathrm{z}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & z & 0 & 0 & g(\mathrm{z}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & z & 0 & 0 & g(\mathrm{z}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \Omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \Omega \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \\ \dfrac{\partial \phi_x}{\partial x} + \Omega \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} + \Omega \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \Omega \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} + \Omega \dfrac{\partial \theta_y}{\partial x} \\ \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \\ \phi_y + \Omega \theta_y + \dfrac{\partial w_0}{\partial y} \\ \phi_x + \Omega \theta_x + \dfrac{\partial w_0}{\partial x} \\ \theta_y \\ \theta_x \end{Bmatrix} \qquad 3.51
$$

The mid-plane derivative variables in $\{\in\}$ can be further expressed in terms of the derivatives of the shape-functions and the generalized nodal coordinates by the following discretized relation.

$$\left\{ \begin{array}{c} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \\ \dfrac{\partial \phi_x}{\partial x} + \Omega \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} + \Omega \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \Omega \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} + \Omega \dfrac{\partial \theta_y}{\partial x} \\ \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \\ \phi_y + \Omega \theta_y + \dfrac{\partial w_0}{\partial y} \\ \phi_x + \Omega \theta_x + \dfrac{\partial w_0}{\partial x} \\ \theta_y \\ \theta_x \end{array} \right\} = [B]_{13x56} \{q\}_{56x1} \qquad 3.52$$

The relation between stresses and strain at any point within the domain of the carbon nanotubes reinforced composite plate can be established using the constitutive equations as follows:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{21} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix}_k \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}_k \qquad 3.53$$

$$\{\sigma\}_k = [Q_{ij}]_k \{\varepsilon\}_k$$
$$\{\sigma\}_{(5x1)} = [Q]_{(5x5)}[H]_{(5x13)}[B]_{(13x56)}\{d^e\}_{(56x1)} \qquad 3.54$$

where,

$$Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}},\ Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}},\ Q_{12} = \frac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}}, Q_{66} = G_{12},\ Q_{44} = G_{23},\ Q_{55} = G_{13}.$$

The discretized stress-strain relationships for the carbon nanotube reinforced composite plates is given by

$$\{\sigma\}_{(5x1)} = [Q]_{(5x5)}[H]_{(5x13)}[B]_{(13x56)}\{d^e\}_{(56x1)}$$

Which discretized the carbon nanotubes reinforced composite plate in the "*n*" number of parts using an eight nodded isoperimetric serendipity biquadratic quadrilateral element. The shape function associated with the eight nodded isoperimetric serendipity biquadratic quadrilateral element is as follows:

$$N_1 = \frac{1}{4}(1+\xi\xi_1)(1+\eta\eta_1)(\xi\xi_1+\eta\eta_1-1) \qquad 3.55$$

$$N_2 = \frac{1}{4}(1+\xi\xi_2)(1+\eta\eta_2)(\xi\xi_2+\eta\eta_2-1) \qquad 3.56$$

$$N_3 = \frac{1}{4}(1+\xi\xi_3)(1+\eta\eta_3)(\xi\xi_3+\eta\eta_3-1) \qquad 3.57$$

$$N_4 = \frac{1}{4}(1+\xi\xi_4)(1+\eta\eta_4)(\xi\xi_4+\eta\eta_4-1) \qquad 3.58$$

$$N_5 = \frac{1}{2}(1-\xi^2)(1+\eta\eta_5) \qquad 3.59$$

$$N_6 = \frac{1}{2}(1+\xi\xi_6)(1-\eta^2) \qquad 3.60$$

$$N_7 = \frac{1}{2}\left(1-\xi^2\right)\left(1+\eta\eta_7\right) \quad 3.61$$

$$N_8 = \frac{1}{2}\left(1+\xi\xi_8\right)\left(1-\eta^2\right) \quad 3.62$$

In the case of the trigonometric shear deformation theory based on secant function and inverse hyperbolic sine function, the total degrees of freedom increase from five to seven corresponding to each node after the addition of two new degrees of freedom. Thus further, the degree of freedom of the eight nodded isoperimetric serendipity biquadratic quadrilateral element goes to 56. Shape function and generalised nodal coordinates are used to transform the main variables associated with the problem.

$\Delta = \sum_{k=1}^{8} N_k \Delta_k$ Where, k is the node number 3.63

3.9.1 Hamilton's principle

The discretized governing equations of motion are derived in this section using Hamilton’s principle.

$$\int_{t_2}^{t_1} (\delta L)\,\mathrm{dt} = 0 \quad 3.64$$

The energy stores in the carbon nanotubes reinforced composite plate due to different parameters are derived and discusses in detail. To derive the equations of motion with Hamilton's principle, the variation in the strain energy, work potential and kinetic energy are required to be discretized.

3.9.1.1. Strain energy due to linear strains

The discretized equation for the variation in the strain energy store in the system due to linear strains is expressed as follows:

$$U = \frac{1}{2}\sum_{k=1}^{n}\iiint\{\sigma\}\{\in\}dxdydz$$ 3.65a

Substituting the discretized strain-displacement relations and the stress-strain constitutive relations we get,

$$U = \frac{1}{2}\sum_{k=1}^{n}\iiint\{\in\}^{T}\left[Q\right]_{k}\{\in\}dxdydz$$ 3.65b

$$U = \frac{1}{2}\sum_{k=1}^{n}\iiint\{[H]\{\Subset\}\}^{T}\left[Q\right]_{k}[H]\{\Subset\}dxdydz$$ 3.65c

$$U = \frac{1}{2}\sum_{k=1}^{n}\iiint\{\Subset\}^{T}[\mathrm{H}]^{T}\left[Q\right]_{k}[H]\{\Subset\}dxdydz$$ 3.65d

$$[D] = \int[\mathrm{H}]^{T}\left[Q\right]_{k}[H]dz$$ 3.65e

$$U = \frac{1}{2}\sum_{k=1}^{n}\iint\{[B]\{q\}\}^{T}[D][B]\{q\}dxdy$$ 3.65f

$$U = \frac{1}{2}\sum_{k=1}^{n}\iint\{q\}^{T}\{B\}^{T}[D][B]\{q\}dxdy$$ 3.65g

$$U = \frac{1}{2}\{q\}^{T}[K]\{q\}$$ 3.65h

where, [K] is the global elastic stiffness matrix. The strain energy stored in the system due to linear strains is the algebraic sum of the energy stored in each node due to every degree of freedom.

3.9.1.2. Strain energy due to non-linear strains

The discretized equation for the variation in the strain energy stored in the system due to non-linear strains is due to the geometry, in plane normal force, and in plane shear force.

$$U_{nl} = \frac{1}{2}\sum_{k=1}^{n}\iiint\{\sigma\}\{\in_{nl}\}dxdydz \tag{3.66a}$$

Substituting the discretized strain-displacement relations and the stress-strain constitutive relations we get,

$$U_{nl} = \frac{1}{2}\sum_{k=1}^{n}\iiint\{\in_{nl}\}^{T}\left[S\right]_{k}\{\in_{nl}\}dxdydz \tag{3.66b}$$

$$U_{nl} = \frac{1}{2}\sum_{k=1}^{n}\iiint\{[H_{nl}]\{\overline{\in}_{nl}\}\}^{T}\left[S\right]_{k}[H_{nl}]\{\overline{\in}_{nl}\}dxdydz \tag{3.66c}$$

$$U_{nl} = \frac{1}{2}\sum_{k=1}^{n}\iiint\{\overline{\in}_{nl}\}^{T}[\mathrm{H}_{nl}]^{T}\left[S\right]_{k}[H_{nl}]\{\overline{\in}_{nl}\}dxdydz \tag{3.66d}$$

$$[D_{nl}] = \int[\mathrm{H}_{nl}]^{T}\left[S\right]_{k}[H_{nl}]dz \tag{3.66e}$$

$$U_{nl} = \frac{1}{2}\sum_{k=1}^{n}\iint\{[B_{nl}]\{q\}\}^{T}[D_{nl}][B_{nl}]\{q\}dxdy \tag{3.66f}$$

$$U_{nl} = \frac{1}{2}\sum_{k=1}^{n}\iint\{q\}^{T}\{B_{nl}\}^{T}[D_{nl}][B_{nl}]\{q\}dxdy \tag{3.66g}$$

$$U_{nl} = \frac{1}{2}\{q\}^{T}[K_{nl}]\{q\} \tag{3.66h}$$

where, $[K_{nl}]$ is the global geometric stiffness matrix.

$$[S]_{k} = \begin{bmatrix} S_{xx} & S_{xy} \\ S_{xy} & S_{yy} \end{bmatrix}_{k} \tag{3.67}$$

The inplane normal forces (S_{xx} and S_{yy}) and inplane shear forces (S_{xy}) are represented in the term of single inplane normal force with different constant values.

$$[S]_{k} = \begin{bmatrix} \lambda_{1}S_{xx} & \lambda_{2}S_{xx} \\ \lambda_{2}S_{xx} & \lambda_{1}S_{xx} \end{bmatrix}_{k} \tag{3.68}$$

3.9.1.3. The variation in the strain energy of the elastic foundation

$$\delta U^{EF(\mathrm{e})} = \int_0^{I^{(\mathrm{e})}} \int_0^{b^e} \left(\left\{ \delta w_0 \quad \frac{\partial \delta w_0}{\partial x} \quad \frac{\partial \delta w_0}{\partial y} \right\} \begin{bmatrix} \beta_w & 0 & 0 \\ 0 & \beta_s & 0 \\ 0 & 0 & \beta_S \end{bmatrix} \left\{ \begin{matrix} w_0 \\ \frac{\partial w_0}{\partial x} \\ \frac{\partial w_0}{\partial y} \end{matrix} \right\} \right) dx^{(\mathrm{e})} dy^{(\mathrm{e})} \tag{3.71}$$

$$\delta U^{EF(\mathrm{e})} = \int_0^{I^{(\mathrm{e})}} \int_0^{b^{(\mathrm{e})}} (\{\delta \mathrm{W}_{EF}\}^T \{\mathrm{K}_{EF}\} \{\mathrm{W}_{EF}\}) dx^{(\mathrm{e})} dy^{(\mathrm{e})} \tag{3.72}$$

Where,

$$\{\mathrm{W}_{EF}\} = [\mathrm{B}_{EF}]\{\mathrm{q}^{(\mathrm{e})}\} \tag{3.73}$$

Where, the element of [B_{EF}] is as follow

$$[\mathrm{B}_{EF}] = \begin{bmatrix} 0 & 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & \partial N_i / \partial x & 0 & 0 & 0 \\ 0 & 0 & \partial N_i / \partial y & 0 & 0 & 0 \end{bmatrix} \tag{3.74}$$

The variation in the strain energy of the elastic foundation can be finally obtained as

$$\delta U^{EF(\mathrm{e})} = \left\{ \delta q^{(\mathrm{e})} \right\}^T \left(\int_0^{I^{(\mathrm{e})}} \int_0^{b^{(\mathrm{e})}} (\{\delta \mathrm{W}_{EF}\}^T \{\mathrm{K}_{EF}\} \{\mathrm{W}_{EF}\}) dx^{(\mathrm{e})} dy^{(\mathrm{e})} \right) \left\{ \delta q^{(\mathrm{e})} \right\} \tag{3.75}$$

And in the global form,

$$\delta U^{EF(\mathrm{e})} = \left\{ \delta q^{(\mathrm{e})} \right\}^T \left[K^{(\mathrm{F})^{(\mathrm{e})}} \right] \left\{ \delta q^{(\mathrm{e})} \right\} \tag{3.76}$$

3.9.1.4. Kinetic energy stored in carbon nanotubes reinforced composite plate

The kinetic energy stored for the k^{th} layer in the carbon nanotube reinforced composite plate.

$$T = \frac{1}{2}\sum_{k=1}^{n}\{q_k\}^T[M_k]\{q_k\} \qquad 3.77a$$

And in the global form, the total kinetic stored is

$$T = \frac{1}{2}\{q\}^T[M]\{q\} \qquad 3.77b$$

where, [M] is the global mass matrix. The kinetic energy stored in the system due to inertia is the algebraic sum of the function of inertia at each node.

3.9.1.5. Work done by the applied transverse load

$$W = \sum_{k=1}^{n} W_k \qquad 3.78$$

where,

$$W_k = \iint p(x,y)w\,dxdy \qquad 3.79$$

where,

$$\iint p(x,y)w\,dxdy = \{f\}_k\{q\}_k \qquad 3.80$$

$$\sum_{k=1}^{n}\{f\}_k\{q\}_k = \{F\}\{q\} \qquad 3.81$$

$$\{F\} = \begin{bmatrix} 0 & 0 & p_0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad 3.82$$

where, p_0 is the transverse load applied on the system.

3.9.1.6. Strain energy stored due to artificial constraints

Finally, all the discretized equations required for deriving the governing equations of motion using Hamilton's principle are derived. We are now only left with the satisfaction of the constraint equations using the penalty approach. In the penalty approach, a penalty function is created with the constraint equations and added to the total potential energy of an element. The strain energy stored due to the artificial constraints which are imposed on the system in order to make the problem mathematically economical and the C^1 continuity is changed to the C^0 continuity by imposing the new degrees of freedom to the system. The penalty parameter "γ" is considered as 10^6. The strain energy stored due to artificial constraints is as follow:

$$U_C = \frac{\gamma}{2}\sum_{k=1}^{n}\iiint\left[\left(\frac{\partial w_0}{\partial x}-\phi_x\right)_k^T\left(\frac{\partial w_0}{\partial x}-\phi_x\right)_k+\left(\frac{\partial w_0}{\partial y}-\phi_y\right)_k^T\left(\frac{\partial w_0}{\partial y}-\phi_y\right)_k\right]dxdydz \quad 3.69$$

And in the global form,

$$U_C = \frac{\gamma}{2}\{q\}^T[K_C]\{q\} \quad 3.70$$

where, $[K_C]$ is the global stiffness matrix due to artificial constraints

3.9.2 Governing equations

The governing equation for the bending, free vibration, and buckling analysis of carbon nanotubes reinforced composite plate resting on elastic foundation is develop using the Hamilton's principle. The equation consists of different terms which are required to develop the governing equation for the structural analysis of the plate.

$$\frac{\partial U}{\partial\{q_k\}}+\frac{\partial U^{EF(e)}}{\partial\{q_k\}}+\lambda\frac{\partial U_{nl}}{\partial\{q_k\}}+\frac{\partial}{\partial\{q_k\}}\left[\frac{\lambda}{2}\iiint\left(\left(\frac{\partial w_0}{\partial x}-\phi_x\right)_k^T\left(\frac{\partial w_0}{\partial x}-\phi_x\right)_k+\left(\frac{\partial w_0}{\partial y}-\phi_y\right)_k^T\left(\frac{\partial w_0}{\partial y}-\phi_y\right)_k\right)dxdydz\right]$$
$$+\frac{\partial W}{\partial\{q_k\}}+\frac{\partial T}{\partial\{q_k\}}=0$$

3.83

By incorporating the corresponding values, we get

$$[[K]+\gamma[K_c]]\{q\}+\lambda[K_G]+[M]\{\ddot{q}\}=\{F\} \quad 3.84$$

For the static analysis of carbon nanotubes reinforced composite plate resting on elastic foundation the Eq. (3.84) is transformed to Eq. (3.85) by neglecting the global geometrical stiffness matrix, global mass matrix and the inertia. The force vectors are also not time-dependent. The governing equations describing the bending responses of carbon nanotubes reinforced composite plate resting on elastic foundation is as follows:

$$[[K]+\gamma[K_c]]\{q\}=\{F\} \quad 3.85$$

For the free vibration analysis of carbon nanotubes reinforced composite plate the Eq. (3.34) is transformed to Eq. (3.86) by neglecting the global stiffness matrix and force vector. The equation for the free vibration analysis of carbon nanotubes reinforced composite plate is as follow:

$$\lambda[K]\{q\}+[M]\{\ddot{q}\}=0 \quad 3.86$$

For the buckling analysis of carbon nanotubes reinforced composite plate the Eq. (3.84) is transformed to Eq. (3.87) by neglecting the global mass stiffness matrix. The equation for the buckling analysis of carbon nanotubes reinforced composite plate is as follow:

$$\{[K]-\lambda[G]\}\{\Delta_i\}=\{0\} \quad 3.87$$

Eq. (3.85-3.87) cannot be solved now as the stiffness matrix is invertible due to the non-availability of the constraint conditions. The constraint conditions or the boundary conditions are required to be imposed on the system to remove the rigid-body motion which makes the matrices invertible. The boundary conditions of the problem are presented as follows for Simply-Supported boundary condition

For boundaries parallel to y axis, $x = 0, l$

$v = w = \beta_y = \theta_y = 0$

For boundaries parallel to x axis, $y = 0, b$

$u = w = \beta_x = \theta_x = 0$

After imposing the boundary conditions in Eq. 3.85, it is solved for the unknown field variables. The stresses and strains are then calculated with the results of the field variables at any desired location in the plate. The stresses are first evaluated at the gauss points and then extrapolated to the nodes with extrapolation functions (Cook et al., 2007). The nodes are shared by the adjacent elements in a FE mesh and the stresses at the common node from the adjacent element are not the same. Therefore, a nodal averaging technique is applied to get an average value of the stresses from the adjacent elements at the common nodes. Similarly, after imposing the boundary conditions, Eq. 3.86 is solved as an individual eigen-value problem in which the eigen-values denote the natural frequencies and the eigen vectors denote the mode shape of the vibration. Similarly, after imposing the boundary conditions, Eq. 3.87 is solved as an eigen-value problem in which the eigen-values denote the critical buckling load and the eigen vectors denote the mode shape of the buckling.

3.10 Material properties

The material properties used in the CNTRC plate and sandwich plates are listed in the tabular form. Different distributions of CNTs are considered for the analyses which are mentioned in Table 3.1 and Table 3.2. The material properties are discussed in the Table 3.3. The efficiency parameter η_i (where i=1, 2 and 3) for PMMA and PmVA (M1) for different volume fraction of CNTs which are given in Table 3.4 and Table 3.5 respectively.

The stacking sequences considered for the carbon nanotube reinforced sandwich plate is given in Table 3.6.

Table 3.1: Distribution relationship for volume fraction of CNTRC plates.

Distribution type	Distribution relationship
UD CNTRC plate	$V_{CNT} = V^*_{CNT}$
FG-V CNTRC plate	$V_{CNT} = \left(1 + \frac{2z}{h}\right) V^*_{CNT}$
FG-O CNTRC plate	$V_{CNT} = 2\left(1 - \frac{2z}{h}\right) V^*_{CNT}$
FG-X CNTRC plate	$V_{CNT} = 2\left(\frac{2\lvert z\rvert}{h}\right) V^*_{CNT}$

Table 3.2: Distribution relationship for volume fraction of CNTs in carbon nanotube reinforced sandwich plate.

Distribution type	Distribution relationship
UD CNTRC plate	$V_{CNT} = V^*_{CNT}$
FG-T CNTRC plate	$V_{CNT} = 2\left(\frac{z_1 - z}{z_1 - z_0}\right) V^*_{CNT}$
FG-B CNTRC plate	$V_{CNT} = 2\left(\frac{z - z_2}{z_3 - z_2}\right) V^*_{CNT}$
FG-V-C CNTRC plate	$V_{CNT} = 2\left(1 + \frac{\lvert z\rvert}{h}\right) V^*_{CNT}$
FG-V CNTRC plate	$V_{CNT} = \left(1 + \frac{2}{h^k}\left(z - \frac{z_t^k + z_b^k}{2}\right)\right) V^*_{CNT}$
FG-/\ CNTRC plate	$V_{CNT} = \left(1 - \frac{2}{h^k}\left(z - \frac{z_t^k + z_b^k}{2}\right)\right) V^*_{CNT}$
FG-X CNTRC plate	$V_{CNT} = \left(\frac{4}{h^k}\left\lvert z - \frac{z_t^k + z_b^k}{2}\right\rvert\right) V^*_{CNT}$
FG-O CNTRC plate	$V_{CNT} = 2\left(1 - \frac{2}{h^k}\left\lvert z - \frac{z_t^k + z_b^k}{2}\right\rvert\right) V^*_{CNT}$

Table 3.3: Material properties used for CNTRC plates.

Material	E_1 (GPa)	$E_2 = E_3$ (GPa)	$G_{12} = G_{12} = G_{12}$ (GPa)	$_{12} = \upsilon_{23} = \upsilon_{13}$	(Kg/m^3)
CNTs	5646.6	7080	1944.5	0.175	1400
PmPV(M2)	2.1	2.1	0.7835	0.34	1150
PMMA(M1)	2.5	2.5	0.7835	0.34	1150
T	105.7	105.7	40.97	0.29	4429

Table 3.4: CNT efficiency parameter which is affiliated to V_{CNT}^{*} for PMMA (M1)/CNT.

V_{CNT}^{*}	η_1	η_2	η_3
0.12	0.137	1.022	0.715
0.17	0.142	1.626	1.381
0.28	0.141	1.585	1.109

Table 3.5: CNT efficiency parameter which is affiliated to V_{CNT}^{*} for PmPV (M2)/CNT.

V_{CNT}^{*}	η_1	η_2	η_3
0.11	0.149	0.934	0.934
0.14	0.150	0.941	0.941
0.17	0.149	1.381	1.381

Table 3.6: Selected stacking sequences

Laminate	Normalized thickness	Lamina materials	Orientation
L1	0.25/0.5/0.25	FG(M1/CNT)/T/ FG(M1/CNT)	(0/Core/0)
L2	0.1667/0.6667/0.1667	FG(M1/CNT)/T/ FG(M1/CNT)	(0/Core/0)
L3	0.125/0.75/0.125	FG(M1/CNT)/T/ FG(M1/CNT)	(0/Core/0)
L4	0.25/0.5/0.25	UD(M2/CNT)/M2/ UD(M2/CNT)	(0/Core/0)
L5	0.25/0.5/0.25	FG(M2/CNT)/M2/ FG(M2/CNT)	(0/Core/0)
L6	0.25/0.5/0. 25	M2/UD(M2/CNT)/M2	(0/Core/0)
L7	0.25/0.5/0. 25	M2/FG(M2/CNT)/M2	(0/Core/0)

3.11. Non-dimensional parameter

The various non-dimensional parameters used for presenting the results are given below:

$$\bar{w}_1 = \frac{h^2 E^m w_1}{q_0 a^3}\left(\frac{a}{2},\frac{b}{2}\right); \quad \sigma_{xx}\frac{h^2}{a^2 q_0}\left(\frac{a}{2},\frac{b}{2},\frac{h}{2}\right); \quad \bar{\sigma}_{yz} = \sigma_{yz}\frac{h^2}{a^2 q_0}\left(0,0,\frac{h}{2}\right); \quad \bar{\varpi} = \omega\frac{a^2}{h}\sqrt{\frac{\rho^m}{E^m}}$$

$$D_0 = \frac{E^p h^3}{12[1-(\nu^p)^2]}; \; \bar{N}_{cr} = N_{cr}\frac{a^2}{\pi^2 D_0}$$

3.12. Summary

The goal of this chapter is to present the steps required for developing an analytical model and a FE model for the structural responses of functionally graded carbon nanotube reinforced composite and sandwich plate resting on a Pasternak elastic foundation in the framework of non-polynomial shear deformation theories based on secant function and inverse hyperbolic sine function. The elastic soil is modelled using the Pasternak's foundation model. In trigonometric shear deformation theory, the 3D displacements are expressed in terms of 2D deformation modes defined at the midplane and non-polynomial mathematical functions that are defined globally for the overall thickness of the plates like the ESL models. The non-polynomial higher order shear deformation theory based upon different non-polynomial trigonometric function is used for modelling the in-plane and transverse displacements of any point inside the plate. The non-polynomial trigonometric shear deformation theories used here contains non-polynomial shear strain functions such as "inverse hyperbolic sine function" and "secant function" to introduce the non-linearity of transverse shear stresses through thickness at the cost of less number of field variables with respect to the higher order shear deformation theories available in the literature which are generally of polynomial nature. First order shear deformation theory does not have the required deformation modes to model thick carbon nanotube reinforced sandwich plates and is usually preferred to study the thin ones where shear deformation is not dominant. While the higher-order deformation modes (membrane and bending) are present in the polynomial based higher order shear deformation theories, yet their inclusion is only possible with a large number of higher-order terms which increases computational costs. In non-polynomial shear deformation theories, the non-linearity of shear deformation is accommodated with the aid of a single non-polynomial function "secant function" and "inverse hyperbolic sine function" in the kinematic field. Hence, efficient results are obtained at the cost of lesser computational efforts. Next, the trigonometric shear deformation theory inherently satisfied the traction free conditions of transverse shear stresses at the top and bottom surfaces of the plate while in most of the polynomial based higher order shear deformation theory, this condition are generally not taken into consideration and in some cases, these conditions are artificially enforced. The assumptions made in the formulations along with the basic equations like the kinematic field, strain-displacement relations, reaction-deflection relationship of the foundation model and the stress-strain constitutive model for functionally graded carbon nanotube reinforced composite and sandwich plate resting on a Pasternak elastic foundation which form the basis of the present formulation are presented. Hamilton's principle is employed to form the governing equations and the solutions of the equations are carried out using Navier-based analytical method and FEM. Three classes of problems are mainly discussed like the bending, free vibration and buckling analysis for functionally graded carbon nanotube reinforced composite and sandwich plate resting on a Pasternak elastic foundation. A detailed discussion on the development of the governing equations for the above-mentioned problems and the solution strategies in the form of closed-form analytical and FE solutions is presented.