Chapter 2

Preliminaries

2.1 Notations

Throughout this thesis work, the symbol $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^{n\times n}$ is the set of all $n \times n$ real matrices. The sign $|\cdot|$ and $\|\cdot\|$ represent the Euclidean vector norm and induced matrix norm respectively. The matrix $P > 0$ (< 0) for $P \in \mathbb{R}^{n\times n}$ means that P is a symmetric positive (negative) definite matrix. The notations $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are the largest and smallest eigenvalues of the symmetric matrix P. The sign $Sym(A)$ represents the $A + A^T$ where A^T is the transpose of a matrix A. The symbol $*$ stands for symmetric off-diagonal terms in the matrix inequalities.
The signum function is defined as follows:

$$\text{sign}(z) = \begin{cases} -1 & \text{if } z(t) < 0 \\ [-1, 1] & \text{if } z(t) = 0 \\ 1 & \text{if } z(t) > 0 \end{cases}$$

The sign $\phi : [-h, 0] \to \mathbb{R}^n$ denotes the space of functions, which are absolutely continuous on $[-h, 0)$, have a finite $\lim\limits_{h\to 0^-} \phi(h)$. $\mathbb{W}$ is the square integrable first-order derivatives of ϕ defined as

$$\|\phi\|_{\mathbb{W}} = \max_{-h\le t\le 0} |\phi(h)| + \left(\int_{-h}^{0} |\dot{\phi}(s)|^2 ds \right)^{\frac{1}{2}}.$$

$\mathbb{L}_2([-h, 0]; \mathbb{R}^n)$ represents the space of square integrable functions $\phi : [-h, 0] \to \mathbb{R}^n$.

A continuous function $\alpha : [0, r] \to [0, \infty)$ is class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$. The function is class $\mathcal{K}_\infty$ if $r = \infty$ and $\alpha(s) \to \infty$ when $s \to \infty$. A function

$\beta : [0, \infty)$ is class $\mathcal{KL}$ if $\beta(\cdot, s)$ is class $\mathcal{K}$ for each fixed $s > 0$ and $\beta(r, \cdot)$ monotonically decreasing to zero for each fixed $r > 0$. $O(h)$ denote a matrix/scalar function of $h \in \mathbb{R}_+$ such that $\lim\limits_{h \to 0^+} \frac{|O(h)|}{h} = M$, where $M > 0$ is constant.

For a set $\left\{R_1 \quad R_2 \quad \cdots \quad R_N\right\}$ of matrices, we use diag$\left\{R_1 \quad R_2 \quad \cdots \quad R_N\right\}$ to denote the block diagonal matrix with R_is along the diagonal, and the matrix $\begin{bmatrix} R_1^T & R_2^T & \cdots & R_N^T \end{bmatrix}^T$ is denoted by col$\begin{bmatrix} R_1 & R_2 & \cdots & R_N \end{bmatrix}$. The Kronecker product of the matrices R_1 and R_2 is denoted by $R_1 \otimes R_2$.

We also denote $z_t(\theta) = z(t + \theta),\ \theta \in [-h, 0]$. The bounding ellipsoid is represented as $\mathcal{E}(W, \delta) = \left\{z(t) \in \mathcal{R}^n | z^T(t) W z(t) \leq \delta\right\}$. The ball of the radius r with the center at $z_0(t)$ is represented as

$$\mathcal{B}(z_0, r) := \{z(t) \in \mathbb{R}^n | \|z(t) - z_0(t)\| \leq r,\ z_0(t) \in \mathbb{R}^n, r > 0\}.$$

2.2 Norms

In many physical systems, disturbances are present inherently. Due to these disturbances, the system's stability is affected, and the performance of the system deteriorated. The robust control theory provides the control design techniques which eliminate the disturbances from the system. These control problems can be expressed in terms of norms of input and output signals.

Norm of a Vector Space [21]: A norm on a vector space V is defined as a non-negative function $||\cdot||$ that satisfies the following equations,

$$\begin{aligned} ||z(t)|| = 0 \ \text{ if and only if } z(t) = 0, \\ ||\alpha z(t)|| = |\alpha| \cdot ||z(t)||, \\ ||z(t) + y(t)|| \geq ||z(t)|| + ||y(t)||, \end{aligned} \tag{2.1}$$

for all $\alpha \in \mathbb{R}$ and $z(t), y(t) \in V$.

The theory of vector norms can be extended to signals. Some important signal norms are presented below.

$\mathcal{L}_1$-Norm: [21] The $\mathcal{L}_1$-norm of a signal is described as the integration of the absolute value of the signal,

$$||z(t)||_1 = \int_0^\infty |z(t)| dt. \tag{2.2}$$

$\mathcal{L}_2$-Norm: [21] The $\mathcal{L}_2$-norm of a signal is described as the integration of squared value of the signal,

$$||z(t)||_2 = \int_0^\infty |z(t)|^2 dt. \tag{2.3}$$

$\mathcal{L}_\infty$-Norm: [21] The $\mathcal{L}_\infty$-norm of a signal is described as the maximum absolute value of the signal,

$$||z(t)||_\infty = \sup_{t \geq 0} |z(t)|. \tag{2.4}$$

2.3 Time-delay Systems

Consider the following linear systems with a delay:

$$\begin{aligned} \dot{z}(t) &= Az(t) + A_d z(t-h), \\ z(t) &= \varphi(t), \quad t \in [-h, 0], \end{aligned} \tag{2.5}$$

where $z(t) \in \mathbb{R}^n$ is the state vector, $h > 0$ is a delay in the state of the system, $\varphi(t)$ is the initial condition and $A \in \mathbb{R}^{n \times n}$ and $A_d \in \mathbb{R}^{n \times n}$ are the system matrices. The future evolution of system trajectories depends not only on its present state, but also on its history. The primary methods of examining its stability can be classified as two types: frequency domain and time-domain.

Frequency-domain methods: Frequency-domain techniques provide the most simplest method of examining a system's stability without delay ($h = 0$). The necessary and sufficient conditions for the stability of such a system is $\lambda(A + A_h) < 0$. When $h > 0$, frequency-domain techniques yield the result that system (2.5) is stable if and only if all the roots of its characteristics equation,

$$f(\lambda) = det(\lambda I - A - A_d e^{-h\lambda}) = 0, \tag{2.6}$$

have negative real parts. However, this equation is inherent, which makes it challenging to solve. Furthermore, if the system has uncertainties and a time-varying delay, the solution is even more complicated. Therefore, frequency-domain techniques for the stability of time-delay systems have severe limitations.

Time-domain methods: Time-domain techniques are primarily based on two famous theorems: the Lyapunov-Krasovskii and the Lyapunov-Razumikhin theorem. These techniques were established in the 1950s by the Russian mathematician Krasovskii and Razumikhin, respectively. The original intention was to obtain a sufficient condition for the stability of the system (2.5) by constructing an appropriate Lyapunov-Krasovskii functional or a suitable Lyapunov function. The methods mentioned above are explained here in detail.

Consider a retarded functional differential equation

$$\dot{z}(t) = f(t, z_t), \tag{2.7}$$

where $z(t) \in \mathbb{R}^n$ is the state, $z_t = z(t+\theta), -h \leq \theta \leq 0, h > 0$ is the time-delay; $f(t, z_t) : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$, where $\mathcal{C}$ is the set of continuous functions mapping form $\mathbb{R}^n$ in the time-interval $t - h \leq t$ to $\mathbb{R}^n$. If the evaluation of $z(t)$ is sought at time instant $t \geq t_0$, then z_t for $-h \leq \theta \leq 0$, which therefore defines the initial conditions and is denoted as $z_{t_0} \in \mathcal{C}$.

Lyapunov-Krasovskii stability theorem [1, 109]. The system (2.7) is uniformly stable if there exists a continuous differentiable function $V(z_t), V(0) = 0$, such that

$$u(\|z(t)\|) \leq V(z_t) \leq v(\|z_t\|_{\mathcal{C}}) \tag{2.8}$$

and

$$\dot{V}(z_t) \leq -w(\|z(t)\|), \tag{2.9}$$

where u, v, w are continuous nondecreasing scalar functions with $u(0) = v(0) = w(0) = 0$ and $u(\alpha) > 0, v(\alpha) > 0, w(\alpha) \geq 0$ for $\alpha > 0$. If $w(\alpha) > 0$ for $\alpha > 0$, then it is uniformly asymptotically stable and if, moreover, $\lim_{\alpha \to \infty} u(\alpha) = \infty$, then it is globally uniformly asymptotically stable.

Lyapunov-Razumikhin stability theorem [1, 109]. The system (2.7) is uniformly stable if there exists a continuous differentiable function $V(z), V(0) = 0$, such that

$$u(\|z(t)\|) \leq V(z) \leq v(\|z_t\|_{\mathcal{C}}) \tag{2.10}$$

and

$$\dot{V}(z(t)) \leq -w(\|z(t)\|), \tag{2.11}$$

where u, v, w are continuous nondecreasing scalar functions with $u(0) = v(0) = w(0) = 0$ and $u(\alpha) > 0, v(\alpha) > 0, w(\alpha) \geq 0$ for $\alpha > 0$. If $w(\alpha) > 0$ for $\alpha > 0$, then it is uniformly asymptotically stable and if, moreover, $\lim_{\alpha \to \infty} u(\alpha) = \infty$, then it is globally uniformly asymptotically stable.

These two methods are extensively studied, and also the development of these techniques is described further. In both techniques, two types of sufficient conditions are broadly examined. The first type is delay-independent stability, and the second type is delay-dependent stability conditions.

Consider the Lyapunov-Krasovskii functional candidate

$$V_1(z_t) = z^T(t)Pz(t) + \int_{t-h}^{t} z^T(s)Qz(s)ds, \tag{2.12}$$

where $P > 0$ and $Q > 0$ are Lyapunov matrices and to be determined; and z_t denotes the translation operator acting on the trajectory: $z_t(\theta) = z(t + \theta)$ for some interval $[-h, 0]$. Then the derivative of $V_1(z_t)$ along the the solutions of system (2.5) is defined and limiting it to less than the zero yields the following delay-independent stability conditions of the system:

$$\begin{bmatrix} PA + A^T P + Q & PA_d \\ * & -Q \end{bmatrix} < 0. \tag{2.13}$$

Since the inequalities are linear concerning matrix variables P and Q, it is defined as an LMI. If it has feasible solution, then the Lyapunov-Krasovskii stability theorem provides asymptotic stability of the system (2.5) for all $h \geq 0$.

The delay-independent stability conditions have no information on delay and provide conservative results, especially when it is small. Another type of condition overcomes this conservativeness: delay-dependent conditions, which contain delay h. Since the solution of system (2.5) is continuous function of h, there must exist an upper bound, $\overline{h}$, on the delay such that the system (2.5) is stable for all $h \in [0, \overline{h}]$. Thus, the upper bound on the delay is the main criterion for judging a delay-dependent condition's conservativeness.

Next, the delay-dependent stability includes the addition of a quadratic double-integral term to the Lyapunov-Krasovskii functional (2.12)

$$V(z_t) = V_1(z_t) + V_2(z_t), \tag{2.14}$$

where

$$V_2(z_t) = \int_{-h}^{0} \int_{t+\theta}^{t} z^T(s) Z z(s) ds d\theta.$$

The derivative of $V_2(z_t)$ is

$$\dot{z}_t = hz^T(t)Zz(t) - \int_{t-h}^{t} z^T(s)Zz(s)ds. \tag{2.15}$$

Delay-dependent conditions can be obtained from the Lyapunov-Krasovskii stability theorem. However, dealing with the integral term of (2.15) is difficult. Three methods of studying delay-dependent problems have been studied: the discretized Lyapunov-Krasovskii functional method, fixed model transformations, and parametrized model transformations.

The discretized Lyapunov-Krasovskii functional techniques are used to find the linear systems and neutral systems stability with a constant delay [13, 15, 152]. This method estimates the maximum allowable delay, which guarantees the system's stability. However, it is not simple, straightforward, and cannot efficiently handle systems with time-varying delays. Thus, this method has not been extensively investigated or employed [13].

The following inequalities play an essential role in obtaining the stability conditions.

Basic inequality: $\forall a, b \in \mathbb{R}^n$ and $\forall R > 0$,

$$-2a^T b \leq a^T Ra + b^T R^{-1} b. \tag{2.16}$$

Park's inequality [153]: $\forall a, b \in \mathbb{R}^n$, $\forall R > 0$, and $\forall M \in \mathbb{R}^{n\times n}$,

$$-2a^T b \leq \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} R & RM \\ * & (M^T R + I)R^{-1}(RM + I) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{2.17}$$

Moon et al.'s inequality [154]: $\forall a \in \mathbb{R}^n_a$, $\forall b \in \mathbb{R}^n_b$, $\forall N \in \mathbb{R}^{n_a \times n_b}$, and for $\forall X \in \mathbb{R}^{n_a \times n_a}$, $\forall Y \in \mathbb{R}^{n_a \times n_b}$, and $\forall Z \in \mathbb{R}^{n_b \times n_b}$, if $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} \geq 0$, then

$$-2a^T Nb \leq \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} X & Y - N \\ * & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{2.18}$$

The fundamentals of the typical model transformations are explained in [155, 164].

2.4 Linear Matrix Inequalities (LMIs)

Many control theory problems such as stability analysis of the linear systems and controller design problems can be expressed in terms of LMIs [11]. This is due to the valuable properties of the LMIs, inventions in mathematical programming, and the development of practical algorithms and methods of using them to solve problems. Previously, Riccati equations and inequalities were used to represent and solve most control problems, but that involved many parameters, and symmetric positive definite matrices needed to be adjusted previously. So, even though a solution might exist, it might not necessarily be found. It's a significant disadvantage when dealing with real-world problems. LMIs do not suffer from these issues and require no modification of parameters.
An LMI is an expression of the form

$$E(z) = E_0 + z_1 E_1 + \cdots + z_m E_m < 0, \tag{2.19}$$

where $z_1, z_2, \cdots, z_m$ are real variables, and known as a decision variables of the LMI (2.19); $z = (z_1, z_2, \cdots, z_m)^T \in \mathbb{R}^m$ is a vector consisting of decision variables, which is called the decision vector; and $E_i = E_i^T \in \mathbb{R}^{n \times n},\ i = 0, 1, \cdots, m$ are given symmetric matrices.

The property of LMIs makes it possible to solve some LMI problems by methods commonly used to solve convex optimization problems.

2.4.1 Standard LMI Problems

This section describes three generic LMI problems for which the MATLAB LMI toolbox has solvers. Let E, F and G be symmetric matrix affine functions; and let c be a given constant vector.

LMI problem (LMIP): For the LMI $E(z) < 0$, the problem is to determine whether or not there exists an z^* such that $E(z^*) < 0$ satisfies. This is called a feasibility problem. That is, if there exists such an z^*, then the LMI is feasible; otherwise, it is infeasible.

Eigenvalue problem (EVP): The problem is to minimize the maximum eigenvalue of a matrix subject to an LMI constraint (or prove that the constraint is infeasible). The general form of an EVP is:

$$\begin{aligned} &\text{Minimize} \;\; \lambda \\ &\text{subject to} \;\; F(z) < \lambda I, \\ &\qquad\qquad\;\; G(z) < 0. \end{aligned}$$

EVPs can also appear in the equivalent form of minimizing a linear function subject to an LMI:

$$\begin{aligned} &\text{Minimize} \;\; c^T z \\ &\text{subject to} \;\; F(z) < 0. \end{aligned}$$

This is standard form for the EVP solver in the LMI toolbox.

The feasibility problem for the LMI $E(z) < 0$ can also be written as an EVP:

$$\begin{aligned} &\text{Minimize} \;\; \lambda \\ &\text{subject to} \;\; F(z) - \lambda I < 0. \end{aligned}$$

Clearly, for any z, if λ is selected large enough, (z, λ) is a feasible solution to the above problem. Thus, the problem certainly has a solution. If the minimum λ, λ^* satisfies $\lambda* \leq 0$, then the LMI $E(z) < 0$ is feasible.

Generalized eigenvalue problem (GEVP): The problem is to minimize the maximum generalized eigenvalue of a pair of affine matrix functions, subject to an LMI constraint. For two given symmetric matrices F and E of the same order and a scalar λ, if there exists a nonzero vector y such that $Fy = \lambda Ey$, then λ is called the generalized eigenvalue of matrices F and E.

The problem of finding the maximum generalized eigenvalue of F and E can be transformed into an optimization problem subject to an LMI constraint. Suppose that E is positive definite and that λ is a scalar. If λ is sufficiently large, $F - \lambda E < 0$. As λ decreses, $F - \lambda E < 0$ will become singular at some point. So there exists a nonzero vector y such that $Fy = \lambda Ey$. This λ is the generalized eigenvalue of matrices F and E. Using this idea, one can obtain the generalized eigenvalue of matrices F and E by solving the following optimization problem:

$$\begin{aligned} &\text{Minimize} \ \ \lambda \\ &\text{subject to} \ \ F - \lambda E < 0. \end{aligned}$$

If F and E are affine functions of z, the general form of the problem of minimizing the maximum generalized eigenvalue of the matrix functions $F(z)$ and $E(z)$ subject to an LMI constraint is

$$\begin{aligned} &\text{Minimize} \ \ \lambda \\ &\text{subject to} \ \ F(z) < \lambda E(z), \\ &\qquad\qquad\quad E(z) > 0, \\ &\qquad\qquad\quad G(z) < 0. \end{aligned}$$

It is important to note that, in this problem, the constraints are not linear in z and λ simultaneously.

2.5 Important Lemmas

This section presents essential lemmas, which are used in the upcoming chapters in order to get the desired results. These lemmas are common to chapters. For this reason, we have accumulated the lemmas here.

Lemma 1 *(Schur Complement [1]) For a given symmetric matix* $M = M^T = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, *where* $A \in \mathbb{R}^{r \times r}$, *the following conditions are equivalent:*

1. $M < 0$;
2. $A < 0, \quad C - B^T A^{-1} B < 0;$ *and*
3. $C < 0, \quad A - B C^{-1} B^T$.

Lemma 2 *[1] For given matrices* $Q = Q^T, H,$ *and* E *with appropriate dimensions,*

$$Q + HF(t)E + E^T F^T(t) H^T < 0$$

holds for all $F(t)$ satisfying $F^T(t)F(t) \leq I$ if and only if there exists $\epsilon > 0$ such that

$$Q + \epsilon^{-1}HH^T + \epsilon E^T E < 0.$$

Lemma 3 *[1] Consider $A, D, E, F,$ and Q to be real matrices with appropriate dimensions and assume $F^T F \leq I$ and $Q = Q^T > 0$. Then, the following propositions are true:*
(1) For any $x, y \in \mathbb{R}^n$,

$$-2x^T y \leq x^T Qx + y^T Q^{-1} y.$$

(2) For any $x, y \in \mathbb{R}^n$ and any $\epsilon > 0$,

$$2x^T DEFy \leq \epsilon^{-1} x^T DD^T x + \epsilon y^T E^T Ey.$$

(3) For any $\epsilon > 0$, satisfying $Q - \epsilon DD^T > 0$

$$(A + DEF)^T Q^{-1}(A + DEF) \leq \epsilon^{-1} E^T E + A^T (Q - \epsilon DD^T)^{-1} A.$$

Lemma 4 *(Jensen's Inequality [6]) Consider $L = \int_{\alpha}^{\beta} g(t)z(t)dt$, where $\alpha \leq \beta$, $g : [\alpha, \beta] \in [0, \infty)$, $z(t) \in \mathbb{R}^n$ and the integration concerned is well explained. Then, for any $n \times n$ matrix $R > 0$, the above inequality satisfies*

$$L' RL \leq \int_{\alpha}^{\beta} g(\rho)d\rho \int_{\alpha}^{\beta} g(t)z'(t)Rz(t)dt. \tag{2.20}$$

Lemma 5 *([43]) For a defined matrix $R > 0$, the given inequality satisfies for all continuously differentiable function $z(t)$ in $[a, b] \to \mathbb{R}^n$*

$$\int_a^b \dot{z}^T(s)R\dot{z}(s)ds \geq \frac{1}{b-a}(z(b) - z(a))^T R(z(b) - z(a)) + \frac{3}{b-a}\Phi^T R\Phi \tag{2.21}$$

where $\Phi = z(b) + z(a) - \frac{2}{b-a}\int_a^b z(s)ds$.

Lemma 6 *[42] Consider the autonomous system $\dot{z}(t) = f(z(t), z_h(t), d(t))$ with $z(t) \in \mathbb{R}^n$ and $z_h(t) \in \mathbb{R}^n$ is the system states, $z_h(t) = z(h + t), -h \leq t \leq 0$, $d(t) \in \mathbb{R}^m$ is the disturbance signal, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuously differential function. A normal Lyapunov-Krasovskii functional $V : \mathbb{R} \times \mathbb{W} \times \mathbb{L}_2([-h, 0]; \mathbb{R}^n) \to \mathbb{R}_+$ is said to be*

ISS Lyapunov-Krasovskii functional for the above system if there exists $\alpha_1, \alpha_2 \in \mathbb{K}_\infty$ *and* $\alpha, \xi \in \mathcal{K}$ *such that*

$$\begin{aligned} \alpha_1(|\phi(0)|) &\leq V(t, \phi, \dot{\phi}) \leq \alpha_2(\|\phi\|_{\mathbb{W}}), \\ V(t, \phi, \dot{\phi}) \geq \xi(|d(t)|) &\implies \dot{V}(t, \phi, \dot{\phi}) \leq -\alpha V(t, \phi, \dot{\phi}). \end{aligned}$$

then the autonomous system $\dot{z}(t) = f(z(t), z_h(t), d(t))$ *is uniform globally ISS with* $\gamma = \alpha_1^{-1} \circ \xi$.