Chapter 3

Construction of Matched Wavelets

3.1 Matched Wavelets

In many applications, e.g., signal coding and compression, it is desirable to have maximum correlation between the underlying signal and the analysis wavelet or the corresponding scaling function. This would result in general to a sparse discrete wavelet transform and provides better understanding of the signal structure. The signal-dependent wavelet design is referred to as matched wavelet. The wavelet decomposition with matched wavelet typically uses only one or two wavelet scales.

3.1.1 Motivation: Signal Detection

In a wavelet decomposition, a signal, $f(x)$, is decomposed into a sum of weighted wavelets

$$f(x) = \sum_{j=0}^{\infty} d_k^j \psi_{j,k}(x) \tag{3.1.1}$$

where the wavelet coefficients can be calculated as follows:

$$d_k^j = \langle f(x), \psi_{j,k}(x) \rangle \tag{3.1.2}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ are the wavelets.

In theory of signal detection, one approach for using the wavelet coefficients to detect patterns in a signal is designing a wavelet that matches the signal of interest. Then, one may look for the maximum energy coefficient in the wavelet coefficients at each scale to find the location of patterns.

The projection equation for the wavelet coefficients, given in eq. (3.1.2), can be rewritten in the frequency domain applying Parsevals identity

$$d_k^j = \langle f(x), \psi_{j,k}(x) \rangle = \langle F(\omega), \Psi_{j,k}(2^{-j}\omega) \rangle \tag{3.1.3}$$

where $\Psi_{j,k}(2^{-j}\omega) = 2^{-j/2}e^{-i2^{-j}\omega k}\Psi(2^{-j}\omega)$ is the Fourier Transform of $\psi_{j,k}(x)$. The energy of d_j^k at a particular scale, j_0, and translation, k_0, is given by its squared magnitude

$$|d_{k_0}^{j_0}|^2 = |\langle F(\omega), \Psi_{j_0,k_0}(2^{-j_0}\omega) \rangle|^2 \tag{3.1.4}$$

Applying the CauchySchwarz inequality to the right side of eq. (3.1.4) gives

$$|d_{k_0}^{j_0}|^2 = |\langle F(\omega), \Psi_{j_0,k_0}(2^{-j_0}\omega) \rangle|^2 \leq \langle F(\omega), F(\omega) \rangle \langle \Psi_{j_0,k_0}(2^{-j_0}\omega), \Psi_{j_0,k_0}(2^{-j_0}\omega) \rangle \tag{3.1.5}$$

where the equality holds for

$$F(\omega) = K\Psi_{j_0,k_0}(2^{-j_0}\omega) \tag{3.1.6}$$

where both F and Ψ are complex spectra. Therefore, $|d_{j_0}^{k_0}|^2$ is maximized when the complex frequency spectrum of ψ_{j_0,k_0} is identical to that of $f(x)$. Rewriting (3.1.6) in terms of amplitude and phase gives

$$|F(\omega)|e^{i\theta_F(\omega)} = K2^{-j_0/2}|\Psi(2^{-j_0}\omega)|e^{i(\theta_\psi(2^{-j_0}\omega) - 2^{-j_0}\omega k_0)} \tag{3.1.7}$$

where $\theta_F(\omega)$ and $\theta_\Psi(\omega)$ are the phase of $F(\omega)$ and $\Psi(\omega)$, respectively. The above equation (3.1.7) gives the condition for of maximum projection coefficient at (j_0, k_0). If the signal matches exactly with an orthonormal wavelet, then $d_k^j = \delta_{k,k_0} \cdot \delta_{j,j_0}$. The wavelet decomposition produces only one coefficient at (j_0, k_0). Even if this is not the case, that is, the match is not exact, most of energy of the signal will be captured in few coefficients. This result provides the need for a wavelet matched to a signal of interest in both magnitude and phase. The signal-dependent wavelet design is referred to as *matched wavelet*.

3.1.2 Some Algorithms to Design Matched Wavelets

In general, there are two criterions for matching the wavelet to signal, namely- energy matching and waveform matching. Energy matching means that the wavelet matches the signal by spectrum energy. On the other hand, waveform matching implies the similarity between the scaling (or wavelet) function and input signal. The more similar the scaling $\phi(t)$ or wavelet function $\psi(t)$ is to the input signal, the fewer the items needed to approach the input signal is. It means the similarity between the wavelet function and input signal need to be optimized. The former is used in other applications, such as signal denoising, where the spectrum energy of wavelet matching that of signal is required. The latter is used in some applications, such as image compression, where wavelet with high vanishing moment is required.

Chapa (1995); Chapa *et al.* (2000) have developed two sets of equations for designing a wavelet directly from a signal of interest. The first set derives expressions for continuous matched wavelet spectrum amplitudes, while the second set of equations provides a direct discrete algorithm for calculating close approximations to the optimal complex wavelet spectrum. Gupta *et al.* (2002) obtained the matched wavelet by maximizing the projection of the signal onto the scaling subspace. They proposed several other techniques to construct matched wavelets with some desired properties from a signal of interest (see Gupta *et al.*, 2003a;2003b;2005a;2005b). Misiti *et al.* (2003) approximated a given pattern using least squares optimization under constraints, leading to an admissible wavelet well suited for the pattern detection using continuous wavelet transform(CWT). Bahrampour *et al.* (2008, 2009) used variational methods to design wavelets matching to a specified signal. Mansour (2014) proposed a new construction technique for matched wavelet and matched scaling function that is based on a new parametrization of compactly supported orthonormal wavelets where the coefficients of the wavelet filter are the solution of a linear system of equations and are a continuous function of an arbitrary vector of half its length. Their proposed model provided a more general optimization framework where matched wavelets is a special case.

We have adopted the approach of Chapa *et al.* (2000) for design matched wavelets which is presented in the next section.

3.2 Matching a Wavelet to a Specified Signal by Chapa's Algorithm

In this section, we review the algorithm for designing wavelets matched to a specified signal, which was proposed by Chapa *et al.* (2000). In their technique, wavelets were assumed to be bandlimited. Using the conditions of an orthonormal multiresolution analysis (OMRA), they proposed an algorithm of matching of wavelets to the desired signal in frequency domain. The matching was done on the magnitude and phase independently of one another.

3.2.1 Orthonormal MRAs

For an MRA to be orthonormal: 1) $\{\psi_{j,k} : k \in \mathbb{Z}\}$ and $\{\phi_{j,k} : k \in \mathbb{Z}$ must be orthonormal bases of W_j and V_j, respectively;
2) At any scale j, V_j and W_j must be orthogonal, i.e., $W_j \perp V_j$
3) W_j's must be orthogonal, i.e., $W_j \perp W_k$, for $j \neq k$.
where ϕ is scaling function of MRA and ψ is the associated wavelet function. These requirements lead to the following conditions on ϕ and ψ:

$$\begin{aligned} \langle \phi_{j,k}, \phi_{j,m} \rangle &= \delta_{k,m} \\ \langle \phi_{j,k}, \psi_{j,m} \rangle &= 0 \\ \langle \psi_{j,k}, \psi_{l,m} \rangle &= \delta_{j,l} \cdot \delta_{k,m} \end{aligned} \tag{3.2.1}$$

Taking the Fourier transform of the first condition in eq. (3.2.1) gives,

$$\sum_{m=-\infty}^{\infty} |\Phi(\omega + 2\pi m)|^2 = 1. \tag{3.2.2}$$

This is called Poisson summation formula. Since $\phi(x) \in V_0 \subset V_1$ and $\psi(x) \in W_0 \subset V_1$, they can be expressed as a linear combination of the basis of V_1:

$$\phi(x) = 2 \sum_{k=-\infty}^{\infty} h_k \phi(2x - k)$$

$$\psi(x) = 2 \sum_{k=-\infty}^{\infty} g_k \phi(2x - k)$$

In the frequency domain the above equations are given as

$$\Phi(\omega) = H\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right) \tag{3.2.3}$$

$$\Psi(\omega) = G\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right) \tag{3.2.4}$$

where $H(\omega)$ and $G(\omega)$ are the Fourier transforms of sequences h_k and g_k, the quadrature mirror filters (QMF) for OMRA. Note that both $H(\omega)$ and $G(\omega)$ are 2π-periodic.

For Orthonormal MRAs, we must have

$$|H(\omega)|^2 + |G(\omega)|^2 = 1 \tag{3.2.5}$$

$$H(\omega)\overline{H(\omega+\pi)} + G(\omega)\overline{G(\omega+\pi)} = 0 \tag{3.2.6}$$

Taking $g_k = (-1)^k h_{1-k}$ guarantees eq. (3.2.6) to be satisfied always. Taking the Fourier transform of above relation gives

$$G(\omega) = e^{-i\omega}\overline{H(\omega+\pi)} \tag{3.2.7}$$

3.2.2 Finding the scaling function from a wavelet

In order to derive an expression for $|\Phi|$ in terms of $|\Psi|$, the following conditions are required:

$$\Phi(0) = 1 \tag{3.2.8}$$

$$\langle \phi_{j,k}, \phi_{j,m} \rangle = \delta_{k,m} \tag{3.2.9}$$

$$\phi(x) = 2\sum_{k=-\infty}^{\infty} h_k \phi(2x-k) \tag{3.2.10}$$

Others conditions are required for $\phi(x)$ to generate an orthonormal MRA. Substituting eq. (3.2.3) and eq. (3.2.4) into eq. (3.2.5) gives a relationship between $|\Phi(\omega)|$ and $|\Psi(\omega)|$ in the context of an MRA:

$$|\Phi(\omega)|^2 = |\Psi(2\omega)|^2 + |\Phi(2\omega)|^2$$

Since the matching algorithm is performed on sampled data (in the frequency domain), there is a need to develop an equation for finding the sampled scaling function spectrum from the sampled wavelet spectrum.

Theorem 3.2.1 (Finding $|\Phi(k)|$ from $|\Psi(k)|$) *In an orthonormal MRA, let $\Phi(k\triangle\omega)$ and $\Psi(k \triangle \omega)$ be the sampled scaling function and wavelet spectra, respectively, with sample spacing $\triangle\omega = \pi/2^l$. Any sample of $|\Phi|$ at $\omega = k\pi/2^l$ can be expressed by the following recursive equation:*

$$\left|\Phi\left(\frac{\pi k}{2^l}\right)\right|^2 = \left|\Phi\left(\frac{\pi k}{2^{l-1}}\right)\right|^2 + \left|\Psi\left(\frac{\pi k}{2^{l-1}}\right)\right|^2 \quad \text{for } k \neq 0 \tag{3.2.11}$$

which leads to the following closed form solution

$$\left|\Phi\left(\frac{\pi k}{2^l}\right)\right|^2 = \sum_{p=0}^{l}\left|\Psi\left(\frac{\pi k}{2^{p-1}}\right)\right|^2 \quad \text{for } k \neq 0. \tag{3.2.12}$$

3.2.3 Properties of Matching Spectrum Amplitude

The next step is to derive constraints on $|\Psi|$ that are necessary and sufficient to guarantee $\phi_{j,k}$ is an orthonormal basis of V_j

Theorem 3.2.2 (Guaranteeing Orthonormality) *The following condition on $|\Psi|$ is both necessary and sufficient to guarantee that $\langle\phi_{j,k}, \phi_{j,m}\rangle = \delta_{k,m}$, where $\phi_{j,k}$ is derived from $|\Phi(\omega)|$:*

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\left|\Psi\left((2^{n+1}\omega + 2\pi m)\right)\right|^2 = 1. \tag{3.2.13}$$

While eq. (3.2.12) and eq. (3.2.13) provide a method for deriving an orthonormal function,$\phi(x)$, from a given wavelet, there is no guarantee that $\phi(x)$ generates an MRA. In order for $\phi(x)$ to generate an MRA, it must satisfy its 2-scale relation, which given in the frequency domain is

$$\Phi(\omega) = H\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right) \tag{3.2.14}$$

Repeated substitution of this recursive equation gives

$$\Phi(\omega) = H\left(\frac{\omega}{2}\right)H\left(\frac{\omega}{4}\right)H\left(\frac{\omega}{8}\right)\cdots \tag{3.2.15}$$

Eq. (3.2.15) indicates that there must be a certain structure in ϕ in order for it to be a scaling function. Incorporating the infinite product into the conditions developed thus far would be very difficult. A simple way to guarantee that eq. (3.2.15) is satisfied is to assume $\Phi(\omega)$ is bandlimited. However, one is not free to choose any bandlimits.

Theorem 3.2.3 (Bandlimited Φ and Ψ) *In a multiresolution analysis, the spectrum of a bandlimited scaling function, $\Phi(\omega)$, has maximum support given by*

$$|\omega| \leq \frac{4\pi}{3}$$

. In an orthonormal MRA with a bandlimited scaling function, the corresponding orthonormal wavelet has a maximum bandlimit of

$$\frac{2\pi}{3} \leq |\omega| \leq \frac{8\pi}{3}$$

From eq. (3.2.4), it is clear that a bandlimited scaling function that generates an orthonormal MRA gives rise to a bandlimited wavelet.

In order to complete the groundwork for the spectrum amplitude matching algorithm, a set of equations for sampled spectra, is needed.

Theorem 3.2.4 (Guaranteeing an Orthonormal MRA) *Let $Y(k) = |\Psi(k \triangle \omega)|^2, k \in \mathbb{Z}$, where $\triangle\omega = 2\pi/2^l$. The necessary and sufficient condition on Y to guarantee that $|\Phi(n)|$, found in Theorem 3.2.1, generates an orthonormal MRA is given as follows,:*

$$\sum_{m=-\infty}^{\infty} \sum_{p=0}^{l} Y\left(\frac{2^l}{2^p}(k + 2^{l+1}m)\right) = 1 \tag{3.2.16}$$

where

$$2^l/3 < |\frac{2^l}{2^p}(k + 2^{l+1}m)| < 2^{l+2}/3 \tag{3.2.17}$$

Because the wavelets being designed are assumed to be real, the magnitude of the wavelet spectrum is even, $|\Psi(\omega)| = |\Psi(-\omega)|$, and only the spectra for positive frequency indices, k, in the passband need be matched. The conditions in Theorem 3.2.4 for $k > 0$ generate a set of L linear equality constraints in $Y(k)$ of the form

$$\sum_{i=1}^{L} \alpha_{ik} Y(k) = 1 \quad \text{for } k = \{\lceil 2^l/3 \rceil, \cdots, \lfloor 2^{l+2}/3 \rfloor\} \tag{3.2.18}$$

where $\alpha_{ik} \in \{0, 1, 2\}$ The condition in eq. (3.2.18) can be expressed in vector notation as

$$\mathbf{AY} = \mathbf{1} \tag{3.2.19}$$

where $\mathbf{A}$ is a $L \times 2^l$ matrix given by

$$\mathbf{A} = \{\alpha_{ij} \in \{0, 1, 2\} : i = 1, \cdots, L; j = 1, \cdots, 2^l\}$$

and $\mathbf{1}$ is a $L \times 1$ vector given by $\mathbf{1} = \{1 \quad 1 \cdots 1\}$

3.2.4 Matching Spectrum Amplitude

By virtue of the bandlimits given in Theorem 3.2.3, the desired signal must be dilated in such a way that the energy in this passband is a maximum. This dilated spectrum, $F(\omega)$, is the starting point for the matching algorithm.

Theorem 3.2.5 (Matched Wavelet Amplitude) *Let $\mathbf{W}$ and $\mathbf{Y}$ be vectors containing the samples of $|F(k \bigtriangleup \omega)|^2$ and $|\Phi(k \bigtriangleup \omega)|^2$, respectively, in the passband, i.e.,*

$$\mathbf{W} = \{|F(k \bigtriangleup \omega)|^2; k = \lceil 2^l/3 \rceil, \cdots, \lfloor 2^{l+2}/3 \rfloor\}$$

$$\mathbf{Y} = \{|\Phi(k \bigtriangleup \omega)|^2; k = \lceil 2^l/3 \rceil, \cdots, \lfloor 2^{l+2}/3 \rfloor\}$$

where $F(\omega)$ is the spectrum of the dilated signal for which we desire a matched wavelet and $\Psi(\omega)$ is the matched wavelet spectrum. If the error to be minimized is given by

$$E = \frac{(\mathbf{W} - a\mathbf{Y})^T(\mathbf{W} - a\mathbf{Y})}{\mathbf{W}^T\mathbf{W}}$$

then the optimal wavelet power spectrum is given by the following expression:

$$\mathbf{Y} = \frac{1}{a}\mathbf{W} + \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\left(\mathbf{1} - \frac{1}{a}\mathbf{A}\mathbf{W}\right) \tag{3.2.20}$$

where

$$a = \frac{\mathbf{1}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{W}}{\mathbf{1}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{1}} \tag{3.2.21}$$

where $(\mathbf{A}\mathbf{A}^T)$ is full rank. The match error is given by

$$E = \frac{\left(\mathbf{1} - \frac{1}{a}\mathbf{A}\mathbf{W}\right)^T(\mathbf{A}\mathbf{A}^T)^{-1}\left(\mathbf{1} - \frac{1}{a}\mathbf{A}\mathbf{W}\right)}{\frac{1}{a^2}\mathbf{W}^T\mathbf{W}}.$$

The resultant wavelet is orthonormal, and the scaling function it generates by way of eq.(3.2.12)generates an orthonormal MRA.

3.2.5 Matching Spectrum Phase

The next set of equations provides a closed form solution for finding group delay (negative derivative of the phase) of the matched wavelet from the group delay of the signal of interest. First we will develop an expression for the group delay of $\Psi(\omega)$ in terms of the group delay of the scaling function spectrum, $\Phi(\omega)$. We have

$$\Psi(2\omega) = e^{-i\omega}\frac{\overline{\Phi(2\omega + 2\pi)}}{\overline{\Phi(\omega + \pi)}}\Phi(\omega), \tag{3.2.22}$$

therefore the phase Ψ is given by

$$\theta_\Psi(\omega) = -\frac{\omega}{2} - \theta_\Phi(\omega + 2\pi) + \theta_\Phi(\frac{\omega}{2} + \pi) + \theta_\Phi(\frac{\omega}{2}) \tag{3.2.23}$$

where $\theta_\Psi(\omega)$ and $\theta_\Phi(\omega)$ are the phases of Ψ and Φ, respectively. Denoting the negative of the group delays as Λ_Ψ and Λ_Φ, setting $\Gamma_\Psi(\omega) = \Lambda_\Psi(\omega) + 1/2$ we get

$$\Gamma_\Psi(\omega) = -\Lambda_\Phi(\omega + 2\pi) + \frac{1}{2}\Lambda_\Phi(\frac{\omega}{2} + \pi) + \frac{1}{2}\Lambda_\Phi(\frac{\omega}{2}) \tag{3.2.24}$$

Next, an expression for the group delay of $\Psi(\omega)$ is obtained in terms of the group delay of $H(\omega)$, which we will denote as $\lambda(\omega)$.

By repeated substitutions of equations in (3.2.3), (3.2.4) and (3.2.7) we get the following infinite products:

$$\Psi(\omega) = \prod_{m=1}^{\infty} H\left(\frac{\omega}{2^m}\right) \tag{3.2.25}$$

$$\Psi(\omega) = e^{-i\frac{\omega}{2}}\overline{H\left(\frac{\omega}{2} + \pi\right)} \prod_{m=2}^{\infty} H\left(\frac{\omega}{2^m}\right) \tag{3.2.26}$$

where $H(\omega)$ is 2π-periodic.

Taking the phase of both sides gives

$$\theta_\Phi = \sum_{m=1}^{\infty} \theta_H\left(\frac{\omega}{2^m}\right) \theta_\Psi = -\frac{\omega}{2} - \theta_H\left(\frac{\omega}{2} + \pi\right) + \sum_{m=2}^{\infty} \theta_H\left(\frac{\omega}{2^m}\right) \tag{3.2.27}$$

Taking derivative of both sides the negative of the group delays are found as follows:

$$\Lambda_\Phi(\omega) = \sum_{m=1}^{\infty} 2^{-m}\lambda\left(\frac{\omega}{2^m}\right) \tag{3.2.28}$$

$$\Gamma_\Psi(\omega) = \Lambda_\Psi(\omega) + 1/2 = -\frac{1}{2}\lambda\left(\frac{\omega}{2} + \pi\right) + \sum_{m=2}^{\infty} 2^{-m}\lambda\left(\frac{\omega}{2^m}\right) \tag{3.2.29}$$

where $\Lambda_\Phi(\omega) = \frac{d\theta_\Phi(\omega)}{d\omega}$, $\lambda\Psi(\omega) = \frac{d\theta_\Psi(\omega)}{d\omega}$, $\lambda(\omega) = \frac{d\theta_H(\omega)}{d\omega}$ are 2π-periodic. Moreover, they are even functions. Due to additional periodicity constraint on $\lambda(\omega)$, the group delays of desired signal and wavelet cannot be matched directly. To solve this problem, one period of $\lambda(\omega)$ is modelled as $\lambda_T(\omega)$ and expressed as a polynomial of even order R,

$$\lambda_T(\omega) = \sum_{r=0}^{R/2} c_r\omega^{2r} \prod\left(\frac{\omega}{2\pi}\right) \tag{3.2.30}$$

where

$$\prod(\omega) = \begin{cases} 1 & \text{if } -1/2 \leq \omega < 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{3.2.31}$$

Therefore we can construct $\lambda(\omega)$ by replicating $\lambda_T(\omega)$ every 2π,

$$\lambda(\omega) = \sum_{k=-\infty}^{\infty} \lambda_T(\omega - 2\pi k) = \sum_{k=-\infty}^{\infty} \sum_{r=0}^{R/2} c_r(\omega - 2\pi k)^{2r} \prod \left(\frac{\omega - 2\pi k}{2\pi} \right) \quad (3.2.32)$$

Let N be the number of samples in $F(n \bigtriangleup \omega)$ with sampling frequency $\bigtriangleup\omega = 2\pi/T$, then the discrete form of eq. (3.2.32) is given by

$$\lambda(n) = \sum_{r=0}^{R/2} c_r \sum_{k=-P/2}^{P/2-1} (n - kT)^2 \prod \left(\frac{n - kT}{T} \right) \quad (3.2.33)$$

where $P = N/T$ is the number of periods over N samples and $-N/2 \leq n < N/2$. In vector notation

$$\lambda = Bc \quad (3.2.34)$$

where the elements of B are given as

$$b_{n,r} = \sum_{k=-P/2}^{P/2-1} (n - kT)^2 \prod \left(\frac{n - kT}{T} \right). \quad (3.2.35)$$

Note that λ is a $N \times 1$ vector, B is a $N \times (R/2 + 1)$ matrix and c is a $(R/2 + 1) \times 1$ vector. Substituting eq. (3.2.34) in eq. (3.2.29) and eq. (3.2.28) we get matrix equations for Γ_Ψ and Λ_Φ

$$\Gamma_\Psi = D_\Psi c \Lambda_\Phi = D_\Phi c \quad (3.2.36)$$

where

$$D_\Psi = -\frac{1}{2} B_{(q+T)/2} + \sum_{m=2}^{\infty} 2^{-m} B_{(q/2^m)} \quad (3.2.37)$$

and

$$D_\Phi = \sum_{m=1}^{\infty} 2^{-m} B_{(q/2^m)} \quad (3.2.38)$$

Now, the expression for Γ_Ψ will be derived such that it is closest to the desired signal's group delay,Γ_F, in a least squares sense. Let γ be the unweighted error to be minimized,

$$\gamma = \sum_{n=-N/2}^{N/2-1} (\Gamma_F(n) - \Gamma_\Psi(n))^2. \quad (3.2.39)$$

Since the wavelet phase need only match that of the desired signal in the passband, weight the error function by a normalized weighting function. Let $\Omega(n) = Y(n)/\sum Y(n)$, where $Y(n)$ are the elements of $\mathbf{Y}$ obtained in Theorem 3.2.5. The weighted error function becomes

$$\gamma_\Omega = \sum_{n=-N/2}^{N/2-1} [\Omega(n)(\Gamma_F(n) - \Gamma_\Psi(n))]^2 \quad (3.2.40)$$

Rewriting (3.2.40) in vector notation gives

$$\gamma = (\overline{\Gamma_F} - \overline{D_\Psi}c)^T(\overline{\Gamma_F} - \overline{D_\Psi}c) \tag{3.2.41}$$

where $\overline{\Gamma_F}$ and $\overline{D_\Psi}$ are obtained by taking the non-zero entries only in weighted Γ_F and D_Ψ. Setting $\bigtriangledown_c\gamma = 0$, the vector,$\hat{c}$, which minimizes γ, is found to be equal to

$$\hat{c} = (\overline{D_\Psi}^T\overline{D_\Psi})^{-1}\overline{D_\Psi}^T\overline{\Gamma_F} \tag{3.2.42}$$

With this best estimate of c, the group delay of the wavelet can be obtained as

$$\Gamma_\Psi = D_\Psi\hat{c}. \tag{3.2.43}$$

Also, the best estimates of $\lambda, \Lambda_\Psi, \Lambda_\Phi$ can be calculated from

$$\lambda = B\hat{c} \tag{3.2.44}$$

$$\Lambda_\Psi = \left(D_\Psi\hat{c} - \overline{D_\Psi\hat{c}}\right) - \frac{\triangle\omega}{2} \tag{3.2.45}$$

$$\Lambda_\Phi = D_\Phi\hat{c} - \overline{D_\Phi\hat{c}} \tag{3.2.46}$$

where $\overline{D_\Psi\hat{c}}$ and $\overline{D_\Phi\hat{c}}$ are means of $D_\Psi\hat{c}$ and $D_\Phi\hat{c}$, respectively. Both Λ_Φ and Λ_Ψ can be summed to obtain the discrete phases of Φ and Ψ that when combined with the magnitudes from Theorem 3.2.5 give the full estimate of $\Phi(n\triangle\omega)$ and $\Psi(n\triangle\omega)$ which satisfy all conditions for an orthonormal MRA. The QMF filters, h and g, corresponding to the matched wavelet and its scaling function can be found using eq. (3.2.3) and (3.2.4) and the inverse Fourier Transform.

3.3 Example

In this section, the performance of the magnitude and phase matching algorithm is demonstrated by an example. We take Daubechies wavelet (Db4) as desired signal, f_D (shown in Figure 3.1). The matched wavelet is obtained by taking the inverse Fourier transform of its discrete complex spectra $\Psi(n) = \sqrt{Y(n)}e^{i\theta_\Psi(n)}$. In order to use eq. (3.2.20) to find $Y(n)$, the values of $\mathbf{W}$, $\mathbf{A}$ and a are required.

In this example, we set $N = 512$, $\triangle\omega = 2\pi/16$ so that $l = 4$. With value of $l = 4$, the non-zero frequency indices in eq. (3.2.18) are $k = \{6, 7, \cdots, 21\}$. The constraints in eq. (3.2.16) and (3.2.17) in Theorem 3.2.4 generate $L = 11$ in 16 unknowns, resulting in constraint matrix $\mathbf{A}$.

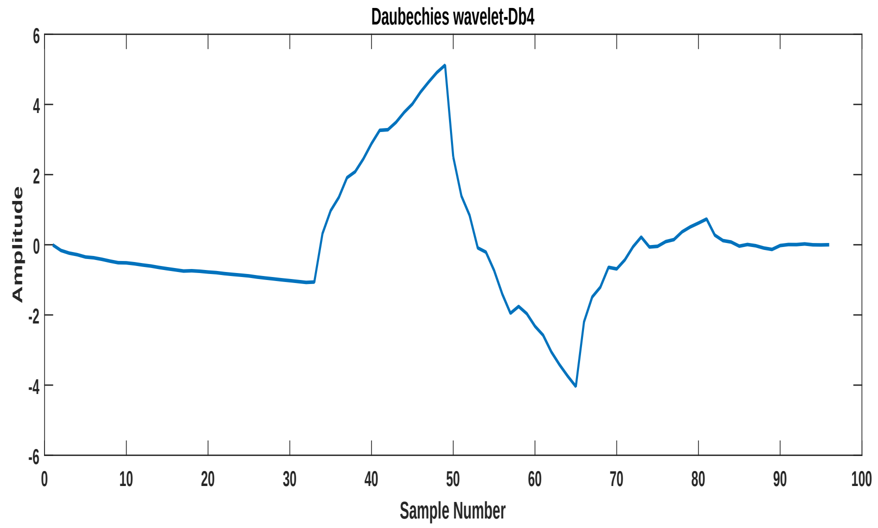

Figure 3.1: Daubechies Wavelet-Db4

The first step of the matching algorithm is to dilate the signal such that there is a maximum amount of energy in the wavelet passband. The passband taken in this example was $2\pi/3 \leq |\omega| \leq 8\pi/3$ to ensure orthonormal wavelets. We have also used suitable zero-padding to the signal to get its spectrum in the passband.

The desired signal power spectrum, $W(n) : n = -256, \cdots, 255$, is given as

$$W(n) = \begin{cases} |F_D(n \triangle \omega)|^2 & \text{for } |n| = \{6, 7, \cdots, 21\} \\ 0 & \text{otherwise.} \end{cases}$$

where $F_D(n \triangle \omega)$ is the Fourier transform of f_D. $Y(k)$ is found using eq. (3.2.20) and (3.2.21) where $W = \{W(k) : k = 6, 7, \cdots, 21\}$. The results of the spectrum match in the passband are shown in Figure (3.2). The full matched wavelet spectrum is constructed by reflecting $Y(k)$ onto the negative axis, and taking its square root.

The first step in finding the matched phase is to find the group delay of the desired signal,Γ_F, which is done using the following process:

1. Calculate $F_D^\theta(n \triangle \omega) = F_D(n \triangle \omega)/|F_D(n \triangle \omega)|$.

2. Interpolate across samples of $F_D^\theta(n \triangle \omega)$ where $|F_D(n \triangle \omega)| = 0$.

3. $\Lambda_F = | \triangle^1 F_D^\theta(n \triangle \omega)|$ where $\triangle^1$ is the first difference operator.

Next, the matrix D_Ψ is calculated from eq. (3.2.37). We have taken $N = 512$ and $R = 16$ for this example. Therefore, D_Ψ is a 512×9 matrix. The polynomial coefficient vector,

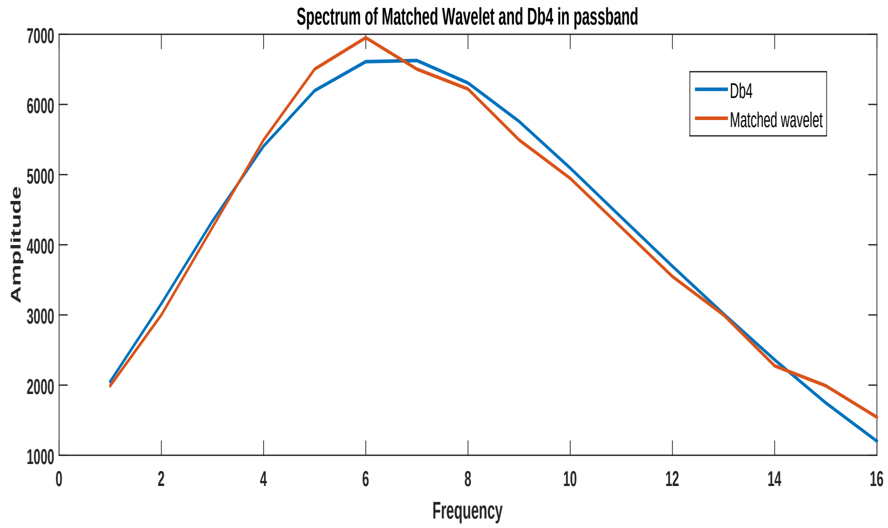

Figure 3.2: Spectrum match of matched wavelt and Db4 in passband

$\hat{c}$, is calculated using eq. (3.2.42) where $\overline{D_\Psi}$ and $\overline{\Lambda_F}$ are weighted by the normalized matched spectrum, Y, calculated above. The group delay of matched wavelet function is calculated from eq. (3.2.45). The matched wavelet phase $\theta_\Psi(n)$ is found by integrating (or summing) $\Lambda_\Psi(n)$. The matched wavelet function are each found by taking the inverse Fourier Transform of its complex spectra. The desired signal (Daubechies wavelet) with its constructed matched wavelet function is shown in Figure 3.3. Few values of filter coefficient g is given in Table (3.1).

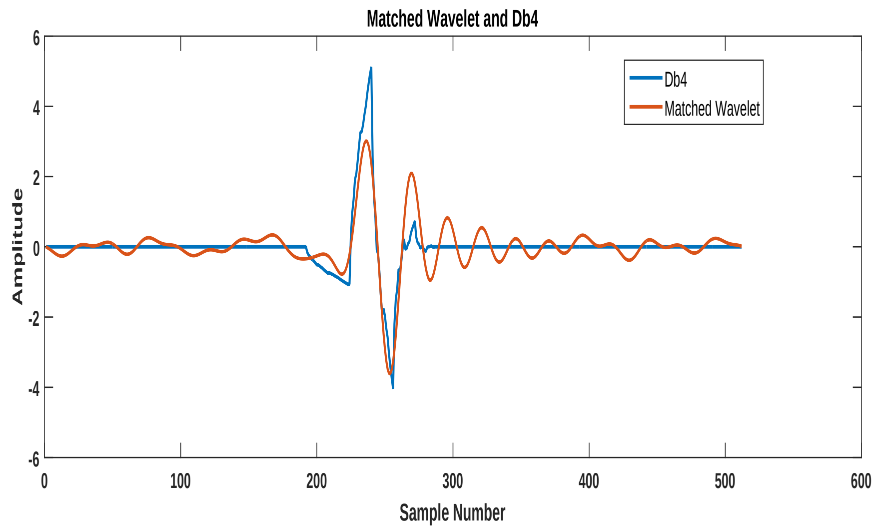

Figure 3.3: Wavelet Matched to Db4 wavelet in passband

k	g(k)	k	g(k)
-8	0.0018	0	-0.0024
-7	0.0015	1	0.0013
-6	-0.0031	2	-0.0012
-5	-0.0037	3	0.0008
-4	0.0124	4	-0.0005
-3	-0.0136	5	-0.0003
-2	0.0061	6	-0.0002
-1	0.0005	7	0.0006

Table 3.1: $g(k)$ for the matched wavelet for D4