# Chapter 2

# The Continuous Wavelet Transform in n-dimensions

## 2.1 Introduction

Wavelets are a simple mathematical tool having exciting applications in signal and numerical analysis. The signal processing of two dimensional video images and the digital three dimensional scans of solid objects are done using multidimensional wavelets. Daubechies (1992) gave several methods to obtain wavelets in higher dimensions. There are $n$ dimensional wavelets that are of separable types in the sense that they can be written as a product of $n$ one dimensional wavelets. From one dimensional Haar wavelet $\psi(x)$, one can easily construct a $n$-dimensional wavelet of separable type by

$$\psi(x) = \psi(x_1)\psi(x_2)...\psi(x_n)$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. But there are multidimensional wavelets which can not be separated. The wavelet

$$\psi(x) = \begin{cases} x_1 x_2 \cdots x_n \, e^{-\frac{1}{(1-\|x\|^2)}} & , \|x\| < 1 \\ 0 & , \|x\| \geq 1 \end{cases}$$

This chapter is based on the following accepted paper: Pandey, J. N., Jha, N. K., and Singh, O. P., The Continuous Wavelet Transform in n-dimensions, ***International Journal of Wavelets, Multiresolution and Information Processing***, 2016.

where $\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ , is a $n$-dimensional wavelet of non-separable type belonging to the Schwartz testing function space $\mathcal{D}(\mathbb{R}^n)$.

The wavelet transform is used to break data, functions or operators into different frequency components, and then analyse each of these components with resolution matched to its scale. During the last two decades several authors [Antoine *et al* (2006); Chui (1992); Daubechies (1992); Meyer (1992); Pandey *et al.* (2015); Pathak (2009); Walter and Shen (2009); Walter (1995)] have worked on multidimensional wavelet transform and its inversion formulae. A function $\psi \in L^2(\mathbb{R}^n)$ is a basic wavelet if it satisfies the admissibility condition

$$\int\limits_{\mathbb{R}^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty, \tag{2.1.1}$$

where $\hat{\psi}(\omega) = (\mathcal{F}\psi)(\omega)$ is the Fourier transform of $\psi$. The Fourier transform of $f \in L^2(\mathbb{R}^n)$ is defined by

$$\begin{aligned}(\mathcal{F}f)(\omega) &= \underset{N\to\infty}{\text{l.i.m.}} \left(\frac{1}{(2\pi)}\right)^{n/2} \int_{-N}^{N} f(t)e^{-i\omega.t}\, dt_1\, dt_2 \cdots dt_n \\ &= \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(t)e^{-i\omega.t}\, dt. \quad \text{(Akhiezer and Glazman , 1966, p.75)} \qquad (2.1.2)\end{aligned}$$

Here, $N = (N_1, N_2, \cdots, N_n)$, $N \to \infty$ implies that each of the components of $N$ tends to $\infty$ independently of each other. The limit in mean (l.i.m.) describes the convergence in $L^2(\mathbb{R}^n)$. It is known that if $f \in L^2(\mathbb{R}^n)$ then $(\mathcal{F}f)(\omega)$ also belongs to $L^2(\mathbb{R}^n)$ [see Akhiezer and Glazman (1966); Bogess and Narcowich (2001); Halmos (1967); Keinert (2003); Treves (1967); Walter and Shen (2009)]. The Fourier transform operator is an isometric homeomorphism from $L^2(\mathbb{R}^n)$ to itself with the inverse operator defined as

$$\mathcal{F}^{-1}\hat{f} = \frac{1}{(2\pi)^{n/2}} \underset{N\to\infty}{\text{l.i.m.}} \int_{-N}^{N} \hat{f}(\omega)e^{+i(\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n)} d\omega_1 d\omega_2 \ldots d\omega_n = f(t).$$

From the function $\psi \in L^2(\mathbb{R}^n)$ satisfying the admissibility condition (2.1.1), a doubly-indexed family of wavelets from $\psi$ is generated by dilating and translating as

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right),$$

where $a, b \in \mathbb{R}^n, a \neq 0$ (i.e. $a_i \neq 0$ for $i = 1, 2, \ldots, n$), $|a| = |a_1 a_2 \ldots a_n|$, and

$$\psi\left(\frac{x-b}{a}\right) \equiv \psi\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \ldots, \frac{x_n-a_n}{a_n}\right).$$

The normalization has been chosen so that $\|\psi_{a,b}\| = 1$, for all $a, b$. We also assume $\|\psi\| = 1$. The continuous wavelet transform with respect to this family of wavelets is defined as

$$(W_\psi f)(a,b) = |a|^{-1/2} \int_{\mathbb{R}^n} f(x)\, \overline{\psi\left(\frac{x-b}{a}\right)}\, dx. \tag{2.1.3}$$

For one dimensional case the inversion formula for (2.1.3) is given as [see Halmos (1967); Meyer (1992)]

$$f = C_\psi^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\psi f)(a,b) \psi_{a,b} \frac{dbda}{a^2}, \quad a \neq 0, \tag{2.1.4}$$

where $C_\psi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(y)|^2}{|y|} dy$.

There are several possible extensions of the above inversion formula (2.1.4) with $n > 1$. (Daubechies , 1992, see p. 33) choose the wavelet $\psi \in L^2(\mathbb{R}^n)$ so that it is spherically symmetric and obtained the inversion formula as

$$f = C_\psi^{-1} \int_0^\infty \frac{da}{a^{n+1}} \int_{\mathbb{R}^n} db (W_\psi f)(a,b) \psi_{a,b}, \tag{2.1.5}$$

with $\psi_{a,b}(x) = a^{-n/2} \psi\left(\frac{x-b}{a}\right)$, $a \in \mathbb{R}_+, a > 0$, and $b \in \mathbb{R}^n$. The convergence in (2.1.5) is interpreted in the $L^2(\mathbb{R}^n)$ sense.

In this chapter we prove the following more general inversion formula for (2.1.3)

$$f(x) = C_\psi^{-1} \int_{\mathbb{R}^n} \frac{da}{|a|^2} \int_{\mathbb{R}^n} db (W_\psi f)(a,b) \psi_{a,b}(x), \tag{2.1.6}$$

where $a, b \in \mathbb{R}^n, a \neq 0$. Though the inversion formula (2.1.6) is similar to (2.1.5) it is more general in the sense that: (i) in equation (2.1.5), $a$ is restricted to real number greater than zero but in (2.1.6), $a \in \mathbb{R}^n$ with none of the components vanishing, (ii) convergence in (2.1.6) is in the sense of convergence in $L^2(\mathbb{R}^n)$ like in (2.1.5) but at points $x$ where both $f(x)$ and $\psi_{a,b}(x)$ are continuous, the inversion formula (2.1.6) also converges pointwise, i.e.,

$$\frac{1}{C_\psi} \int_{\mathbb{R}^n} \frac{da}{|a|^2} \int_{\mathbb{R}^n} db (W_\psi f)(a,b) \psi_{a,b}(x_0) = f(x_0).$$

Pointwise convergence to the wavelet series expansion was done by Walter (1995) and the pointwise convergence for the inversion formula for continuous wavelet transform in one dimension was done by (Chui , 1992, see pp. 62-63). He however assumes the continuity of the function $f$ at the point $x$ and the continuity of $\psi\left(\frac{x-b}{a}\right)$ at the point $x$ which due to the fact that $a \neq 0$,$b$ are arbitrary real numbers implies the continuity of

$\psi(x), \forall x \in \mathbb{R}$. In our proof of the pointwise convergence of the $n-$dimensional inversion formula, we have assumed the continuity of $f$ at the point $x$ and $\psi \in L^2(\mathbb{R}^n)$, i.e., $\psi$ need not be continuous at the point $x$.

## 2.2 The Main Result

We begin this section with a definition.

**Definition 2.2.1** *A function $\psi \in L^2(\mathbb{R}^n)$ is said to be a window function if each of $x_i\psi(x),\ x_ix_j\psi(x), x_ix_jx_k\psi(x), \cdots, x_ix_jx_k\cdots x_l\psi(x), \cdots, x_1x_2\cdots x_n\psi(x)$ belongs to $L^2(\mathbb{R}^n)$ and two or more than two lower suffixes appearing in a term are all different.*

Clearly $e^{-(x_1^2+x_2^2+\cdots x_n^2)}$ is a window function. A window function $\psi(x_1, x_2, \cdots x_n)$ which also satisfies the condition that

$$\int_{-\infty}^{\infty} \psi(x_1, x_2, \cdots x_i, x_{i+1}, \dots x_n)\, dx_i = 0, \forall i = 1, 2, \cdots n$$

is a function satisfying admissibility condition and so is a wavelet. Therefore, $\psi(x_1, x_2, x_3) = x_1x_2x_3e^{-(x_1^2+x_2^2+x_3^2)}$ is a wavelet Pandey *et al.* (2015); Pathak (2009); Strang and Nguyen (1997).

We now state and prove the main theorem.

**Theorem 2.2.2** *Let $\psi$ be a wavelet in $L^2(\mathbb{R}^n)$ satisfying the admissibility condition, then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W_\psi f(a,b)\overline{W_\psi g(a,b)}\,\frac{da}{|a^2|}db = C_\psi\langle f, g\rangle \tag{2.2.1}$$

*for all $f, g \in L^2(\mathbb{R}^n)$ and*

$$f(x) = \frac{1}{C_\psi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (W_\psi f)(a,b)\psi_{a,b}(x)\frac{da}{|a^2|}db, \tag{2.2.2}$$

*in $L^2(\mathbb{R}^n)$ sense, where $C_\psi = (2\pi)^n \int_{\mathbb{R}^n} \frac{|\hat{\psi}(y)|^2}{|y|}dy$. However, if $f$ and $\psi_{a,b}$ are continuous in $\mathbb{R}^n$, then the convergence in (2.2.2) is pointwise for all $x \in \mathbb{R}^n$.*

The following two lemmas will be used to prove Theorem 2.2.2.

All the variables $t$, $x$, $\omega$, $a$ and $b$ are $n$ dimensional with real components. The variable $a \neq 0$ means that no component of $a$ assumes zero value.

**Lemma 2.2.3** *Let $f, \psi \in L^2(\mathbb{R}^n)$ and assume that $\psi$ is an $n$-dimensional wavelet. Then*

$$\int_{\mathbb{R}^n} f(t)\bar{\psi}\left(\frac{t-b}{a}\right) dt = |a|\, \overline{\mathcal{F}[\bar{\hat{f}}(\omega)\hat{\psi}(a\omega)]}(2\pi)^{n/2}. \tag{2.2.3}$$

**Proof.**

$$\begin{aligned}
\int_{\mathbb{R}^n} f(t)\bar{\psi}\left(\frac{t-b}{a}\right) dt &= \langle f(t), \psi(\frac{t-b}{a})\rangle_t \\
&= \langle \mathcal{F}f(t), \mathcal{F}\psi(\frac{t-b}{a})\rangle_\omega \\
&= \langle \hat{f}(\omega), |a|\hat{\psi}(a\omega)e^{-i\omega b}\rangle_\omega \\
&= |a| \left[\overline{\mathcal{F}[\bar{\hat{f}}\hat{\psi}(a\omega)](b)}\right](2\pi)^{n/2}.
\end{aligned}$$

In the second equality we used the fact that the Fourier transform is an isometric homeomorphism on $L^2(\mathbb{R}^n)$.

This Fourier transform is a function of $b$ here

$$= |a|\overline{(\mathcal{F}\,F(\omega)}(b)(2\pi)^{n/2},$$

where

$$F(\omega) = \overline{\hat{f}(\omega)}\hat{\psi}(a\omega).$$

**Lemma 2.2.4** *Let $g$ and $\psi$ both belong to the space $L^2(\mathbb{R}^n)$ and $\psi$ be a wavelet in $\mathbb{R}^n$ as considered in lemma 2.2.3. Then*

$$\begin{aligned}
\int_{\mathbb{R}^n} \overline{g(s)}\psi\left(\frac{s-b}{a}\right) ds &= |a|\mathcal{F}[\bar{\hat{g}}\hat{\psi}(a\omega)](2\pi)^{n/2} \\
&= |a|\mathcal{F}(G)(2\pi)^{n/2}
\end{aligned}$$

*where $G = \bar{\hat{g}}(\omega)\hat{\psi}(a\omega)$ is a function of $\omega$. So the parameter for the Fourier transformation of $F$ and $G$ is $b$ here.*

**Proof.** Our proof is very similar to the proof of lemma 2.2.3.

$$\begin{aligned}
\int_{\mathbb{R}^n} \overline{g(s)}\psi\left(\frac{s-b}{a}\right) ds &= \langle \psi\left(\frac{s-b}{a}\right), g(s)\rangle \\
&= \langle \mathcal{F}\psi\left(\frac{s-b}{a}\right), \hat{g}(\omega)\rangle \\
&= \langle |a|\hat{\psi}(a\omega)e^{-i\omega b}, \hat{g}(\omega)\rangle \\
&= |a| \int_{\mathbb{R}^n} \overline{\hat{g}(\omega)}\hat{\psi}(a\omega)e^{-i\omega b}dw = |a|\mathcal{F}G(2\pi)^{n/2}
\end{aligned}$$

where $G = \overline{\hat{g}}(\omega)\hat{\psi}(a\omega)$.

We now proceed to prove our main theorem.

**Proof.**[Proof of Theorem 2.2.2] Let $\psi$ be a basic wavelet and let $W_\psi$ be the corresponding integral wavelet transform. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (W_\psi f)(a,b)\overline{(W_\psi g)(a,b)}\,\frac{dadb}{a^2} = C_\psi\langle f, g\rangle \tag{2.2.4}$$

for all $f, g \in L^2(\mathbb{R}^n)$.

Let us define function $F$ and $G$ as follows:

$$\begin{aligned} F(x) &= \overline{\hat{f}}(x)\hat{\psi}(ax) \\ G(x) &= \overline{\hat{g}}(x)\hat{\psi}(ax). \end{aligned} \tag{2.2.5}$$

Now we proceed to prove the main theorem as follows:

$$\begin{aligned} &\int_{\mathbb{R}^n} (W_\psi f)(a,b)\overline{(W_\psi g)(a,b)}db \\ &= \frac{1}{|a|}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}dt\int_{\mathbb{R}^n}\overline{g(s)}\psi\left(\frac{s-b}{a}\right)ds\right\}db. \end{aligned} \tag{2.2.6}$$

Here

$$|a| = |a_1a_2\ldots a_n|, \psi\left(\frac{t-b}{a}\right) = \psi\left(\frac{t_1-b_1}{a_1}, \frac{t_2-b_2}{a_2}\cdots\frac{t_n-b_n}{a_n}\right).$$

So the integral (2.2.6) now is

$$\begin{aligned} &= \frac{|a|^2}{|a|}(2\pi)^n\int_{\mathbb{R}^n}\left[\overline{\mathcal{F}(F)}(b)\mathcal{F}(G)(b)\right]db \quad \text{in view of lemma 1 and lemma 2.} \\ &= \frac{|a|^2}{|a|}(2\pi)^n\langle\mathcal{F}(G)(b), \mathcal{F}(F)(b)\rangle \\ &= \frac{a^2}{|a|}(2\pi)^n\langle G(x), F(x)\rangle \\ &= \frac{a^2}{|a|}(2\pi)^n\int_{\mathbb{R}^n} G(x)\bar{F}(x)dx \\ &= \frac{a^2}{|a|}(2\pi)^n\int_{\mathbb{R}^n}\overline{\hat{g}(x)}\hat{\psi}(ax)\hat{f}(x)\overline{\hat{\psi}(ax)}dx \\ &= \frac{a^2}{|a|}(2\pi)^n\int_{\mathbb{R}^n}\hat{f}(x)\overline{\hat{g}}(x)|\hat{\psi}(ax)|^2dx. \end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W_\psi f(a,b)\overline{W_\psi g(a,b)}\,\frac{dadb}{a^2} &= (2\pi)^n \int_{\mathbb{R}^n} \langle \hat{f}(x), \overline{\hat{g}}(x)\rangle \frac{|\hat{\psi}(ax)|^2}{|a|} da \\
&= (2\pi)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(x)\overline{\hat{g}}(x)\left[\frac{|\hat{\psi}(ax)|^2}{|a|} da\right] dx \\
&= (2\pi)^n \langle f, g\rangle \int_{\mathbb{R}^n} \frac{|\hat{\psi}(ax)|^2}{|a|} da \\
&= (2\pi)^n \langle f, g\rangle \int_{\mathbb{R}^n} \frac{|\hat{\psi}(y)|^2}{|y|} dy \\
&= \langle f, g\rangle C_\psi \quad \text{where } C_\psi = (2\pi)^n \int_{\mathbb{R}^n} \frac{|\hat{\psi}(y)|^2}{|y|} dy.
\end{aligned}$$

**Corollary 2.2.5** *Let $f$, $g$, $\psi$ and $C_\psi$ be as defined in Theorem 2.2.2, then*

$$f(x) = \frac{1}{C_\psi} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (W_\psi f)(a,b)\psi_{a,b}(x)\frac{dadb}{a^2} \tag{2.2.7}$$

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right), \quad |a| = |a_1 a_2 \dots a_n|.$$

The equation (2.2.4) with careful computation can be brought to the form

$$\left\langle \frac{1}{C_\psi} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (W_\psi f)(a,b)\frac{1}{\sqrt{(a)}}\,\psi\left(\frac{x-b}{a}\right)\frac{dadb}{a^2},\, g \right\rangle = \langle f, g\rangle. \tag{2.2.8}$$

Hence (2.2.7) follows.

The topology over the space $\mathbb{R}^n$ is described by the norm

$$|x| \equiv \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}, \quad x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n.$$

The same topology can as well be generated by the norm

$$\|x\|_1 = \max(|x_1|, |x_2|, \dots |x_n|)$$

for

$$\|x\|_1 \le \|x\| \le n\|x\|_1.$$

$[x : \|x\| \le 1]$ is a closed unit sphere in $\mathbb{R}^n$ with radius 1 and the centre at the origin and $[x : \|x\|_1 \le 1]$ is a closed unit $n$-cube with centre at the origin and each of the sides of length 2 and sides being parallel to the coordinate axes. We will call this also as a closed unit $n$-cube (centre at $x = 0$).

It is easy to figure out that the closed unit $n$-cube contains the closed unit sphere in $\mathbb{R}^n$.

**Corollary 2.2.6** *Let $f$ and $\psi$ both belong to $L^2(\mathbb{R}^n)$. Assume that $f$ is continuous at a point $x \in \mathbb{R}^n$ and $\psi$ is continuous for all $x \in \mathbb{R}^n$. Then,*

*(i) The convergence in Theorem 2.2.2 takes place in the pointwise sense at the point $x$.*

*(ii) If however $f$ and $\psi$ both belong to $L^2(\mathbb{R}^n)$ and both are continuous for all $x \in \mathbb{R}^n$, then the convergence in Theorem 2.2.2 takes place in the pointwise sense for all $x \in \mathbb{R}^n$. We will only prove (i) as (ii) is an immediate consequence of (i).*

**Proof.** Let $\{x_m\}_{m=1}^{\infty}$ be a sequence in $\mathbb{R}^n$ converging to the point $x_0 = (x_0^1, x_0^2, \dots x_0^n)$ and $x_m = (x_m^1, x_m^2, \dots x_m^n)$ i.e. as $m \to \infty$, $\|x_m - x_0\|_1 \to 0$. We construct one such sequence $\{x_m\}_{m=1}^{\infty}$ as follows

$$\|x_m - x_0\|_1 \leq \frac{1}{2^n m}, \quad m = 1, 2, \dots$$

$$x_m = \left(x_0^1 + \frac{1}{2^n m}, x_0^2 + \frac{1}{2^n m}, \dots + x_0^m + \frac{1}{2^n m}\right).$$

Since $f$ is continuous at $x_0$, so for $\varepsilon > 0 \,\exists\, \delta > 0$ so that $|f(x) - f(x_0)| < \varepsilon$, whenever $\|x - x_0\|_1 < \delta$. Consider $\{g_m(x)\}_{m=1}^{\infty}$ is a sequence of functions in $L^2(\mathbb{R}^n)$ such that

$$g_m(x) \quad = \quad \begin{cases} m^n & \text{if } \|x_m - x_0\|_1 \leq \frac{1}{2^n m^n} \\ 0 & \text{if } \|x_m - x_0\|_1 > \frac{1}{2^n m^n} \end{cases}.$$

So, clearly,

$$\int_{\mathbb{R}^n} g_m(x) dx = 1$$

$$\langle f(x), g_m(x) \rangle = \int_{\mathbb{R}^n} f(x) \bar{g}_m(x) dx = \int_{\mathbb{R}^n} (f(x) - f(x_0)) g_m(x) dx + f(x_0),$$

$$\bar{g}_m(x) = g_m(x),$$

Now choose $m > 0$ so large that $\frac{1}{2^n m^n} < \delta$.

Therefore,

$$\begin{aligned} |\langle f(x), g_m(x) \rangle - f(x_0)| &\leq \varepsilon \int_{\mathbb{R}^n} |g_m(x)| dx \\ &\leq \varepsilon \int_{\mathbb{R}^n} g_m(x) dx \\ &\leq \varepsilon. \end{aligned}$$

Since $\varepsilon$ is arbitrary and $f$ is continuous at $x_0$ it follows that

$$\lim_{m \to \infty} \langle f(x), g_m(x) \rangle = f(x_0) \tag{2.2.9}$$

Now assume that for fixed $a \neq 0$ and $b \in \mathbb{R}^n$, $\psi\left(\frac{x-b}{a}\right)$ is a continuous function for all $x \in \mathbb{R}^n$. Replace $g$ by $g_m(x)$ in (2.2.8). In the region $x, b \in \mathbb{R}^n$ and $\|a\| > \varepsilon$ the switch in the order of integration in (2.2.8) with respect to $x$ and $b, a$ is justified in view of Fubini's theorem. So integration with $db\, da\, dx$ is put in the form $dx\, db\, da$ and then we let $\varepsilon \to 0$. Thereafter letting $m \to \infty$ in the left hand side expression of (2.2.8) we have

$$\begin{aligned}
&\frac{1}{C_\psi} \lim_{m\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_\psi f)(a,b) \int_{-\infty}^{\infty} g_m(x)\psi_{a,b}(x) dx \frac{dbda}{a^2} \\
&= \frac{1}{C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_\psi f)(a,b)\psi_{a,b}(x_0)\frac{dbda}{a^2}
\end{aligned} \tag{2.2.10}$$

Hence, in the sense of pointwise convergence we have, using (2.2.9) and (2.2.10)

$$\frac{1}{C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_\psi f)(a,b)\psi_{a,b}(x_0)\frac{dbda}{a^2} = f(x_0). \tag{2.2.11}$$

## 2.3 Some Stronger Results

Corollaries 2.2.5 and 2.2.6 to Theorem 2.2.2 and their generalisations can be proved very simply by using a transformation used by Vladimirov in his book (Vladimirov , 1979, p. 8). We state our result as follows.

**Theorem 2.3.1** *Let $f \in L^2(\mathbb{R}^n), n \geq 1$ be continuous at a point $x = x_0$, then*

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^n} f(x) w_\epsilon(x)\, dx = f(x_0) \tag{2.3.1}$$

where the function $w_\epsilon(x) \in \mathcal{D}(\mathbb{R}^n)$ and is defined as follows

$$w_\epsilon(x) = \begin{cases} C_\epsilon\, e^{-\frac{\epsilon^2}{\epsilon^2 - \|x-x_0\|^2}} & , \|x - x_0\| < \epsilon \\ 0 & , \|x - x_0| \geq \epsilon \end{cases}$$

the constant $C_\epsilon$ is chosen so that

$$\int_{\mathbb{R}^n} w_\epsilon(x)\, dx = 1.$$

**Proof.** The integral of $w_\epsilon(x)$ over $\mathbb{R}^n$ is 1, so

$$C_\epsilon \int_{\|x-x_0\|<\epsilon} e^{-\frac{\epsilon^2}{\epsilon^2-\|x-x_0\|^2}}\, dx = 1$$

Putting $\frac{x-x_0}{\epsilon} = \xi, dx = \epsilon^n d\xi$,

$$C_\epsilon \epsilon^n \int_{\|\xi\|<1} e^{-\frac{1}{1-\|\xi\|^2}} \, d\xi = 1 \tag{2.3.2}$$

This determines $C_\epsilon$.

Now,

$$\int_{\mathbb{R}^n} f(x) w_\epsilon(x) \, dx = \int_{\|x-x_0\|<\epsilon} f(x) C_\epsilon e^{-\frac{\epsilon^2}{\epsilon^2 - \|x-x_0\|^2}} \, dx$$

Using $\frac{x-x_0}{\epsilon} = \xi$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) w_\epsilon(x) \, dx &= C_\epsilon \epsilon^n \int_{\|\xi\|<1} f(x_0 + \epsilon\xi) e^{-\frac{1}{1-\|\xi\|^2}} \, d\xi \\ &= C_\epsilon \epsilon^n \int_{\|\xi\|<1} \left[ f(x_0 + \epsilon\xi) - f(x_0) \right] e^{-\frac{1}{1-\|\xi\|^2}} \, d\xi \\ &\quad + f(x_0) \, C_\epsilon \epsilon^n \int_{\|\xi\|<1} e^{-\frac{1}{1-\|\xi\|^2}} \, d\xi \\ &= C_\epsilon \epsilon^n \int_{\|\xi\|<1} \left[ f(x_0 + \epsilon\xi) - f(x_0) \right] e^{-\frac{1}{1-\|\xi\|^2}} \, d\xi + f(x_0) \end{aligned} \tag{2.3.3}$$

where (2.3.2) has been used to obtain the last equality.

Since $f$ is continuous at $x_0$ and $\|\xi\| < 1$, we can choose a sufficiently small $\epsilon > 0$ so that $|f(x_0 + \epsilon\xi) - f(x_0)| < \eta$ where $\eta$ is arbitrarily chosen small positive number. Therefore, using (2.3.2) we get,

$$|C_\epsilon \epsilon^n \int_{\|\xi\|<1} \left[ f(x_0 + \epsilon\xi) - f(x_0) \right] e^{-\frac{1}{1-\|\xi\|^2}} \, d\xi| < \eta$$

Since $\eta$ is an arbitrarily chosen small number we have,

$$\lim_{\epsilon \to 0} C_\epsilon \epsilon^n \int_{\|\xi\|<1} \left[ f(x_0 + \epsilon\xi) - f(x_0) \right] e^{-\frac{1}{1-\|\xi\|^2}} \, d\xi = 0 \tag{2.3.4}$$

Therefore, using (2.3.3) and (2.3.4), (2.3.1) follows.

Since $w_\epsilon(x) \in \mathcal{D}(\mathbb{R}^n)$ it also belongs to $L^2(\mathbb{R}^n)$. Our this result can be used to prove parts (i) and (ii) of corollary 2.2.6 to Theorem 2.2.2. Some of these results are found in Dirac (1930); Postnikov *et al.* (2007); Vladimirov (1979) as well in some form but our results are more detailed and rigorous.

**Theorem 2.3.2** *Let $f \in L^2(\mathbb{R}^n)$ be continuous at $x = x_0$ and $\hat{f}(y)$ be its Fourier transform. Let $w_\epsilon(x)$ be the function as defined in theorem 2.3.1, then*

$$\lim_{\epsilon \to 0} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(y) e^{iy \cdot x} w_\epsilon(x) \, dy = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(y) e^{iy \cdot x_0} \, dy \tag{2.3.5}$$

**Proof.** It is easy to verify that $\forall x \in \mathbb{R}^n$, $w_\epsilon^2(x) \leq w_\epsilon(x)$. Therefore,

$$\|w_\epsilon(x)\|^2 = \int_{\mathbb{R}^n} w_\epsilon^2(x)\, dx \leq \int_{\mathbb{R}^n} w_\epsilon(x)\, dx \leq 1$$

Therefore, $\|w_\epsilon(x)\| \leq 1, \forall \epsilon > 0$. Now,

$$\begin{aligned} \left\langle \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(y) e^{iy\cdot x}\, dy, w_\epsilon(x) \right\rangle &= \left\langle \frac{1}{(2\pi)^{n/2}} \int_{\|y\|\leq N} \hat{f}(y) e^{iy\cdot x}\, dy, w_\epsilon(x) \right\rangle \\ &+ \left\langle \frac{1}{(2\pi)^{n/2}} \int_{\|y\|\geq N} \hat{f}(y) e^{iy\cdot x}\, dy, w_\epsilon(x) \right\rangle \end{aligned} \tag{2.3.6}$$

Since $\frac{1}{(2\pi)^{n/2}} \int_{\|y\|\leq N} \hat{f}(y) e^{iy\cdot x}\, dy$ is a continuous function of $x$ , $\forall x \in \mathbb{R}^n$ it follows in view of theorem 2.3.1 that

$$\lim_{\epsilon\to 0} \left\langle \frac{1}{(2\pi)^{n/2}} \int_{\|y\|\leq N} \hat{f}(y) e^{iy\cdot x}\, dy, w_\epsilon(x) \right\rangle = \frac{1}{(2\pi)^{n/2}} \int_{\|y\|\leq N} \hat{f}(y) e^{iy\cdot x_0}\, dy$$

Therefore,

$$\lim_{N\to\infty} \lim_{\epsilon\to 0} \left\langle \frac{1}{(2\pi)^{n/2}} \int_{\|y\|\leq N} \hat{f}(y) e^{iy\cdot x}\, dy, w_\epsilon(x) \right\rangle = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(y) e^{iy\cdot x_0}\, dy \tag{2.3.7}$$

Now consider the other expression on the R.H.S. of (2.3.6).

$$\begin{aligned} \left| \left\langle \frac{1}{(2\pi)^{n/2}} \int_{\|y\|\geq N} \hat{f}(y) e^{iy\cdot x}\, dy, w_\epsilon(x) \right\rangle \right| &\leq \left\| \frac{1}{(2\pi)^{n/2}} \int_{\|y\|\geq N} \hat{f}(y) e^{iy\cdot x}\, dy \right\| \|w_\epsilon(x)\| \\ &\leq \left\| \frac{1}{(2\pi)^{n/2}} \int_{\|y\|\geq N} \hat{f}(y) e^{iy\cdot x}\, dy \right\| \\ &\to 0 \text{ as } N \to \infty \end{aligned}$$

Therefore,

$$\lim_{N\to\infty} \lim_{\epsilon\to 0} \left\langle \frac{1}{(2\pi)^{n/2}} \int_{\|y\|\geq N} \hat{f}(y) e^{iy\cdot x}\, dy, w_\epsilon(x) \right\rangle = 0 \tag{2.3.8}$$

Using (2.3.7) and (2.3.8), our result follows.

**Corollary:** Now using theorems 2.3.1 and 2.3.2 and the fact that

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(y) e^{iy\cdot x}\, dy = f(x) \text{ in } L^2(\mathbb{R}^n)$$

we get

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(y) e^{iy\cdot x_0}\, dy = f(x_0).$$

**Discussion:** Our $n$-dimensional wavelet inversion formula is ( see Bogess and Narcowich (2001); Pandey *et al.* (2015))

$$\frac{1}{C_\psi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |a|^{-1/2} \psi\left(\frac{x-b}{a}\right) (W_\psi f)(a,b) \frac{db\, da}{|a|^2} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(y) e^{iy\cdot x}\, dy$$

We now derive a pointwise inversion formula as follows: Let

$$F(x_0) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |a|^{-1/2} \psi\left(\frac{x_0 - b}{a}\right) (W_\psi f)(a, b) \frac{db\, da}{a^2}$$

Using Plancherel theorem we have

$$F(x_0) = \int_{-\infty}^{\infty} \frac{da}{\sqrt{|a|}a^2} \int_{-\infty}^{\infty} \overline{F_b \left\{ \overline{\psi\left(\frac{x_0 - b}{a}\right)} \right\}} (y)\, F_b\{(W_\psi f)(a, b)\}(y)\, dy$$

Now

$$\overline{F_b \left\{ \overline{\psi\left(\frac{x_0 - b}{a}\right)} \right\}} (y) = a e^{iyx_0} \hat{\psi}(ay)$$

So

$$\begin{aligned} F(x_0) &= \frac{1}{C_\psi} \int_{-\infty}^{\infty} \frac{\sqrt{2\pi}a}{|a|} \int_{-\infty}^{\infty} |\hat{\psi}(ay)|^2 \hat{f}(y) e^{iyx_0}\, dy \\ &= \frac{1}{C_\psi} \int_{-\infty}^{\infty} \sqrt{2\pi} \hat{f}(y) e^{iyx_0}\, dy \int_{-\infty}^{\infty} \frac{|\hat{\psi}(ay)|^2}{|a|}\, da \end{aligned}$$

But

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(ay)|^2}{|a|}\, da = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(u)|^2}{|u|}\, du = \frac{C_\psi}{2\pi}$$

Therefore,

$$\begin{aligned} F(x_0) &= \frac{1}{C_\psi} \int_{-\infty}^{\infty} \sqrt{2\pi} \hat{f}(y) e^{iyx_0} \frac{C_\psi}{2\pi}\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(y) e^{iyx_0}\, dy \end{aligned}$$

or,

$$\frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |a|^{-1/2} \psi\left(\frac{x_0 - b}{a}\right) (W_\psi f)(a, b) \frac{db\, da}{a^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(y) e^{iyx_0}\, dy \quad (2.3.9)$$

In $\mathbb{R}^n$ this formula takes the form (Pandey *et al.* , 2015, see)

$$\frac{1}{C_\psi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |a|^{-1/2} \psi\left(\frac{x_0 - b}{a}\right) (W_\psi f)(a, b) \frac{db\, da}{|a|^2} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(y) e^{iy\cdot x_0}\, dy \quad (2.3.10)$$

Here, $a = (a_1, a_2, \cdots, a_n)$, $b = (b_1, b_2, \cdots, b_n)$, $|a| = |a_1 a_2 \cdots a_n|$,

$x_0 = (x_0^1, x_0^2, \cdots, x_0^n)$, $y \cdot x_0 = y_1 x_0^1 + y_2 x_0^2 + \cdots + y_n x_0^n$.

So by Theorem 2.3.2 we have

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(y) e^{iy\cdot x_0}\, dy = f(x_0) \quad (2.3.11)$$

Therefore, using (2.3.10) and (2.3.11) our pointwise $n$-dimensional wavelet inversion formula becomes

$$\frac{1}{C_\psi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|a|^{-1/2}\psi(\frac{x_0-b}{a})(W_\psi f)(a,b)\,\frac{db\,da}{|a|^2}=f(x_0) \tag{2.3.12}$$

The advantage of this pointwise inversion formula is that it only uses the fact that $\psi \in L^2(\mathbb{R}^n)$ and that $f \in L^2(\mathbb{R}^n)$ is continuous at $x = x_0$.

Our result (2.3.12) is more general than the result (2.2.11).

## 2.4 Conclusion

The work presented in this chapter generalizes the conventional approach to the multidimensional wavelet transform with positive scales on the case of both positive and negative scales with respect to its inversion. Its importance is principally connected with the consideration of the wavelet transform applied to real-valued functions, which have both positive and negative Fourier components. The standard cut-off of negative frequencies (which is required to apply CWT with $a > 0$) may result in a loss of information, if the transformed function were a non-symmetric (in the Fourier space) mixture of real and imaginary frequency components. The proposed and proven inversion formula is free from the mentioned defect. The proved result significantly enhances the possible further practical utility of the wavelet inversion formula to the image processing and other areas.