Chapter 1

Introduction

In recent years, wavelet analysis has attracted much attention due to its wide range of applications in various disciplines of science and engineering. The concept of wavelet analysis is not new in the sense that similar ideas under different names have been in place since the beginning of twentieth century. The Littlewood-Paley technique (see Littlewood and Paley (1937)) and Calderon Zygmund theory (see Calderon (1964)) in Harmonic analysis and digital filter bank theory in signal processing may be considered as fore runners of wavelet analysis. The wavelet theory provides a unified framework and coherent theory for a number of different ideas and techniques that have been independently developed in several fields. In its present form, the wavelets attracted attention in the 1980s through the works of researchers from various fields (see Grossmann and Morlet (1984); Daubechies (1988); Mallat (1989); Chui (1992)). There are opportunities for further development of both the mathematical understanding of wavelets and a wide range of applications in science and engineering.

In this chapter, we present an overview of wavelet transform.

1.1 Evolution of Wavelet Transform

1.1.1 The Fourier Series and Fourier Transform

The Fourier analysis is one of the most popular technique for signal analysis. It transforms the signal from one domain (time) to another domain (frequency) in which many characteristics of the signal are revealed.

In 1808, Joseph Fourier proposed that any 2π-periodic function (signal) can be represented as the sum of infinite number of sinusoids. Such a representation is known as the Fourier series, and can be expressed as

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} \sin(nt) \tag{1.1.1}$$

In complex exponential notation eq. (1.1.1) can be represented as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \tag{1.1.2}$$

where the constants c_n, called Fourier coefficients of f, are given by

$$c_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(t) e^{-int}\, dt. \tag{1.1.3}$$

The above fact can be summarised as "every 2π-periodic square-integrable function is generated by a superposition" of integral dilations of the basic function $w(t) = e^{it}$ [Chui]".

The above approach of decomposing a signal into integer frequency components was not well suited for the non-periodic signals. To overcome this shortcoming the Fourier transform of a signal $f(t)$ is defined as

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt \tag{1.1.4}$$

with its inversion formula given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega \tag{1.1.5}$$

The Fourier transform can be thought of as a continuous form of Fourier series [Narcowich]. The above equations can be summed up as " a non-periodic signal can be represented by the integral sum of complex exponentials with weights given by the Frequency spectrum $F(\omega)$".

If we have a scaled version of the original signal, $f_s(t) = f(st)$, then its corresponding Fourier transform will be $F_s(\omega) = \frac{1}{|s|} F(\frac{\omega}{s})$. From last two relations, we can observe that reducing the time spread of f by s(if $s > 1$) results in dilation of the Fourier transform by s. This is interpreted as localization in time can be achieved at the cost of losing frequency localization. The poor localization in time is the main drawback of the Fourier transform. While Fourier analysis was very efficient technique for analysing signals having time-dependent wave-like features, it was suitable for signals which have transitory characteristics.

1.1.2 Short-Time Fourier Transform (STFT)

To see the changes of frequency content of a signal over time, a new dimension of time was to be incorporated in the Fourier transform. This was done by Gabor (1946), who adapted the Fourier transform to analyse only a small section of the signal at a time by cutting it out using a window function. The truncated sections of the signal, which now can be assumed as stationary, were analysed by sliding the window function (of fixed width) along the time axle to obtain the change relationship of frequency over time. The resulted in two-dimensional representation of the signal as function of time and frequency. This time-frequency representation, called Short-time Fourier transform (STFT), can be mathematically described as:

$$F(\tau, \omega) = \int_{-\infty}^{\infty} f(t) w(t-\tau) e^{-i\omega t}\, dt \tag{1.1.6}$$

where $w(t)$ is a window function.

Though the STFT overcame the deficiency of the Fourier transform to some extent in local analysis, it had some limitations due to the fixed size of the window function. As the resolution of window function is restricted by Heisenberg uncertainty principle, the time resolution was lower and the frequency resolution was higher when a long window was used, the situation was reversed when a short window was used. This resulted in fixed time- frequency resolution which makes STFT not suitable for analysing the signals having a variety of difference of scales (see Yansong *et al.* (2011)). The next logical step- a windowing technique with windows of variable size, was the Wavelet transform.

1.1.3 Wavelet Transform

The need of simultaneous representation and localization of both time and frequency for non-stationary signals (e.g. music, speech, images) led toward the evolution of wavelet transform from the popular Fourier transform. To deal with the fixed time-frequency partitioning (resolution) of STFT, a new set of basis functions was required. To accomplish this, a function ψ, which was localized both in time and frequency, was modified by dilations and translations to obtain a family of functions, called wavelets $\psi_{a,b}$, given by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right). \tag{1.1.7}$$

Given this family of wavelets, the wavelet transform of a function $f \in L^2(\mathbb{R})$ is defined as

$$(W_\psi f)(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}\,dt \tag{1.1.8}$$

The parameter a, a non-zero real number, is used for dilation or scale, measures the degree of compression. The parameter b, a real number, is the translation parameter which determining the time location of the wavelet. The factor $\frac{1}{\sqrt{|a|}}$ was to ensure the energy of each wavelet $\psi_{a,b}$ is same as that of original function ψ. With the dilation parameter a, the support of $\psi_{a,b}$ varies proportionally , whereas the frequency interval varies in inverse proportion. In other words, wavelets have time-widths adapted to their frequencies. This enables wavelets enables to capture events local in time by giving up some frequency resolution and vice versa. The mother wavelet can be stretched according to the frequency to provide reasonable window, a long time window is used in low frequency and a short time window is used in high frequency. The variable sized windows of wavelets provides good frequency resolution for low frequencies, and good time resolution for high frequencies. The balance between time and frequency makes wavelets an ideal tool for studying non-stationary signals.

Time-frquency tiling in different transforms are shown in Figure 1.1. (a) The frequency domain after computing the Fourier transform, representing perfect frequency resolution and no time resolution. (b) The time domain representation of the observed time series, representing perfect time resolution and no frequency resolution. (c) Represents balanced resolution between time and frequency by using the short-time Fourier transforms (Gabor transform). (d) The wavelet transform adaptively partitions the time-frequency plane.

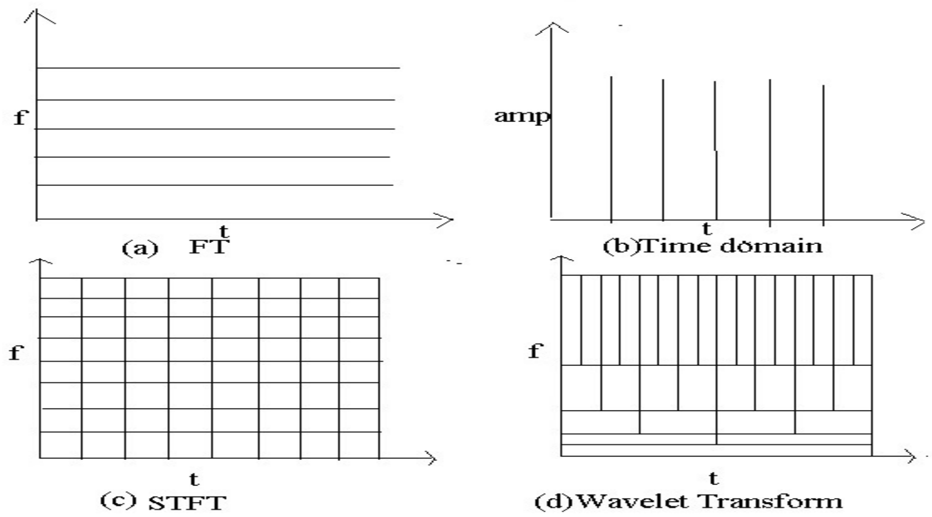


Figure 1.1: Time-Frequency tiling in different transforms

1.2 Wavelet Analysis

1.2.1 Definition and Basic Properties

Consider a real or complex-valued function ψ satisfying the following conditions:

$$\int_{-\infty}^{\infty} |\psi(t)|^2\, dt < \infty \tag{1.2.1}$$

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} < \infty \tag{1.2.2}$$

where $\hat{\psi}$ is the Fourier transform of ψ. The first condition implies finite energy of the function ψ, and the second condition, the admissibility condition, implies that if $\hat{\psi}(\omega)$ is smooth then $\hat{\psi}(0) = 0$ which in turn implies that $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$. This is suggestive of a function that is oscillatory or has wavy appearance. Thus a function satisfying the above properties must be a "small wave" or a wavelet.

Definition 1.2.1 *A wavelet is a function $\psi \in L^2(\mathbb{R})$ which satisfies the admissibility condition*

$$0 < C_\psi := 2\pi \int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\, d\omega < \infty$$

where $\hat{\psi}$ denotes the Fourier transform of the wavelet ψ.

Following are examples of some common wavelets.

1. Haar Wavelet:

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1/2 \\ -1 & \text{if } 1/2 \leq < 1 \\ 0 & \text{else} \end{cases}$$

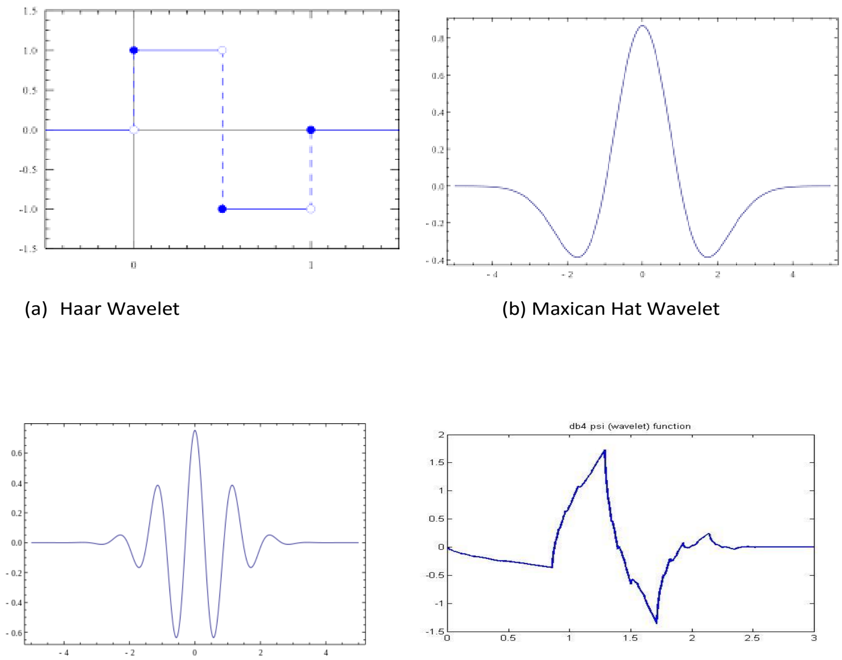
2. Mexican Hat Wavelet: It is defined as the second derivative of a Gaussian.

$$\psi(t) = -\frac{d^2}{dt^2}e^{-t^2/2} = (1 - t^2)e^{-t^2/2}$$

3. Morlet wavelet: The Morlet wavelet is constructed by manipulating a cosine function.

4. Daubechies wavelet: It has no closed form formula. These are obtained by iteration.

These above wavelets are shown in Figure 1.2.



(a) Haar Wavelet

(b) Maxican Hat Wavelet

(c) Morlet Wavelet

(d) Daubechies Wavelet D4

Figure 1.2: Some common Wavelets

1.2.2 Continuous Wavelet Transform (CWT)

For a prototype function $\psi(t) \in L^2(\mathbb{R})$ called the mother wavelet or wavelet function, the family of functions can be obtained by shifting and scaling this $\psi(t)$ as:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right), \text{ where } a, b \in \mathbb{R}, a > 0$$

The parameter b corresponds to the time shift and the parameter a corresponds to the scale of the analyzing wavelet. The factor of $\frac{1}{\sqrt{a}}$ appears for normalization so that $\|\psi(t)\| = \|\psi_{a,b}(t)\|$, that is, the energy remains the same for all a and b.

If ψ satisfies the conditions described above, then for a real valued signal $f(t)$ (a function with finite energy i.e. $f(t) \in L^2(\mathbb{R})$, the set of square integrable functions) the wavelet transform of with respect to the wavelet function $\psi(t)$ at a scale $a \in \mathbb{R}^+$ and at translational value $b \in \mathbb{R}$ is defined as:

$$(W_\psi f)(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t)\psi^*_{a,b}(t)\, dt = \langle f(t), \psi_{a,b}(t)\rangle \tag{1.2.3}$$

where $*$ stands for complex conjugation and $\langle , \rangle$ denotes the inner products.
Alternatively, the CWT can be expressed as the output of a filter matched to $\psi_{a,b}$ at time b

$$(W_\psi f)(a, b) = f * \tilde{\psi}_{a,b} \tag{1.2.4}$$

where $*$ denotes linear convolution and $\tilde{\psi}(t) = \psi^*(-t)$.

If the mother wavelet satisfies the admissibility condition, then the Inversion formula for wavelet transform is given by

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(a, b)\psi_{a,b}(t) \frac{da\, db}{a^2} \tag{1.2.5}$$

Thus the above equation interprets the wavelet transform as providing a weighing function for synthesizing a given function $f(t)$ from the translates and dilates of the mother wavelet $\psi(t)$.
Using the Cauchy-Schwarz inequality in eq. (1.2.3) gives

$$|(W_\psi f)(a, b)|^2 \leq \|f(t)\|^2 \|\psi_{a,b}(t)\|^2. \tag{1.2.6}$$

This implies that $(W_\psi f)(a, b)$ always exists because the function and the wavelet have finite norms. Equality holds in this relation if and only if

$$\psi_{a,b}(t) = \alpha f(t) \tag{1.2.7}$$

for some scalar α. This equation inspires the need of wavelets matched to the signal at hand. Chapter 3-5 of this thesis is based on construction and application of matched wavelets.

1.2.3 Discrete Wavelet Transform (DWT)

The general CWT maps a 1-D signal into 2-D (dilation and position) space. As parameters (a, b) take continuous values, the resulting CWT is a very redundant representation in the sense that the entire support of $W(a, b)$ need not be used to recover $f(t)$ (see Rao and Bopardikar (1998)). It is computationally impossible to analyze a signal using all wavelet coefficients. Therefore, instead to varying the parameters a and b continuously, we analyze the signal with a small number of scales with varying number of translations at each scale. The scale and shift parameters are evaluated on a discrete grid of time-scale plane leading to a discrete set of continuous basis functions. The discretization is performed by setting $a = a_0^{-j}$ and $b = ka_0^{-j}b_0$ for $j, k \in \mathbb{Z}$. The corresponding family of wavelets are now given as

$$\psi_{j,k}(t) = a_0^{j/2}\psi(a_0^j t - kb_0) \tag{1.2.8}$$

With $a_0 = 2$ and $b_0 = 1$, the process is called *dyadic sampling* because consecutive values of the discrete scales as well as the corresponding sampling intervals differ by a factor of two. With this sampling, the discretized version of CWT is given by

$$W(j, k) = \int_{-\infty}^{\infty} f(t)\psi_{j,k}(t)\,dt \tag{1.2.9}$$

$W(j, k)$'s are called wavelet coefficients or *discrete wavelet transform*(DWT) of $f(t)$.

The original function can be reconstructed using

$$f(t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} W(j, k)\psi_{j,k}(t) \tag{1.2.10}$$

The above equation (1.2.10) is called the wavelet series of $f(t)$.

We conclude this section by listing of *some Advantages of wavelet analysis:*

1. Wavelets offer a simultaneous localization in time and frequency domain.

2. Wavelets have the great advantage of being able to separate the fine details in a signal. Very small wavelets can be used to isolate very fine details in a signal, while very large wavelets can identify coarse details.

3. With wavelets, it is often possible to obtain a good approximation of the given function f by using only a few coefficients which is the great achievement compared to Fourier transform.

4. In the case of wavelet transform, the analysing function ψ can be chosen according to the application at hand that is we have more freedom in the selection of basis functions. This flexibility was missed in Fourier transform (and STFT) where exponentials were the only possible basis functions.

5. It is computationally very fast (using fast wavelet transform).

6. Most of the wavelet coefficients vanish rapidly.

7. Wavelet analysis is able to reveal signal aspects that other analysis techniques miss, such as trends, breakdown points, discontinuities, etc.

1.3 Implementation of DWT

Implementation of DWT can be understood with the concept of Multiresolution Analysis.

Definition 1.3.1 (Multiresolution Analysis) *A sequence of closed (nested) subspaces* $\{V_j : j \in \mathbb{Z}\}$ *of* $L^2(\mathbb{R})$ *together with a function* $\phi \in V_0$ *is called a Multiresolution Analysis (MRA) if it satisfies the following conditions;*

1. *(Nested)* $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$*;*

2. *(Density)* $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$*;*

3. *(Separation)* $\cap_{j\in\mathbb{Z}} V_j = \{0\}$*;*

4. *(Scaling)* $f(t) \in V_j$ *if and only if* $f(2t) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$*;*

5. *(Orthonormal basis) The integer translates* $\{\phi(t-n) : n \in \mathbb{Z}\}$ *of* ϕ*, called a scaling function, is an orthonormal basis for* V_0*.*

Condition 5 might be relaxed in general case where it is enough for $\{\phi(t-n) : n \in \mathbb{Z}\}$ *to be a Riesz basis of* V_0*.*

Let W_j be the orthogonal complement of V_j in V_{j+1}, then

$$V_{j+1} = V_j \oplus W_j \tag{1.3.1}$$

$$= V_{j-1} \oplus W_{j-1} \oplus W_j \tag{1.3.2}$$

$$= V_0 \oplus W_0 \oplus \cdots \oplus W_{j-2} \oplus W_{j-1} \oplus W_j \tag{1.3.3}$$

Letting $j \to \infty$ and the use of condition 2 gives,

$$L^2(\mathbb{R}) = V_0 \oplus_{j\geq 0} W_j \tag{1.3.4}$$

Decomposing V_j for $j \leq 0$, and using conditions 2 and 3 of MRA we have

$$L^2(\mathbb{R}) = \overline{\oplus_{j\in\mathbb{Z}} W_j} \tag{1.3.5}$$

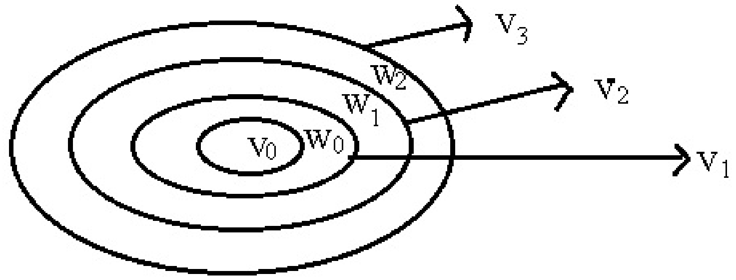


Figure 1.3: Nested spaces spanned by scaling and wavelet bases

Mallat (1989) showed that there exists a function $\psi(t)$ (Mother wavelet) such that $\{\psi(t-n) : n \in \mathbb{Z}\}$ constitutes an orthonormal basis for W_0.

From conditions 4 and 5 of MRA above, it is easy to see that for each fixed $j \in \mathbb{Z}$, the set of functions

$$\{\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k) : k \in \mathbb{Z}\}$$

is an orthonormal basis for V_j. Similarly, the family

$$\{\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) : k \in \mathbb{Z}\}$$

forms an orthogonal basis for W_j.

Now, using condition 4 of MRA, we can express $\psi(t) \in V_0$ in terms of basis elements of V_1, since $V_0 \subset V_1$. This fact is represented by the following relation, called two-scale relation,

$$\phi(t) = 2\sum_{k\in\mathbb{Z}} h(k)\phi(2t-k) \tag{1.3.6}$$

where $h(k) = \langle\phi(t), \phi(2t-k)$. Since $W_0 \subset V_1$, the wavelet function satisfies similar equation,

$$\psi(t) = 2\sum_{k\in\mathbb{Z}} g(k)\phi(2t-k) \tag{1.3.7}$$

where $g(k) = \langle\psi(t), \phi(2t-k)$. We take $g(k) = (-1)^k h(1-k)$ to ensure orthogonality of ϕ and ψ. The coefficients $h(k)$ and $g(k)$ are called filter coefficients.

In the view of eq. (1.3.4), a function $f(t) \in L^2(\mathbb{R})$ can be expressed as

$$f(t) = \sum_{k\in\mathbb{Z}} a_k\phi(t-k) + \sum_{j\geq 0} w_j(t) \tag{1.3.8}$$

and eq. (1.3.5) gives

$$f(t) = \sum_{j\in\mathbb{Z}} w_j(t) \tag{1.3.9}$$

with $w_j(t) \in W_j$ for all $j \in \mathbb{Z}$. Since $\{\psi_{j,k}(t) : k \in \mathbb{Z}\}$ is an orthonormal basis of W_j, we can represent $w_j(t)$ as

$$w_j(t) = \sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(t) \tag{1.3.10}$$

where $d_{j,k} = \langle f(t), \psi_{j,k}(t)\rangle$ Therefore, the equations (1.3.8) and (1.3.9) take the form

$$f(t) = \sum_{k\in\mathbb{Z}} a_k\phi(t-k) + \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(t) \tag{1.3.11}$$

$$f(t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(t) \tag{1.3.12}$$

The eq. 1.3.12 is similar to eq. 1.2.10.

For a signal (or function) $f(t)$ the orthogonal projection onto the space V_j is defined as

$$A_j f(t) = \sum_{k\in\mathbb{Z}} s_(j,k)\phi_{j,k}(t) \tag{1.3.13}$$

where $s_{j,k} = \langle f(t), \phi_{j,k}(t)\rangle$, called *scaling coefficients*. Since $A_j f(t)$ is the orthogonal projection of $f(t)$, it is the best approximation of $f(t)$ in V_j. We can see $\lim_{j\to-\infty} A_j f(t) = 0$.

From the relation $V_j \subset V_{j+1}$, we can conclude that $A_{j+1}f(t)$ is a better approximation of $f(t)$ than $A_j f(t)$. In view of eq. 1.3.1, the difference of two approximations, known as detail signal, is given by

$$D_j f(t) = A_{j+1} f(t) - A_j f(f) = \sum_{k\in\mathbb{Z}} \langle f(t), \psi_{j,k}(t)\rangle \psi_{j,k}(t) = \sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(t) \quad (1.3.14)$$

that is, the wavelet coefficients $d_{j,k}$ describes the information loss when going from projection of $f(t)$ onto the space V_{j+1}, to the projection onto the lower resolution space V_j.

DWT Decomposition and Reconstruction:

We have

$$s_{j,k} = \langle f(t), \phi_{j,k}(t)\rangle \quad (1.3.15)$$

But, from two-scale relation 1.3.6 between levels j and $j+1$ we get

$$\phi_{j,k}(t) = 2^{-1/2} \sum_{m\in\mathbb{Z}} h(m-2k)\phi_{j+1,m}(t) \quad (1.3.16)$$

Taking inner product with $f(t)$ on both sides of equation (1.3.16) and the use of equation (1.3.15) gives

$$s_{j,k} = 2^{-1/2} \sum_{m\in\mathbb{Z}} h(m-2k) s_{j+1,m} \quad (1.3.17)$$

The corresponding relation for the wavelet coefficients is:

$$d_{j,k} = 2^{-1/2} \sum_{m\in\mathbb{Z}} g(m-2k) s_{j+1,m} \quad (1.3.18)$$

The equations 1.3.17 and 1.3.18 are called *decomposition formula*.

Using orthogonality and scaling relation we get *reconstruction formula*

$$s_{j+1,k} = 2^{-1/2} \sum_{m\in\mathbb{Z}} s_{j,m} h(k-2m) + 2^{-1/2} \sum_{m\in\mathbb{Z}} d_{j,m} g(k-2m) \quad (1.3.19)$$

The decomposition and reconstruction formula relate the wavelet and approximation coefficients at two successive levels. Decomposition can be interpreted as the convolution of approximation coefficients is with filter coefficient h (and g), followed by downsampling by two whereas in reconstruction, upsampling by two is applied on the approximation and detail coefficients at each level. These coefficients are fed into the low pass and high pass synthesis filters and added afterward. This process continues until the number of levels become same as the number of levels in decomposition process. Reconstruction process is necessary to achieve the original signal. The decomposition and reconstruction formula are shown by block diagrams in Figures 1.4 and 1.5, respectively.

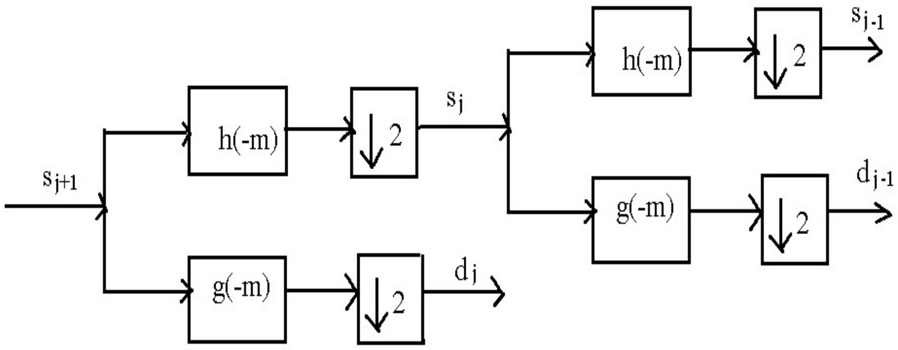


Figure 1.4: Block diagram for Multiresolution Decomposition

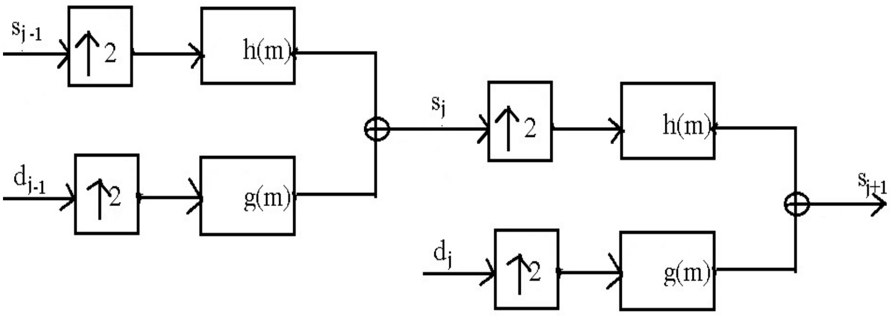


Figure 1.5: Block diagram for Multiresolution Reconstruction

1.4 Literature Review

Multidimensional Wavelet Transform

During the last two decades several authors [Antoine *et al* (2006); Chui (1992); Daubechies (1992); Meyer (1992); Cohen and Daubechies (1993); Pandey *et al.* (2015); Pathak (2009); Walter and Shen (2009); Walter (1995)] have worked on multidimensional wavelet transform and its inversion formulae. The most notable amongst them are the works of Daubechies (1992) and Meyer (1992). (Daubechies , 1992, see pp. 33-34) chose the wavelet $\psi \in L^2(\mathbb{R}^n)$ so that it is spherically symmetric. Some authors [Murenzi (1989); Argoul *et al.* (1989)] chose a ψ that is not spherically symmetric, and introduced rotations as well as dilations and translations.

Pointwise convergence to the wavelet series expansion was done by Walter (1995) and the pointwise convergence for the inversion formula for continuous wavelet transform in one dimension was done by (Chui , 1992, see pp. 62-63). He however assumes the continuity of the function f at the point x and the continuity of $\psi(x), \forall x \in \mathbb{R}$.

The formulas derived by Daubechies and Meyer are valid only at $\mathbb{R} \times \mathbb{R}^n$. Pandey *et al.* (2015) proved a more general formula that is valid in $\mathbb{R}^n \times \mathbb{R}^n$.

Matched Wavelet Construction

In applications where the output of the Wavelet Transform is to be maximized, it is necessary to use wavelets that are specifically matched to the signal of interest.In the last decade or so, a large number of researchers have focused their attention on estimation of wavelet that is matched to the signal or provides best representation of the signal. Tewfik *et al.* (1992) developed a technique for finding an orthonormal wavelet with compact support that provides the ”best” signal representation of a specified signal over a finite number of scales by minimizing an upper bound of the error performance. Mallat and Zhang (1993) pointed out that a single wavelet basis function is not flexible enough to represent a complicated non-stationary signal. Aldroubi *at al.* (1993) proposed method to find matched wavelet by projecting the signal onto an existing basis. Gopinath *et al.* (1994) expanded the work of Tewfik, et. al., by assuming bandlimited wavelets. Krim *et al.* (1999) searched for best basis for signal enhancement in white Gaussian noise.

Chapa (1995); Chapa *et al.* (2000) proposed a generalized technique to design a wavelet such that a single wavelet could provide the best match for the signal of interest achieved orthonormality for bandlimited wavelets. In their approach, they separated the matching procedure into matching spectra amplitude and matching spectra phase of signal and bandlimited wavelet.

Gupta *et al.* (2002) obtained the matched wavelet by maximizing the projection of the signal onto the scaling subspace. They proposed several other techniques to construct matched wavelets with some desired properties from a signal of interest (see Gupta *et al.*, 2003a;2003b;2005a;2005b). Misiti *et al.* (2003) approximated a given pattern using least squares optimization under constraints, leading to an admissible wavelet well suited for the pattern detection using continuous wavelet transform(CWT). Bahrampour *et al.* (2008, 2009) used variational methods to design wavelets matching to a specified signal. Mansour (2014) proposed a new construction technique for matched wavelet and matched scaling function that is based on a new parametrization of compactly supported orthonormal wavelets where the coefficients of the wavelet filter are the solution of a linear system of equations and are a continuous function of an arbitrary vector of half its length. Their proposed model provided a more general optimization framework where matched wavelets is a special case.

Wavelet based Transformer Protection

The Power Transformer is a major equipment in power systems. It requires highly reliable protective devices. When Power Transformer internal faults occur, immediate disconnection of the faulted transformer is necessary to avoid extensive damage and/or preserve power system stability. Many methods for protection of transformer have been proposed. A comparative study of algorithms for protection of power transformers has been presented by Rahman *et al.* (1988) and Habib *et al.* (1988). The ability to extract information from the transient signals simultaneously in both time and frequency domains has made wavelet transform an excellent tool for analysis of such signals which include the inrush and fault currents in transformer. Some wavelet based transformer protection methods are presented in (Moises *et al.* (1999); Rajoub (2002); Faiz *et al.* (2006); Miaou *et al.* (2002)).

ECG signal Compression Using Wavelet Transform

ECG signals are collected both over long periods of time and at high resolution. This results in substantial volumes of data for storage and subsequent transmission. The main goal of data compression is to reduce the number of bits of information required to store or transmit digitized ECG signals without significant loss of signal quality. Many schemes have been proposed for ECG signal compression. These can be categorized as either direct methods or transform methods. In direct methods, compression is performed directly on the ECG signal. In transform methods, the ECG signal is first transformed into another domain, compression is done afterward. Some traditional transform methods are Fourier, Walsh, Kahunen Loeve and discrete cosine transforms. The wavelet transform has also been used for ECG signal compression (Jalaleddine et al. 1990). The ability to provide simultaneous time and frequency resolution makes wavelet transform an excellent tool for analysis of non-stationary, high frequency component signals, like ECG. An early paper by Crowe et al (1992) suggested the wavelet transform as a method for compressing both ECG and heart rate variability data sets. Two methods of data reduction on a dyadic scale for normal and abnormal cardiac rhythms has been compared by Thakor et al (1993), detailing the errors associated with increasing data reduction ratios.

Chen et al (1993) used discrete orthonormal wavelet transforms and Daubechies D10 wavelets for ECG compression. They obtained compression ratios up to 22.9:1 while retaining clinically acceptable signal quality. In a later paper (Chen and Itoh 1998), again using D10 wavelets, they incorporate an adaptive quantization strategy which allows a predetermined desired signal quality to be achieved. Miaou and Lin (2000, 2001) also propose quality driven compression methodology based on Daubechies and biorthogonal wavelets. The set partitioning of hierarchical tree (SPIHT) coding strategy was adopted in the latter algorithm. Bradie (1996) suggested the use of a wavelet-packet-based algorithm for compression of the ECG. When compared to the KarhunenLoeve transform (KLT) applied to the same data the WP method generated significantly lower data rates at less than one-third the computational effort with generally excellent reconstructed signal quality. However, Blanchett et al (1998) report at least as good compression results for a KLT-based method. A comparison of the performance of the many ECG compression methodswavelets and othercan be found in the paper by Cardenas-Barrera and Lorenzo-Ginori (1999). More recent data compression schemes for the ECG include the method using non-orthogonal wavelet transforms by Ahmed et al (2000) and the set partitioning in hierarchical trees (SPIHT) algorithm employed by Lu et al (2000).

1.5 Outline of the thesis

The chapter 1 is introductory containing an overview of wavelet transforms. The work presented in chapter 2 generalizes the conventional approach to the multidimensional wavelet transform with positive scales to the case of both positive and negative scales with respect to its inversion. In chapter 3, review of the algorithm proposed by Chapa *et al.* (2000) for designing a wavelet matching to a specified signal has been presented. Chapter 4 presents a method for inrush and fault detection for differential protection of transformer. A wavelet matched to the inrush and fault waveforms have been constructed and used for analysing the output from a power transformer. Chapter 5 presents an algorithm for compression of ECG signal. A matching wavelet has been designed for such a signal obtained from MIT-BIH database and used for compression.