# Chapter 4

# New Lyapunov-Krasovskii functional for Stability Analysis of Time-delay System

## 4.1 Introduction

In many dynamical systems delays are usually time varying in nature. Two approaches available for stability analysis for such systems are Lyapunov-Krasovskii (LK) and Lyapunov-Razumikhin (LR). Both approaches can be useful to handle dynamical systems with time-varying delay. Less conservative results can be obtained using LK method as compared to LR one, since it takes the advantage of using additional information on the derivative of time varying delay [76]. Therefore, in robust stability analysis of system with time-varying delay using LK approach gets lot of attention.

With the aid of Linear matrix inequalities (LMIs), a variety of stability conditions have been proposed to find larger upper bound delay value by ensuring negative definiteness of derivative of the LK functional (LKF). For obtaining less conservative stability criteria of time-delay systems, it is crucial to find a precise bound of the quadratic integral function in the derivative of LKF. For this purpose, there has been a great amount of effort to find an effective inequality and as a result use of some inequalities have been proposed in the literature such as the Jensen inequality [1], Wirtinger based inequality [77], auxiliary function related inequality [78], Bessel's-Legendre inequality [79] and free matrix related inequalities [80, 81].

Another way to obtain the allowable maximum time-delay value retaining stability is to formulate a suitable LKF. Mainly, following types of functionals have been proposed to contain more information such as (i) augmented LKF [77,82], where delay term is included in the state vector (ii) delay partitioning approach [83], which divides the delay interval into several segments and (iii) Cross-term variables based LKF [84, 85], where $x(t), x(s)$ and $\dot{x}(s)$ are used to create quadratic terms, and (iv) multiple integral LKF [86, 87]. These LKFs helps to obtain less conservative results to a certain extent. In addition matrix based function is employed to the existing LKF have been developed, which leads to provide fruitful results. Recently, delay product type LKF [88,89] have been introduced such that the information of delays and its derivative has been fully utilized. In this DPF, non-integral quadratic terms have been constructed by augmenting the state vectors used in the WI to exploit newly introduced single integral states. Similar idea has been used in [90] and [91] to form the functional by using the signals present in AFI by transforming into delay and its inverse dependent matrices for passivity analysis. In [92] a new form of functional has been reported in which the integral inequalities such as Jensen inequality and Wirtinger inequality are utilized to form DPF, such that its derivative includes delay variations based integral functions to yield better results.

On the basis of above discussion, this chapter further investigates delay-dependent stability analysis for linear systems with time-varying delay. The contribution of this chapter is that two new states are introduced in the augmented vectors of DPF and in the Lyapunov matrix based quadratic term to formulate the LKF. Using this LKF and second order BLI, two stability criteria are proposed.Finally,Two numerical examples are provided to show the effectiveness of the proposed criteria.

## 4.2 System Description and Preliminaries

Consider the time-delay system as:

$$\dot{x}(t) = Ax(t) + A_\tau x(t - \tau(t)), \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $A, A_\tau \in \mathbb{R}^{n\times n}$ are the constant system matrices, with continuously differentiable initial condition.The delay function $\tau(t)$ and its derivative satisfy

$$0 \le \tau(t) \le h, \quad \mu_0 \le \dot{\tau}(t) \le \mu_1 \le 1 \tag{4.2}$$

where $h, \mu_0$ and $\mu_1$ are constants. To obtain the main results, the following lemmas are needed.

**Lemma 6** *[82] For real scalar $\alpha \in (0,1)$, symmetric matrices $R_i(i=0,1) \geq 0$, and any matrices $S_0$ and $S_1$ such that the following inequality holds*

$$\begin{bmatrix} \frac{1}{\alpha}R_0 & 0 \\ 0 & \frac{1}{1-\alpha}R_1 \end{bmatrix} \geq \begin{bmatrix} R_0 + (1-\alpha)X_0 & (1-\alpha)S_0 + \alpha S_1 \\ * & R_1 + \alpha X_1 \end{bmatrix} \tag{4.3}$$

*where $X_0 = R_0 - S_1 R_1^{-1} S_1^T$ and $X_1 = R_1 - S_0^T R_0^{-1} S_0$*

**Lemma 7** *[93] For any symmetric matrices $[R_i(i=0,1)]_{n\times n}$ and any matrices $[Y_i(i=0,1)]_{2n\times n}$, the following inequality holds $\alpha \in (0,1)$*

$$\begin{bmatrix} \frac{1}{\alpha}R_0 & 0 \\ 0 & \frac{1}{1-\alpha}R_1 \end{bmatrix} \geq Sym\left(Y_0\begin{bmatrix} I_n & 0_{n\times n}\end{bmatrix} + Y_1\begin{bmatrix} 0_{n\times n} & I_n\end{bmatrix}\right) - \alpha Y_0 R_0^{-1} Y_0^T - (1-\alpha) Y_1 R_1^{-1} Y_1^T \tag{4.4}$$

**Lemma 8** *[94] For any constant matrix $R \geq 0$, the following inequality holds for all continuously differentiable function $x \in [a,b] \to \mathbb{R}^n$;*

$$(b-a)\int_a^b \dot{x}^T(s) R \dot{x}(s) ds \geq \theta_1^T R \theta_1 + 3\theta_2^T R \theta_2 + 5\theta_3^T R \theta_3 \tag{4.5}$$

$$(b-a)\int_a^b x^T(s) R x(s) ds \geq \vartheta_1^T R \vartheta_1 + 3\vartheta_2^T R \vartheta_2 \tag{4.6}$$

*where*

$\theta_1 = x(b) - x(a), \theta_2 = x(b) + x(a) - \frac{2}{(b-a)}\int_a^b x(s)ds,$

$\theta_3 = x(b) - x(a) - \frac{6}{(b-a)}\int_a^b \delta_{a,b}(s)x(s)ds,$ $\vartheta_1 = \int_a^b x(s)ds, \vartheta_2 = \int_a^b \delta_{a,b}(s)x(s)ds$ *and*

$\delta_{a,b}(s) = 2\left(\frac{s-a}{b-a}\right) - 1.$

**Remark 5** *The integral inequalities (4.5) is a particular case of second order Bessel-Legendre inequality of [94].This inequality overcomes the conservatism provides by Wirtinger-based integral inequality by introducing an extra quadratic term $\theta_3^T R \theta_3$. This additional term gives improvement by considering a new state $\int_a^b \delta_{a,b}(s)x(s)ds$, which contains both single and double integral terms. In this paper this state is used to form the DPF defined later.*

## 4.3 Main Results

In this section, we construct a new DPF to derive a delay-dependent stability result for Linear system with time varying delay (4.1) with constraints (4.2). To simplify the representation, we introduce some notations as follows:

$$h_\tau(t) = h - \tau(t),\; x_\tau(t) = x(t-\tau(t)), x_h(t) = x(t-h),\; \bar{\tau} = 1 - \dot{\tau}(t)$$

$$w_1(t) = \frac{1}{\tau(t)}\int_{-\tau(t)}^{0} x_t(s)ds,\; w_2(t) = \frac{1}{h_\tau(t)}\int_{-h}^{-\tau(t)} x_t(s)ds$$

$$w_3(t) = \frac{1}{\tau(t)}\int_{-\tau(t)}^{0} \delta_1(s)x_t(s)ds,\; w_4(t) = \frac{1}{h_\tau(t)}\int_{-h}^{-\tau(t)} \delta_2(s)x_t(s)ds$$

$$\xi(t) = col[x(t), x_\tau(t), x_h(t), \dot{x}_\tau(t), w_1(t), w_2(t), w_3(t), w_4(t)]$$

$$e_i = [0_{n\times(i-1)}, I_n, 0_{n\times(8-i)}], i = 1, 2, ..., 8,$$

$$e_s = Ae_1 + A_\tau e_2, \qquad e_0 = 0_{n\times 8n}$$

According to the function $\delta_{a,b}$ given in Lemma 3, the functions $\delta_i, i = 1, 2$ can be expressed as

$$\delta_1(s) = 2\left(\frac{s+\tau(t)}{\tau(t)}\right) - 1, \qquad \delta_2(s) = 2\left(\frac{s+h}{h-\tau(t)}\right) - 1$$

On the basis of above $\delta_1(s)$ and $\delta_2(s)$ as augmented terms, delay product type quadratic terms has been constructed by extending the idea to construct LKF in [88]. Also based on $x(s), \dot{x}(s)$ and $\int_{t-\tau(t)}^{s} \dot{x}(s)ds$, a cross-term based quadratic term is being constructed similar to [84]. Using the combination of these delay product type and cross-term based terms a new candidate LKF is constructed as

$$V(t) = V_0(t) + V_1(t) + V_2(t) + V_3(t) \tag{4.7}$$

where

$$V_0(t) = \varpi_1^T(t)P\varpi_1(t)$$

$$V_1(t) = \tau(t)\varpi_2^T(t)Q_1\varpi_2(t) + h_\tau(t)\varpi_3^T(t)Q_2\varpi_3(t)$$

$$V_2(t) = \int_{t-\tau(t)}^{t} \varpi_4^T(s)Q_3\varpi_4(s)ds + \int_{t-h}^{t-\tau(t)} x^T(s)Zx(s)ds$$

$$V_3(t) = \int_{-\tau(t)}^{0}\int_{t+u}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsdu + \int_{-h}^{-\tau(t)}\int_{t+u}^{t} \dot{x}^T(s)R_2\dot{x}dsdu$$
$$+ \int_{-\tau(t)}^{0}\int_{t+u}^{t} x(s)^T(s)M_1x(s)dsdu + \int_{-h}^{-\tau(t)}\int_{t+u}^{t} x^T(s)M_2x(s)dsdu$$

with

$$\varpi_1(t) = col[x(t), x_\tau(t), \tau(t)w_1, h_\tau(t)w_2, \tau(t)w_3, h_\tau(t)w_4],\ \varpi_2(t) = col[x(t), x_\tau(t), w_1, w_3]$$
$$\varpi_3(t) = col[x(t), x_\tau(t), w_2, w_4],\ \varpi_4(s) = col[x(s), \dot{x}(s), \int_{t-\tau(t)}^{s} \dot{x}(s)ds]$$

**Remark 6** *The terms in $V_3(t)$ are similar to the [95]. The $\dot{x}(s)$ and $x(s)$ dependent double integral quadratic terms are considered separately for the intervals $[t-\tau(t), t]$ and $[t-h, t-\tau(t)]$ respectively to exploit the delay range. The time derivative of $V_3(t)$ provides $\tau(t)$ and $\dot{\tau}(t)$ related single-integral functions. The estimation of these delay variation based integral terms introduces new and more $\tau(t)$ and $\dot{\tau}(t)$ dependent terms, which helps to get improved result.*

By employing LKF (4.7), a delay-dependent stability criterion for system (4.1) with conditions (4.2) is as follows:

**Theorem 4** *For positive definite matrices $0 < P \in \mathbb{R}^{6n\times 6n}, 0 < Q_1, Q_2 \in \mathbb{R}^{4n\times 4n}, 0 < Q_3 \in \mathbb{R}^{3n\times 3n}, 0 < Z \in \mathbb{R}^{n\times n}, 0 < R_1, R_2, M_1, M_2 \in \mathbb{R}^{n\times n}$, and matrices $S_1, S_2 \in \mathbb{R}^{3n\times 3n}$ with given scalars $h, \mu_0$ and $\mu_1$, system (4.1) is asymptotically stable, if following LMIs satisfy for all $\dot{\tau}(t) \in [\mu_0, \mu_1]$ and $i = 1, 2$.*

$$\begin{bmatrix} \Phi_0(0, \mu_i) - \Phi_1(0, \mu_i) & E_1^T S_2 \\ * & -h\mathcal{R}_2 \end{bmatrix} < 0 \tag{4.8}$$

$$\begin{bmatrix} \Phi_0(h, \mu_i) - \Phi_1(h, \mu_i) & E_2^T S_1^T \\ * & -h\mathcal{T}_1(\mu_i) \end{bmatrix} < 0 \tag{4.9}$$

*such that following conditions must satisfy*

$$T_1(\mu_i) \geq 0 \tag{4.10}$$

$$T_2(\mu_i) \geq 0 \tag{4.11}$$

*where*

$$\begin{aligned} \Phi_0(\tau(t), \dot{\tau}(t)) = {} & Sym\{G_0^T P G_1\} + \dot{\tau}(t) G_2^T Q_1 G_2 + Sym\{G_2^T Q_1 G_3\} - \dot{\tau}(t) G_4^T Q_2 G_4 \\ & + Sym\{G_4^T Q_2 G_5\} + G_6^T Q_3 G_6 - \bar{\tau} G_7^T Q_3 G_7 + Sym\{G_8^T(\tau(t)) Q_3 G_9\} \\ & + e_s^T[\tau(t)R_1 + h_\tau(t)R_2]e_s + e_1^T[\tau(t)M_1 + h_\tau(t)M_2]e_1 \end{aligned} \tag{4.12}$$

$$\begin{aligned} \Phi_1(\tau(t), \dot{\tau}(t)) = {} & E_1^T \frac{(2-\alpha)}{h} \mathcal{T}_1 E_1 + E_2^T \frac{(1+\alpha)}{h} \mathcal{R}_2 E_2 + \frac{2}{h} E_1^T [(1-\alpha)S_1 + \alpha S_2] E_2 \\ & + E_3^T \tau(t) \mathcal{T}_2 E_3 + E_4^T h_\tau(t) \mathcal{M}_2 E_4 \end{aligned} \tag{4.13}$$

$$G_0 = [e_1^T, e_2^T, \tau(t)e_5^T, h_\tau(t)e_6^T, \tau(t)e_7^T, h_\tau(t)e_8^T]^T$$

$$G_1 = [e_s^T, \bar{\tau}e_4^T, e_1^T - \bar{\tau}e_2^T, \bar{\tau}e_2^T - e_3^T, e_1^T + \bar{\tau}e_2^T - (2 - \dot{\tau}(t))e_5^T - \dot{\tau}(t)e_7^T,$$
$$\bar{\tau}e_2^T + e_3^T - (2 - \dot{\tau}(t))e_6^T + \dot{\tau}(t)e_8^T]^T$$

$$G_2 = [e_1^T, e_2^T, e_5^T, e_7^T]^T, G_4 = [e_1^T, e_2^T, e_6^T, e_8]^T$$

$$G_3 = [\tau(t)e_s^T, \tau(t)\bar{\tau}e_4^T, e_1^T - \bar{\tau}e_2^T - \dot{\tau}(t)e_5^T, e_1^T + \bar{\tau}e_2^T - (2 - \dot{\tau}(t))e_5^T - 2\dot{\tau}(t)e_7^T]^T$$

$$G_5 = [h_\tau(t)e_s^T, h_\tau(t)\bar{\tau}e_4^T, \bar{\tau}e_2^T - e_3^T + \dot{\tau}(t)e_6^T, \bar{\tau}e_2^T + e_3^T - (2 - \dot{\tau}(t))e_6 + 2\dot{\tau}(t)e_8^T]^T$$

$$G_6 = [e_1^T, e_s^T, e_1^T - e_2^T]^T, \; G_7 = [e_2^T, e_4^T, 0_{8n\times n}]^T,$$

$$G_8 = [\tau(t)e_5^T, e_1^T - e_2^T, \tau(t)(e_5^T - e_2)^T]^T, \; G_9 = [0_{8n\times n}, 0_{8n\times n}, -\bar{\tau}e_4^T]^T$$

**Proof :** The derivative of $\int_{-\tau(t)}^{0} \delta_1(s)x_t(s)ds$ and $\int_{-h}^{-\tau(t)} \delta_2(s)x_t(s)ds$ can be expressed as

$$\frac{d}{dt}[\int_{-\tau(t)}^{0} \delta_1(s)x_t(s)ds] = x(t) + \bar{\tau}x_\tau(t) - (2 - \dot{\tau}(t))w_1(t) - \dot{\tau}(t)w_3(t) \tag{4.14}$$

$$\frac{d}{dt}[\int_{-h}^{-\tau(t)} \delta_2(s)x_t(s)ds] = \bar{\tau}x_\tau(t) + x_h - (2 - \dot{\tau}(t))w_2(t) + \dot{\tau}(t)w_4(t) \tag{4.15}$$

Similarly, the derivative of $Q_1, Q_2$ and $Q_3$ dependent quadratic terms in (4.7) can be obtained as

$$\frac{d}{dt}[\tau(t)\varpi_2^T(t)Q_1\varpi_2(t)] = \dot{\tau}(t)\varpi_2^T(t)Q_1\varpi_2(t) + 2\tau(t)\varpi_2^T(t)Q_1 \begin{bmatrix} \dot{x}(t) \\ \bar{\tau}\dot{x}_\tau(t) \\ \frac{x(t)-\bar{\tau}x_\tau(t)-\dot{\tau}(t)w_1}{\tau(t)} \\ \frac{x(t)+\bar{\tau}x_\tau(t)-(2-\dot{\tau}(t))w_1-2\dot{\tau}(t)w_3}{\tau(t)} \end{bmatrix} \tag{4.16}$$

$$\frac{d}{dt}[h_\tau(t)\varpi_3^T(t)Q_2\varpi_3(t)] = -\dot{\tau}(t)\varpi_3^T(t)Q_2\varpi_3(t) + 2h_\tau(t)\varpi_3^T(t)Q_2 \begin{bmatrix} \dot{x}(t) \\ \bar{\tau}\dot{x}_\tau(t) \\ \frac{\bar{\tau}x_\tau(t)-x_h+\dot{\tau}(t)w_2}{h_\tau(t)} \\ \frac{\bar{\tau}x_\tau(t)+x_h-(2-\dot{\tau}(t))w_2+2\dot{\tau}(t)w_4}{h_\tau(t)} \end{bmatrix} \tag{4.17}$$

$$\frac{d}{dt}\left[\int_{t-\tau(t)}^{t} \varpi_4^T(s)Q_3\varpi_4(s)\right] = \varpi_4^T(t)Q_3\varpi_4(t) - \bar{\tau}\varpi_4^T(t-\tau(t))Q_3\varpi_4(t-\tau(t)) + 2\begin{bmatrix} \tau(t)w_1 \\ x(t) - x_\tau(t) \\ \tau(t)(w_1 - x_\tau(t)) \end{bmatrix} Q_3 \begin{bmatrix} 0_{n\times 1} \\ 0_{n\times 1} \\ -\bar{\tau}\dot{x}_\tau(t) \end{bmatrix} \tag{4.18}$$

The differentiation of $V_3(t)$ can be expressed as

$$\dot{V}_3(t) = \dot{x}^T(t)[\tau(t)R_1 + h_\tau(t)R_2]\dot{x}(t) + x^T(t)[\tau(t)M_1 + h_\tau(t)M_2]x(t) - \wp_1(t) - \wp_2(t) \tag{4.19}$$

where

$$\begin{aligned}
\wp_1(t) &= \int_{t-\tau(t)}^{t} \dot{x}^T(s)T_1(\dot{\tau}(t))\dot{x}(s)ds + \int_{t-h}^{t-\tau(t)} \dot{x}^T(s)R_2\dot{x}(s)ds,\\
\wp_2(t) &= \int_{t-\tau(t)}^{t} x^T(s)T_2(\dot{\tau}(t))x(s)ds + \int_{t-h}^{t-\tau(t)} x^T(s)M_2x(s)ds,\\
T_1(\dot{\tau}(t)) &= \bar{\tau}R_1 + \dot{\tau}(t)R_2, \text{ and} T_2(\dot{\tau}(t)) = \bar{\tau}M_1 + \dot{\tau}(t)M_2
\end{aligned} \tag{4.20}$$

Then, using (4.14)-(4.19) one can write derivative of $V(t)$ with respect to time along the trajectory of system (4.1) as

$$\dot{V}(t) = \dot{V}_0(t) + \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) = \xi^T(t)\Phi_0(\tau(t), \dot{\tau}(t))\xi(t) - \wp_1(t) - \wp_2(t) \tag{4.21}$$

where $\Phi_0(\tau(t), \dot{\tau}(t))$ is defined in (4.12).

The condition (4.10) ensures the positive definiteness of $T_1(\dot{\tau}(t))$, then by applying integral inequality (4.5) of Lemma 8 to approximate the integral terms in $\wp_1(t)$, one can write

$$\wp_1(t) \geq \xi^T(t)\left[\frac{1}{\tau(t)}E_1^T\mathcal{T}_1E_1 + \frac{1}{h_\tau(t)}E_2^T\mathcal{R}_2E_2\right]\xi(t) \tag{4.22}$$

where

$$\begin{aligned}
E_1 &= col\{e_1 - e_2, e_1 + e_2 - 2e_5, e_1 - e_2 - 6e_7\}\\
E_2 &= col\{e_2 - e_3, e_2 + e_3 - 2e_6, e_2 - e_3 - 6e_8\}\\
\mathcal{T}_1 &= diag\{T_1, 3T_1, 5T_1\}, \ \mathcal{R}_2 = diag\{R_2, 3R_2, 5R_2\}
\end{aligned}$$

By using Lemma 6 with $\alpha = \frac{\tau(t)}{h}$, $\wp_1(t)$ can be expressed as

$$\begin{aligned}
\wp_1(t) \geq \frac{1}{h}\xi^T(t)[&E_1^T(2-\alpha)\mathcal{T}_1E_1 + E_2^T(1+\alpha)\mathcal{R}_2E_2 + 2E_1^T\{(1-\alpha)S_1 + \alpha S_2\}E_2\\
&- \alpha E_2^TS_1^T\mathcal{T}_1^{-1}S_1E_2 - (1-\alpha)E_1^TS_2\mathcal{R}_2^{-1}S_2^TE_1]\xi(t)
\end{aligned} \tag{4.23}$$

Similarly, the condition (4.11) guaranteed the positive definiteness of $T_2(\dot{\tau}(t))$, hence utilizing inequality (4.6) of Lemma 8, $\wp_2(t)$ can be estimated as

$$\wp_2(t) \geq \xi^T(t)\begin{bmatrix} E_3 \\ E_4 \end{bmatrix}^T \begin{bmatrix} \tau(t)\mathcal{T}_2 & 0 \\ * & h_\tau(t)\mathcal{M}_2 \end{bmatrix} \begin{bmatrix} E_3 \\ E_4 \end{bmatrix}\xi(t) \tag{4.24}$$

where

$$E_3 = [e_5^T, e_7^T]^T, E_4 = [e_6^T, e_8^T]^T, \ \mathcal{T}_2 = diag\{T_2, 3T_2\}, \mathcal{M}_2 = diag\{M_2, 3M_2\}$$

Replacing (4.24) and (4.25) in (4.22) one can write

$$\begin{aligned} \dot{V}(t) &\leq \xi^T(t)[\Phi_0(\tau(t), \dot{\tau}(t)) - \Phi_1(\tau(t), \dot{\tau}(t)) + \mathit{\Pi}(\tau(t), \dot{\tau}(t))]\xi(t) \\ &= \xi^T(t)\Psi\xi(t) \end{aligned} \tag{4.25}$$

where $\Phi_1(\tau(t), \dot{\tau}(t))$ is defined in (4.13).

$$\mathit{\Pi} = \frac{\alpha}{h} E_2^T S_1^T \mathcal{T}_1^{-1} S_1 E_2 + \frac{(1-\alpha)}{h} E_1^T S_2 \mathcal{R}_2^{-1} S_2^T E_1$$

*and*

$$\Psi = \Phi_0(\tau(t), \dot{\tau}(t)) - \Phi_1(\tau(t), \dot{\tau}(t)) + \mathit{\Pi}$$

Finally, $\dot{V}(t) < 0$, if the matrix $\Psi$ is negative definite for all $\tau(t) \in [0, h]$ and $\dot{\tau}(t) \in [\mu_0, \mu_1]$ It is clear that the matrix $\Psi$ in (4.26) is linear with respect to $\tau(t)$ and $\dot{\tau}(t)$. So, if the condition (4.26) is satisfied at the vertices $[0, \mu_0]$, $[0, \mu_1]$, $[h, \mu_0]$ and $[h, \mu_1]$, then it can also satisfy at all the vertices of $(\tau(t), \dot{\tau}(t)) \in [0, h] \times [\mu_0, \mu_1]$. Then using Schur complement the condition (4.26) can be transformed into linear matrix inequalities of (4.8) and (4.9). This completes proof. □

The criterion of Theorem 4 are obtained by utilizing Lemmas 6 and 8. Alternatively, a similar criterion can be developed by using Lemma 7 instead of Lemma 6, which results in next theorem.

**Theorem 5** *For positive definite matrices* $0 < P \in \mathbb{R}^{6n\times 6n}, 0 < Q_1, Q_2 \in \mathbb{R}^{4n\times 4n}, 0 < Q_3 \in \mathbb{R}^{3n\times 3n}, 0 < Z \in \mathbb{R}^{n\times n}, 0 < R_1, R_2, M_1, M_2 \in \mathbb{R}^{n\times n}$*, and matrices* $Y_1, Y_2 \in \mathbb{R}^{8n\times 3n}$ *with given scalars* $h, \mu_0$ *and* $\mu_1$*, system (4.1) is asymptotically stable,such that following LMIs satisfy for* $i = 0, 1$.

$$\begin{bmatrix} \Phi_0(0, \mu_i) - \bar{\Phi}_1(0, \mu_i) & Y_2 \\ * & -h\mathcal{R}_2 \end{bmatrix} < 0 \tag{4.26}$$

$$\begin{bmatrix} \Phi_0(h, \mu_i) - \bar{\Phi}_1(h, \mu_i) & Y_1^T \\ * & -h\mathcal{T}_1(\mu_i) \end{bmatrix} < 0 \tag{4.27}$$

*with conditions*

$$T_1(\mu_i) \geq 0, T_2(\mu_i) \geq 0 \tag{4.28}$$

*where* $\Phi_0(\tau(t), \dot{\tau}(t))$ *is defined in Theorem 4 and*

$$\bar{\Phi}_1(\tau(t), \dot{\tau}(t)) = \frac{1}{h} Sym\{Y_1E_1 + Y_2E_2\} + E_3^T\tau(t)\mathcal{T}_2E_3 + E_4^T h_\tau(t)\mathcal{M}_2E_4 \tag{4.29}$$

**Proof :** The proof follows the same lines of reasoning as the proof of Theorem 4. The main modification is the application of Lemma 7 in place of Lemma 6 for estimation of $\wp_1(t)$ term in (4.22). So, based on Lemma 7, $\wp_1(t)$ can be expressed as

$$\wp_1(t) \geq \frac{1}{h}[Sym\{Y_1E_1 + Y_2E_2\} - \alpha Y_1^T\mathcal{T}_1^{-1}Y_1 - (1-\alpha)Y_2\mathcal{R}_2^{-1}Y_2^T] \tag{4.30}$$

Substituting (4.31) and (4.25) in (4.22), one can write

$$\dot{V}(t) \leq \xi^T(t)[\Phi_0(\tau(t), \dot{\tau}(t)) - \bar{\Phi}_1(\tau(t), \dot{\tau}(t)) + \frac{1}{h}\alpha Y_1^T\mathcal{T}_1^{-1}Y_1 + \frac{1}{h}(1-\alpha)Y_2\mathcal{R}_2^{-1}Y_2^T]\xi(t) \tag{4.31}$$

Then using Schur complement the condition (4.31) can be transformed into linear matrix inequalities of (4.27) and (4.28). This completes the proof. □

Table 4.1: Possible upper bound delay $h$ with a variety of $\mu$

| **Methods** | $\mu = -\mu_0 = \mu_1$ | | | | **NLVs** |
|---|---|---|---|---|---|
| | **0.1** | **0.2** | **0.5** | **0.8** | |
| Theorem 1 [96] | 4.8313 | 4.1428 | 3.1487 | 2.7135 | $142n^2 + 18n$ |
| Theorem 8(N=2) [79] | 4.93 | 4.22 | 3.09 | 2.66 | $67n^2 + 5n$ |
| Proposition 1 [82] | 4.910 | 4.2166 | 3.233 | 2.789 | $57.5n^2 + 17n$ |
| Theorem 2(N=2) [97] | 4.9036 | 4.1906 | 3.1652 | 2.7357 | $65n^2 + 8n$ |
| Proposition 1(N=1) [98] | 4.9192 | 4.2116 | 3.1978 | 2.7656 | $72n^2 + 9n$ |
| Proposition 2(N=2) [98] | 4.9217 | 4.2157 | 3.2211 | 2.7920 | $104n^2 + 9n$ |
| Theorem 4 | 4.9229 | 4.2205 | 3.2059 | 2.7678 | $58n^2 + 10n$ |
| Theorem 5 | 4.9301 | 4.2361 | 3.2396 | 2.8072 | $88n^2 + 10n$ |

## 4.4 Illustrative examples

In this section, two numerical example are considered to show the improvement made in the proposed Theorems.

**Example 1** *Consider the system (4.1) with*

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \tag{4.32}$$

*The system (4.33) has analytical bound* $h_{max} = 6.17258$ *for constant delay* $(\dot{\tau}(t) = 0, \forall t \geq 0)$ *case [1]. The largest upper bound* $h$ *for different* $\mu$ *using theorem 4 and 5 are listed in Table 4.1 along with similar existing approaches in the literature. Moreover, the number of LMI variables (NLVs) involved in solving stability criterion are shown in Table 4.1. It is observed that Theorem 4 yields better results than all the others except Theorem 8(N=2) of [79] for slow varying delays* $(\mu = 0.2)$*. However, for fast varying delays Theorem 4 provides better results except Proposition 2(N=2) of [98]. But Theorem 8(N=2) of [79] and Proposition 2(N=2) of [98] utilizes more LMI variables as compared to Theorem 4. Theorem 5 provides better results in comaprison to all the approaches listed in Table 4.1.*

Table 4.2: Possible upper bound delay $h$ with a variety of $\mu$

| **Methods** | $\mu = -\mu_0 = \mu_1$ | | | | **NLVs** |
|---|---|---|---|---|---|
| | **0.1** | **0.2** | **0.5** | **0.8** | |
| Theorem 1 [80] | 7.148 | 4.466 | 2.352 | 1.768 | $65n^2 + 11n$ |
| Theorem 1 [96] | 7.167 | 4.517 | 2.415 | 1.838 | $142n^2 + 18n$ |
| Theorem 2(N=2) [97] | 7.2633 | 4.5914 | 2.5750 | 2.0115 | $65n^2 + 8n$ |
| Proposition 1 [82] | 7.230 | 4.556 | 2.5090 | 1.940 | $57.5n^2 + 17n$ |
| Theorem 1 | 7.2477 | 4.5571 | 2.5266 | 2.0018 | $57n^2 + 9n$ |
| Theorem 2 | 7.2707 | 4.5979 | 2.5885 | 2.0441 | $87n^2 + 9n$ |

**Example 2** *Consider the system (4.1) with*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \tag{4.33}$$

*For ex. 2, the obtained values of* $h$ *for* $\mu \in [0.1, 0.2, 0.5, 0.8]$ *are tabulated in Table 4.2. One can observed that Theorem 1 gives better results as comparison to all approaches of Table 4.2 except Theorem 4 [96]. But Theorem 5 provides less conservative results than all approaches in Table 4.2.*

## 4.5 Summary

The problem of stability analysis of linear systems with time-varying delay has been investigated. By constructing a new delay product type Lyapunov-Krasovskii functional using the state vectors of second order Bessel-Legendre integral inequality, two stability criteria have been derived for the system under study in combinations with reciprocal convex approaches. Two examples has been considered to validate the improvement offered by the proposed criteria.

Further study focus on the comparative analysis between the first and second order polynomial based stability criteria, derived based the on application of Lyapunov-Krasovskii functional approach.