Chapter 5

Interval Observer design

5.1 Introduction

The structure of interval observer comprises of two dynamical equations that estimate upper and lower bounds (interval) of the state vector at all times when the initial conditions are unknown and there are uncertainties present in the system, however, these initial conditions and uncertainties are bounded.

In this chapter, an interval observer for nonlinear systems considering inputs, outputs, and disturbances is designed using vector framework based contraction theory. In particular, this theory is utilized to prove the convergence property of the interval observer through a *comparison system* with specified properties. It provides the advantage over other existing approaches in the way that it does not require the formulation of error dynamics and need not require the Lyapunov candidate function to show convergence. In addition, this theory is exploited to design dynamic feedback control by the bounds obtained from the constructed interval observer and the system outputs to make the interval observer to be globally asymptotically stable. In the end, several examples with simulation outcomes are illustrated to validate the developed results. It can be observed that many practical systems such as TORA ([111]) and electromechanical system ([113]) belong to the family of systems affine in the unmeasured part of the state variables.
The further part of this chapter is framed as follows. Section 5.2 provides the main results of interval observer design. Examples with respective simulations are presented in Section 5.3. Finally, the conclusions end this chapter in Section 5.4.

5.2 Main Results of Interval Observer design

This section provides the main results. We consider following class of nonlinear systems for our development.

$$\begin{aligned} \dot{x} = & \ A(y)x + f(y,u) + Bd(t) \\ y = & \ Cx \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$ represents the states, $u \in \mathbb{R}^q$ is the control input, $y \in \mathbb{R}^p$ is the system output, f is a general smooth nonlinear function, $d(t) \in \mathbb{R}^l$ is the unknown bounded disturbance with $\underline{d}(t) \leq\leq d(t) \leq\leq \overline{d}(t)$ and initial conditions are also unknown but bounded between two bounds, that is, $\underline{x}_0 \leq\leq x_0 \leq\leq \overline{x}_0$. $A(y)$, B and C are the matrices with appropriate dimensions with $A(y)$ as a Metzler matrix for all $y \in \mathbb{R}^p$, $B \geq 0$ and $C \geq 0$. Let $x(t)$ be the unique solution of system (5.1).

Lemma 5.1 (Framer design) *Consider two dynamics*

$$\dot{\overline{x}} = A(y)\overline{x} + K(y - \overline{y}) + f(y,u) + B\overline{d}, \quad \overline{x}(t_0) = \overline{x}_0 \tag{5.2}$$

$$\dot{\underline{x}} = A(y)\underline{x} + K(y - \underline{y}) + f(y,u) + B\underline{d}, \quad \underline{x}(t_0) = \underline{x}_0 \tag{5.3}$$

where $\overline{y} = C\overline{x}$, $\underline{y} = C\underline{x}$ *and* K *is a gain matrix with appropriate dimension such that* $-KC \geq 0$. *Let* $\overline{x}(t)$ *and* $\underline{x}(t)$ *be the unique solutions of the systems (5.2) and (5.3) respectively, then* $\overline{x}(t)$ *and* $\underline{x}(t)$ *are the upper and lower bounds for the state* $x(t)$.

Proof: Let $\overline{e} = \overline{x} - x$ and $\underline{e} = x - \underline{x}$ be the upper observation and lower observation errors, respectively. The aim is to prove that $\overline{e}(t)$ and $\underline{e}(t)$ are non-negative. The dynamics of the upper error follow

$$\dot{\overline{e}} = (A(y) - KC)\overline{e} + B(\overline{d} - d). \tag{5.4}$$

Similarly, the dynamics of the lower error are described by

$$\dot{\underline{e}} = (A(y) - KC)\underline{e} + B(d - \underline{d}). \tag{5.5}$$

Because $\underline{d} \leq\leq d \leq\leq \overline{d}$ and $B \geq 0$, we have $B(\overline{d} - d) \geq\geq 0$ and $B(d - \underline{d}) \geq\geq 0$. Bearing in mind $A(y)$ Metzler for all $y \in \mathbb{R}^p$ and $-KC \geq 0$, we deduce that $A(y) - KC$ Metzler for all $y \in \mathbb{R}^p$. Moreover, from the fact that $\overline{e}_0 = \overline{x}_0 - x_0 \geq\geq 0$ and $\underline{e}_0 = x_0 - \underline{x}_0 \geq\geq 0$, it follows that, for all $t \geq 0$, $\overline{e}(t) \geq\geq 0$ and $\underline{e}(t) \geq\geq 0$. Thus, $\underline{x}(t) \leq\leq x(t) \leq\leq \overline{x}(t)$. This allows us to conclude that (5.2)-(5.3) is a framer for (5.1). □

Remark 5.2 *If $A(y)$ is not a Metzler matrix, we can use a change of coordinates or any other transformations existing in the literature to make it Metzler as shown in particular in ([55, 57, 58]). We illustrate the case where $A(y)$ is not a Metzler matrix in Example 5.8 of Section 5.3.*

Usually, it is not very difficult to achieve the framer property, which is the notion of providing intervals in which state variables stay, if one does not care the length of estimated intervals. In fact, one can use an artificial system that over-bounds terms to secure the positivity (non-negativity, more precisely) ensuring the framer property [114]. Therefore, tools and component ideas of the framer design are not necessarily novel. The framer design proposed in Lemma 5.1 is the same as in [57] and as in many other studies on interval observers for nonlinear systems in the literature. The goal of this work is to present a reasonable convergence property of the interval, which is not always clear. In Assumption 3 of [57], the authors clarify how they can qualify the convergence property by employing Lyapunov stability, which requires complex and difficult computations, as the authors illustrated them in the numerical example. The approach we propose in the present work is very different from [57] since it is based on ideas from contraction theory, and hardly any computation is needed to expose it.

We introduce the following theorem as guidelines for selecting K in Lemma 5.1 to make framer (5.2)-(5.3) become interval observer (i.e., framer satisfies a convergence property) for (5.1).

Theorem 5.3 *Consider the system (5.1) without disturbance $d(t)$. Let us suppose that there exists the gain matrix K in Lemma 5.1 such that the squared vector-valued distance derivative along virtual dynamics of the system (5.2) follows*

$$\frac{d}{dt}\|\delta\overline{x}\|_v^2 \;<<\; \phi\left(P\ (diag(\delta\overline{x}))^2\mathbb{1}\right), \qquad \forall\, t \geq t_0$$

for a nonzero matrix $P = [p_{ij}]_{n\times n}$ with all real $p_{ij} \geq 0$, function $\phi \in C\left[\mathbb{R}^n,\ \mathbb{R}^n\right]$ satisfies the quasi-monotonicity non-decreasing property, and the comparison system obtained from this inequality is contracting, then the error $\|e(t)\| = \|\overline{x}(t) - x(t)\|$ converges exponentially to zero, that is, for some constants $s > 0$ and $k > 0$

$$\|e(t)\| \leq k e^{s(t-t_0)}\|e(t_0)\|.$$

Proof: Consider the squared vector-valued distance function as defined by (4.1) for the system (5.2). The squared vector- valued distance derivative along the virtual dynamics of the system (5.2) is given by

$$\frac{d}{dt}(\|\delta\overline{x}\|_v^2) = 2P\ \mathrm{diag}(\delta\overline{x})\ (\mathrm{diag}\delta\dot{\overline{x}})\mathbb{1}$$

The gain matrix K is designed such that the above equality transforms to the following inequality

$$\frac{d}{dt}(\|\delta\overline{x}\|_v^2) << \ \phi\left(P\ (\mathrm{diag}(\delta\overline{x}))^2\mathbb{1}\right) \qquad \forall\, t \geq t_0, \tag{5.6}$$

where ϕ is a quasi-monotone non-decreasing function and the comparison system obtained from this inequality (5.6), let's say, $\dot{w} = \phi(w), w \in \mathbb{R}^n$ is contracting. Hence, from Theorem 4.2, the distance between any pair of trajectories $\|\delta\overline{x}\|$ of the estimated system (5.2) converges exponentially to zero since the comparison system trajectories converge exponentially to zero (comparison system is contracting). This means that, for some constants $k > 0$ and $s > 0$, we have

$$\|\overline{x}_1(t) - \overline{x}_2(t)\| \leq\ ke^{-s(t-t_0)}\|\overline{x}_1(t_0) - \overline{x}_2(t_0)\|,\ \ \forall\, t \geq t_0, \tag{5.7}$$

for any two solutions $\overline{x}_1(t)$ and $\overline{x}_2(t)$ of the system (5.2). Furthermore, when $d(t) = 0$, $x(t)$ is a particular solution of the system (5.2) since system (5.2) and (5.1) only differ in the correction term. Hence, the correction term vanishes when $x(t)$ is the solution of (5.2). Thus, $\overline{x}_2(t)$ and $\overline{x}_1(t)$ can be replaced by $x(t)$ and $\overline{x}(t)$ respectively, therefore the above equation (5.7) becomes

$$\|e(t)\| = \|\overline{x}(t) - x(t)\| \leq ke^{-s(t-t_0)}\|e(t_0)\|.$$

Hence, it is proved that the error of estimation converges exponentially to zero. This completes the proof. □

In a similar way, one can prove the exponential convergence of the error $\|e(t)\| = \|x(t) - \underline{x}(t)\|$ to zero. The proposed method has the advantage that it does not require the error dynamics formulation to show that the estimation error $\|e(t)\| = \|\overline{x}(t) - x(t)\| = \|x(t) - \underline{x}(t)\|$ converges exponentially to zero, when the disturbance $d(t) = 0$. It only requires the virtual dynamics of the estimation dynamics (5.2)-(5.3) to show that the derivative of the squared vector distance along this virtual dynamics follows (5.6).

Now, we provide a result to design feedback control $u(y, \overline{x})$ to make the interval observer (5.2)-(5.3) to be asymptotic stable when $d, \overline{d}$ and $\underline{d}$ are zero.

Theorem 5.4 *Consider the system (5.1) (5.2) (5.3) with* $d(t) = 0$, $\overline{d}$ *and* $\underline{d} = 0$. *Let us assume that the origin is an equilibrium point of the system (5.1). Suppose there exists the control* $u(y, x)$, $u(0, 0) = 0$, u *is a general smooth nonlinear function, such that the derivative of squared vector-valued distance along virtual dynamics of the system (5.1) follows*

$$\frac{d}{dt}(\|\delta x\|_v^2) \;<<\; \phi\left(P\ (diag(\delta x))^2 \mathbb{1}, x\right) \qquad \forall\ t \geq t_0,$$

for $P = [p_{ij}]_{n \times n}$, $p_{ij} \geq 0$, $\phi \in C\left[\mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^n\right]$ *satisfies the property of quasi-monotonicity non-decreasing, and the comparison system obtained from this inequality is contracting. Then, the origin of the system (5.1) is globally asymptotically stable. Further, an interval observer (5.2)-(5.3) is asymptotically stable with the control* $u(y, \overline{x})$, *when* $\overline{d}$ *and* $\underline{d}$ *are zero.*

Proof: The virtual dynamics of the system (5.1) with the control $u(y, x)$ at fixed time t is obtained as

$$\delta \dot{x} = \left(\frac{\partial A(y)}{\partial x} x + A(y) + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right)\delta x \tag{5.8}$$

Consider the squared vector-valued distance function as defined by (4.1) for the system (5.1). The squared vector-valued distance derivative along the trajectories of system (5.8) is given by

$$\frac{d}{dt}(\|\delta x\|_v^2) = 2P\ \mathrm{diag}(\delta x)\ \left(\mathrm{diag}\Big(\frac{\partial A(y)}{\partial x} x + A(y) + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\Big)\delta x\right)\mathbb{1}$$

Now, the control $u(y, x)$ is designed such that the above squared vector valued distance derivative follows the inequality

$$\frac{d}{dt}(\|\delta x\|_v^2) \;<<\; \phi\left(P\ (\mathrm{diag}(\delta x))^2 \mathbb{1}, x\right) \qquad \forall\ t \geq t_0,$$

where ϕ is a quasi-monotone non-decreasing function and the comparison system obtained from this inequality, let's say, $\dot{w} = \phi(w, x), w \in \mathbb{R}^n$ is contracting. Thus, from the results of Theorem 4.2, the distance between any pair of trajectories $\|\delta x\|$ of the system (5.1) converges exponentially to zero, that is, the trajectories converge indeed to an equilibrium point (origin). Hence, it is proved that with the control $u(y, x)$, the origin of the system (5.1) is globally asymptotically stable. Now, the proof goes in a similar way to show that the interval observer (5.2)-(5.3) is also asymptotically stable with the control $u(y, \overline{x})$, when $\overline{d}$ and $\underline{d}$ are zero. □

5.3 Simulation Examples

Example 5.5 *Consider the system*

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 + x_1) + d(t)\\ \dot{x}_2 &= -\eta x_2 + \sin^2(x_1)x_3 + x_1^2\sin^2(x_1 - x_3) + d(t)\\ \dot{x}_3 &= \beta x_3 + \cos^2(x_1 - x_3)x_2 + \rho x_1 + d(t)\end{aligned} \tag{5.9}$$

with outputs $y_1 = x_1$ *and* $y_2 = x_1 - x_3$*, where* $x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$ *is the state vector,* $d(t)$ *is the unknown disturbance with* $\underline{d} \leq d(t) \leq \overline{d}$*. Parameter values are given as* $\sigma = 1, \eta = 5, \rho = 5$ *and* $\beta = 5$.

We write the above system in the form

$$\dot{x} = A(y) + f(y) + Bd(t) \tag{5.10}$$

with

$$A(y) = \begin{bmatrix}\sigma & \sigma & 0\\ 0 & -\eta & \sin^2(y_1)\\ 0 & \cos^2(y_2) & \beta\end{bmatrix},\quad f(y) = \begin{bmatrix}0\\ y_1^2\sin^2(y_2)\\ \rho y_1\end{bmatrix}$$

and $B = [1, 1, 1]^\top$*. It can be observed that* $A(y)$ *is Metzler for all* y_1 *and* y_2 *since* $\cos^2(y_2)$ *and* $\sin^2(y_1)$ *are always non-negative for all* y_1 *and* y_2*, but not Hurwitz matrix. Now, an interval observer is designed as follows*

$$\dot{\overline{x}} = A(y)\overline{x} + K(y - \overline{y}) + f(y) + B\overline{d} \tag{5.11}$$

$$\dot{\underline{x}} = A(y)\underline{x} + K(y - \underline{y}) + f(y) + B\underline{d} \tag{5.12}$$

where $K = \begin{bmatrix}K_1 & 0\\ 0 & 0\\ 0 & -K_2\end{bmatrix}$ *is the gain matrix and* $y = Cx$*,* $C = \begin{bmatrix}1 & 0 & 0\\ 1 & 0 & -1\end{bmatrix}$*. We use vector-based contraction approach with* $d(t) = 0$ *to prove the convergence property of the designed interval observer as the construction of the Lyapunov candidate is not easy for a nonlinear system and also there is no need of any specific attractor to prove convergence. Firstly, we consider the system (5.11), the virtual dynamics of the system (5.11) is given by*

$$\begin{aligned}\delta\dot{\overline{x}}_1 &= (\sigma - K_1)\delta\overline{x}_1 + \sigma\delta\overline{x}_2\\ \delta\dot{\overline{x}}_2 &= -\eta\delta\overline{x}_2 + \sin^2(y_1)\delta\overline{x}_3\\ \delta\dot{\overline{x}}_3 &= K_2\delta\overline{x}_1 + \cos^2(y_2)\delta\overline{x}_2 + (\beta - K_2)\delta\overline{x}_3\end{aligned} \tag{5.13}$$

Let the squared vector-valued norm, defined by (4.1), assuming the matrix P as diag($\mathbb{1}$) *be:* $\|\delta\overline{x}\|_v^2 = [\delta\overline{x}_1^2, \delta\overline{x}_2^2, \delta\overline{x}_3^2]^\top$. *Its derivative along the trajectories of (5.13) is given by*

$$
\begin{aligned}
\frac{d}{dt}(\delta\overline{x}_1)^2 &= 2(\sigma - K_1)\delta\overline{x}_1^2 + 2\sigma\delta\overline{x}_1\delta\overline{x}_2 \\
&\leq (3\sigma - 2K_1)\delta\overline{x}_1^2 + \sigma\delta\overline{x}_2^2 \\
\frac{d}{dt}(\delta\overline{x}_2)^2 &\leq (-2\eta + |\sin^2(y_1)|)\delta\overline{x}_2^2 + |\sin^2(y_1)|\delta\overline{x}_3^2 \\
\frac{d}{dt}(\delta\overline{x}_3)^2 &\leq |K_2|\delta\overline{x}_1^2 + |\cos^2(y_2)|\delta\overline{x}_2^2 + (2\beta - 2K_2 + |K_2| + |\cos^2(y_2)|)\delta\overline{x}_3^2
\end{aligned}
$$

From the above inequalities, we obtain the quasi-monotone non-decreasing (off-diagonal entries non-negative) comparison system, $\dot{w} = Gw$, *where*

$$
G = \begin{bmatrix} (3\sigma - 2K_1) & \sigma & 0 \\ 0 & (|\sin^2(y_1)| - 2\eta) & |\sin^2(y_1)| \\ |K_2| & |\cos^2(y_2)| & (2\beta - 2K_2 + |K_2| + |\cos^2(y_2)|) \end{bmatrix}
$$

We select the gains $K_1 = 10$ *and* $K_2 = 40$ *to make the above matrix G Hurwitz (contracting) and* $-KC \geq 0$. *Hence, the original dynamics (5.11) is contracting. In a similar way, the dynamics (5.12) can be proved to be contracting. Thus, the system (5.11)-(5.12) is an interval observer for the system (5.9). The simulation results are shown in Fig. 5.1 and with and without disturbance with values:* $3 \leq x_1(0) \leq 4.5$, $-10 \leq x_2(0) \leq -5$, $4 \leq x_3(0) \leq 7$, $x(0) = [4, -7, 5]^\top$, $d(t) = 0.25\sin(t)$, $\underline{d} = -0.25$ *and* $\overline{d} = 0.25$. *Figure 5.2 shows the simulation results with disturbance* $d(t) = 0.25\sin(t)$ *and measurement noise* $\nu(t) = [-0.2, 0.2]$ *present in the outputs as* $y_1 = x_1 + \nu(t)$ *and* $y_2 = x_1 - x_3 + \nu(t)$. *Interval observers work well under the effect of unknown measurement noises, however the bound of the noises must be known.*

Example 5.6 *Consider the following self-excited nonlinear oscillator ([112])*

$$
\begin{aligned}
\dot{x}_1 &= x_2 + d(t) \\
\dot{x}_2 &= -\omega_1^2\sin(x_1) - \rho x_2 + k_1\arctan(k_2(x_1 - x_3)) + d(t) \\
\dot{x}_3 &= \omega_2(x_1 - x_3) + d(t)
\end{aligned} \tag{5.14}
$$

with outputs $y_1 = x_1$ *and* $y_2 = x_1 - x_3$, *where* $x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$ *is the state vector,* $d(t)$ *is the unknown disturbance with* $\underline{d} \leq d(t) \leq \overline{d}$. *Parameter values are given as:* $\omega_1 = \omega_2 = 40, k_1 = 5, k_2 = 10$ *and* $\rho = 1$. *We write the above system (5.14) in the form*

$$
\dot{x} = Ax + f(y) + Bd(t) \tag{5.15}
$$

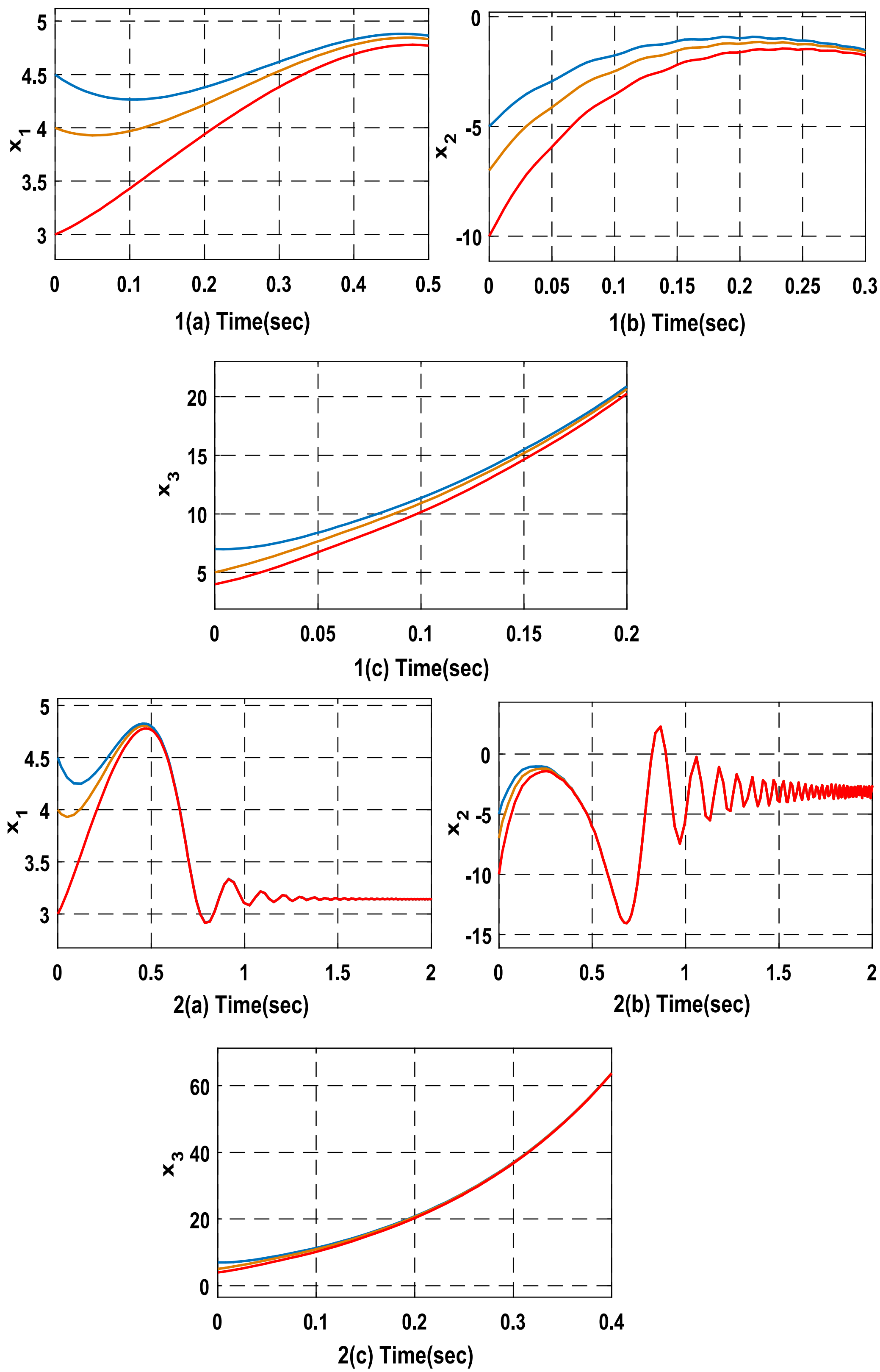

Figure 5.1: Interval observer for system (5.9) (1(a)(b)(c) with disturbance and 2(a)(b)(c) without disturbance)

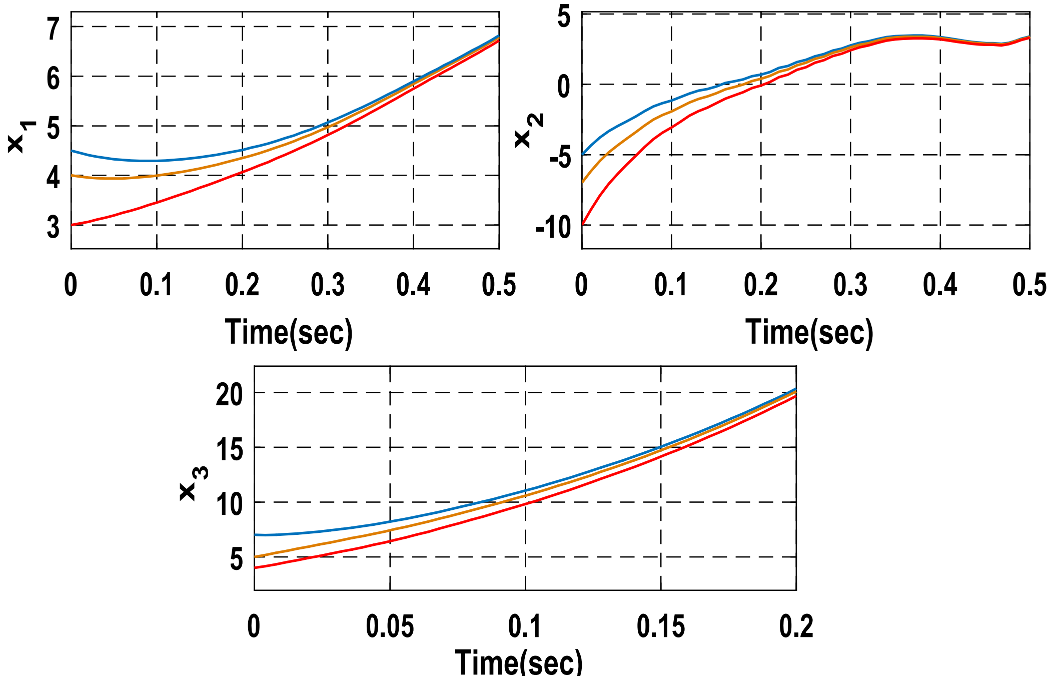

Figure 5.2: Interval observer for system (5.9) with disturbance and measurement noise

with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\rho & 0 \\ \omega_2 & 0 & -\omega_2 \end{bmatrix}, \; f(y) = \begin{bmatrix} 0 \\ -\omega_1^2 \sin(y_1) + k_1 \arctan(k_2 y_2) \\ 0 \end{bmatrix}$$

and $B = [1,1,1]^\top$. *It is observed that the system matrix* A *is Metzler but not Hurwitz matrix. Now, we design an interval observer as follows:*

$$\dot{\overline{x}} = A\overline{x} + K(y - \overline{y}) + f(y) + B\overline{d} \tag{5.16}$$

$$\dot{\underline{x}} = A\underline{x} + K(y - \underline{y}) + f(y) + B\underline{d} \tag{5.17}$$

where $K = \begin{bmatrix} K_1 & 0 \\ 0 & 0 \\ 0 & -K_2 \end{bmatrix}$ *is the gain matrix and* $y = Cx$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. *In a similar way, we consider first the system (5.16), the virtual dynamics of the system (5.16) is given by*

$$\begin{aligned} \delta\dot{\overline{x}}_1 &= -K_1\delta\overline{x}_1 + \delta\overline{x}_2 \\ \delta\dot{\overline{x}}_2 &= -\rho\delta\overline{x}_2 \\ \delta\dot{\overline{x}}_3 &= (\omega_2 + K_2)\delta\overline{x}_1 - (\omega_2 + K_2)\delta\overline{x}_3 \end{aligned} \tag{5.18}$$

Let the squared vector-valued norm, defined by (4.1), assuming the matrix P as diag($\mathbb{1}$) be: $\|\delta\overline{x}\|_v^2 = [\delta\overline{x}_1^2, \delta\overline{x}_2^2, \delta\overline{x}_3^2]^\top$. *Its derivative along the trajectories of (5.18) is given by*

$$\begin{aligned}\frac{d}{dt}(\delta\overline{x}_1)^2 &= -2K_1\delta\overline{x}_1^2 + 2\delta\overline{x}_1\delta\overline{x}_2 \\ &\leq (1-2K_1)\delta\overline{x}_1^2 + \delta\overline{x}_2^2 \\ \frac{d}{dt}(\delta\overline{x}_2)^2 &\leq -2\rho\delta\overline{x}_2^2 \\ \frac{d}{dt}(\delta\overline{x}_3)^2 &\leq |\omega_2 + K_2|\delta\overline{x}_1^2 + (-2\omega_2 - 2K_2 + |\omega_2 + K_2|)\delta\overline{x}_3^2\end{aligned}$$

From the above inequalities, we obtain the quasi-monotone non-decreasing (off-diagonal entries non-negative) comparison system, $\dot{w} = Gw$, *where*

$$G = \begin{bmatrix}(1-2K_1) & 1 & 0 \\ 0 & -2\rho & 0 \\ |\omega_2 + K_2| & 0 & (-2\omega_2 - 2K_2 + |\omega_2 + K_2|)\end{bmatrix}$$

We select the gains $K_1 = 5$ *and* $K_2 = 100$ *to make the above matrix* G *Hurwitz (contracting) and* $-KC \geq 0$. *Hence, the original dynamics (5.16) is contracting. Similarly, the dynamics (5.17) can be proved to be contracting. Thus, system (5.16)-(5.17) is an interval observer for the system (5.14). The simulation results are shown in Fig. 5.3 with* $2 \leq x_1(0) \leq 5$, $1 \leq x_2(0) \leq 4$, $3 \leq x_3(0) \leq 8$, $x(0) = [4, 2, 6]^\top$, $d(t) = \sin(t)$, $\underline{d} = -1$ *and* $\overline{d} = 1$.

Example 5.7 *Consider the modified self-excited nonlinear oscillator*

$$\begin{aligned}\dot{x}_1 &= x_2 + d(t) \\ \dot{x}_2 &= -\omega_1^2\sin(x_1) - \rho x_2 + k_1\arctan(k_2(x_1 - x_3)) + d(t) \\ \dot{x}_3 &= \omega_2 x_3 + d(t)\end{aligned} \tag{5.19}$$

with outputs $y_1 = x_1$ *and* $y_2 = x_1 - x_3$, *where* $x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$ *is the state vector,* $d(t)$ *is the unknown disturbance with* $\underline{d} \leq d(t) \leq \overline{d}$. *Parameter values:* $\omega_1 = \omega_2 = 40, k_1 = 5, k_2 = 10$ *and* $\rho = 1$. *We write the above system in the form*

$$\dot{x} = Ax + f(y) + Bd(t) \tag{5.20}$$

with

$$A = \begin{bmatrix}0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \omega_2\end{bmatrix}, \quad f(y) = \begin{bmatrix}0 \\ -\omega_1^2\sin(y_1) + k_1\arctan(k_2 y_2) \\ 0\end{bmatrix}$$

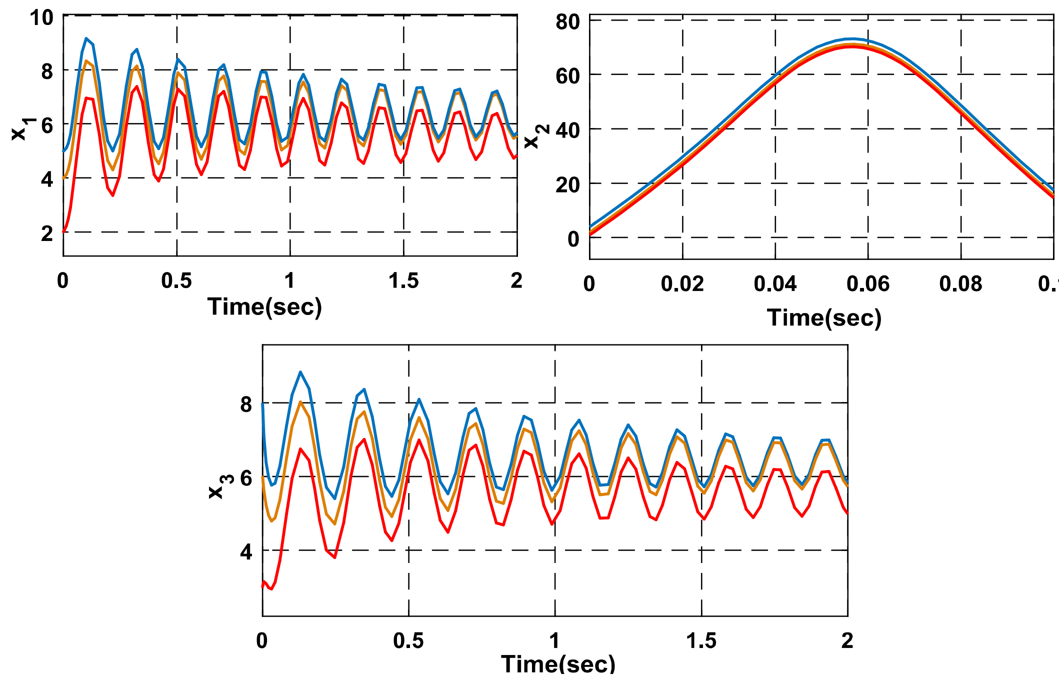

Figure 5.3: Interval observer for system (5.14)

and $B = [1, 1, 1]^\top$. *The system matrix* A *is Metzler but not Hurwitz matrix. Now, we design interval observer as follows:*

$$\dot{\overline{x}} = A(y)\overline{x} + K(y - \overline{y}) + f(y) + B\overline{d} \tag{5.21}$$

$$\dot{\underline{x}} = A(y)\underline{x} + K(y - \underline{y}) + f(y) + B\underline{d} \tag{5.22}$$

where $K = \begin{bmatrix} K_1 & 0 \\ 0 & 0 \\ 0 & -K_2 \end{bmatrix}$ *is the gain matrix and* $y = Cx$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. *We consider first the system (5.21) and obtain the comparison system,* $\dot{w} = Gw$, *where*

$$G = \begin{bmatrix} (1 - 2K_1) & 1 & 0 \\ 0 & -2 & 0 \\ |K_2| & 0 & (2\omega_2 - 2K_2 + |K_2|) \end{bmatrix}$$

We select the gains $K_1 = 5$ *and* $K_2 = 100$ *to make the above matrix* G *Hurwitz (contracting) and* $-KC \geq 0$. *Hence, the original dynamics (5.21) is contracting. In a similar way, the dynamics (5.22) can be proved to be contracting. Thus, the system (5.21)-(5.22) is an interval observer for the system (5.19). The simulation results are shown in Fig. 5.4 with* $2 \leq x_1(0) \leq 5$, $-20 \leq x_2(0) \leq -5$, $3 \leq x_3(0) \leq 8$, $x(0) = [4, -10, 6]^\top$, $d(t) = 0.5\sin(t)$, $\underline{d} = -0.5$ *and* $\overline{d} = 0.5$.

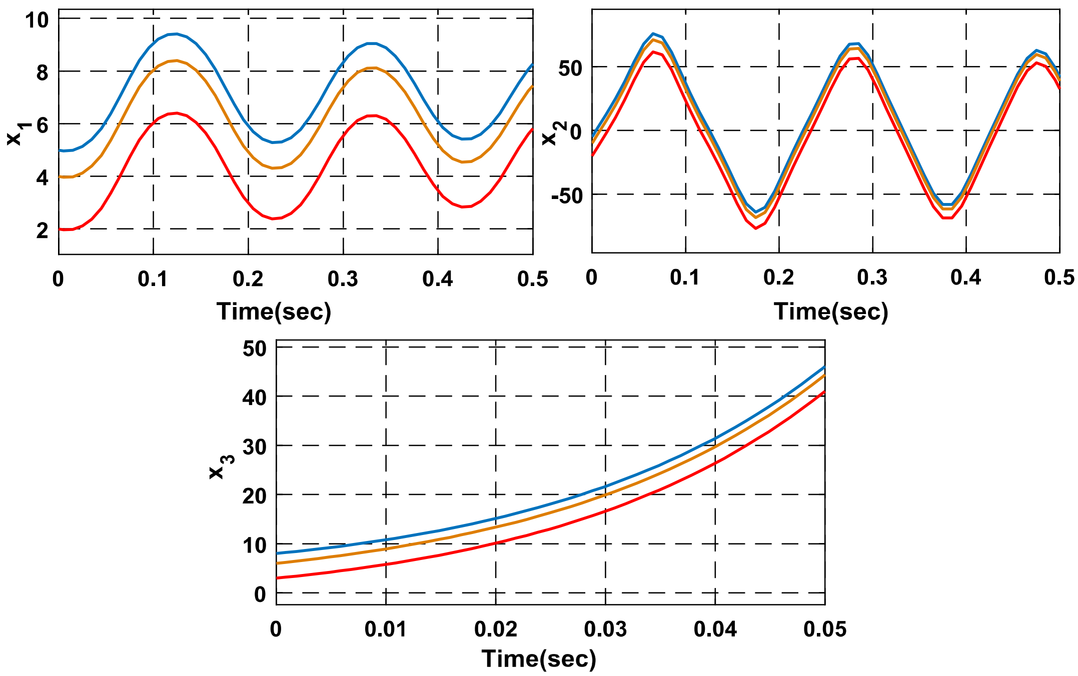

Figure 5.4: Interval observer for system (5.19)

Example 5.8 *Consider a model of an electromechanical system ([113])*

$$\begin{aligned}\dot{x}_1 &= x_2 + d(t)\\ \dot{x}_2 &= b_1 x_3 - a_1 \sin(x_1) - a_2 x_2 + d(t)\\ \dot{x}_3 &= b_0 u - a_3 x_2 - a_4 x_3 + d(t)\end{aligned} \tag{5.23}$$

with output $y = x_1$*, where* $x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$ *is the state vector,* $d(t)$ *is the unknown disturbance with* $\underline{d} \leq d(t) \leq \overline{d}$*. Parameters values are* $b_0 = 40, b_1 = 15, a_1 = 35, a_2 = 0.25, a_3 = 36$ *and* $a_4 = 200$*. We write the above system (5.23) as follows*

$$\dot{x} = Ax + f(y, u) + Bd(t) \tag{5.24}$$

$$\text{with } A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -0.25 & 15 \\ 0 & -36 & -200 \end{bmatrix}, \quad f(y,u) = \begin{bmatrix} 1 \\ -35\sin(y) \\ 40u \end{bmatrix}$$

and $B = [1, 1, 1]^\top$*. It is observed that the system matrix* A *is not a Metzler matrix, but a Hurwitz matrix.*

We design control u *to make the system (5.23) to be asymptotically stable using vector-based contraction theory. To start with, consider the system (5.23) with* $d(t) = 0$*,* *let the input to the first subsystem be* $x_2 = -x_1$ *to obtain the subsystem* $\dot{x}_1 = -x_1$ *to be exponentially stable. Let the deviation variable be* $q_1 = x_2 + x_1$ *to transform the system into the structure*

$$\begin{aligned}\dot{x}_1 &= q_1 - x_1\\ \dot{q}_1 &= b_1 x_3 - a_1\sin(x_1) + 0.75q_1 - 0.75x_1\end{aligned}$$

Let us select the control $x_3 = \frac{1}{b_1}(a_1\sin(x_1) - 2.75q_1 + 0.75x_1)$ *to obtain the subsystem* $\dot{q}_1 = -2q_1$ *to be exponentially stable. Let us again select the deviation variable as* $q_2 = x_3 - \frac{a_1}{b_1}\sin(x_1) + \frac{2.75q_1}{b_1} - \frac{0.75}{b_1}x_1$ *to transform the system into*

$$\begin{aligned}\dot{x}_1 &= q_1 - x_1\\ \dot{q}_1 &= q_2 - 2q_1\\ \dot{q}_2 &= b_0 u - (a_3 + \frac{6.25}{b_1})x_2 - a_4 x_3 - \frac{a_1}{b_1}\cos(x_1)x_2 - \frac{5.5}{b_1}x_1 + \frac{2.75}{b_1}q_2\end{aligned} \tag{5.25}$$

Let the squared vector-valued norm, defined by (4.1), assuming the matrix P *as diag*$(\mathbb{1})$ *be:* $\|\delta x_d\|_v^2 = [\delta x_1^2, \delta q_1^2, \delta q_2^2]^\top$. *Its derivative along the trajectories of the virtual dynamics of (5.25) is given by*

$$\begin{aligned}\frac{d}{dt}(\delta x_1)^2 \leq & -\delta x_1^2 + \delta q_1^2\\ \frac{d}{dt}(\delta q_1)^2 \leq & -3\delta q_1^2 + \delta q_2^2\\ \frac{d}{dt}(\delta q_2)^2 = & 2\delta q_2(b_0\delta u - (a_3 + \frac{6.25}{b_1})\delta x_2 - a_4\delta x_3 - \frac{5.5}{b_1}\delta x_1\\ & - \frac{a_1}{b_1}(-\sin(x_1)x_2\delta x_1 + \cos(x_1)\delta x_2) + \frac{2.75}{b_1}\delta q_2)\end{aligned}$$

We select

$$\begin{aligned}\delta u =& \frac{1}{b_0}((a_3 + \frac{6.25}{b_1})\delta x_2 + a_4\delta x_3 + \frac{5.5}{b_1}\delta x_1 + \frac{a_1}{b_1}(-\sin(x_1)x_2\delta x_1 + \cos(x_1)\delta x_2) - \frac{2.75}{b_1}\delta q_2\\ & - a_4 q_2)\end{aligned}$$

to obtain the following linear comparison system

$$\begin{aligned}\dot{w}_1 =& -w_1 + w_2\\ \dot{w}_2 =& -3w_2 + w_3\\ \dot{w}_3 =& -2a_4 w_3\end{aligned}$$

to be quasi-monotone non-decreasing (off-diagonal entries non-negative) and contracting. Hence, the system (5.23) with $d(t) = 0$ is contracting and thus the system trajectories converge to indeed an equilibrium point (origin) with the obtained control u (obtained by integrating δu),

$$u = \frac{1}{b_0}\Big((a_3 + \frac{(6.25)}{b_1} - 36.7)x_2 + (\frac{5.5}{b_1} - 26.69)x_1 + \frac{2a_1}{b_1}\cos(x_1)x_2 + \frac{a_1}{b_1}(a_4 + \frac{2.75}{b_1})\sin(x_1) \\ - \frac{2.75}{b_1}x_3\Big).$$

Now, the interval observer is formulated using time-invariant transformation as discussed in [57] to transform the system matrix into a Metzler matrix. And, from Theorem 5.4, the formulated interval observer is asymptotically stable with the control $u(y, \bar{x})$. We compare the performance of the interval observer with the control designed in [57] and the proposed control using state bounds from the interval observer itself and the system outputs considering $d(t) = \frac{1}{9}\sin(t)$, $\underline{d} = -\frac{1}{9}$ and $\overline{d} = \frac{1}{9}$.

From Fig. 5.5, it can be noticed that the convergence time is less in the case of the proposed method as compared to the method in [57]. Moreover, it does not require the Lyapunov candidate function formulation to show the asymptotic stability of the interval observer.

5.4 Conclusion

An interval observer for a class of nonlinear systems considering inputs, outputs, and disturbances has been designed by the exploitation of the generalized contraction theory known as vector-based contraction theory. Dynamic output feedback control has been designed using state bounds from the constructed interval observer to prove it to be globally asymptotic stable. In the end, examples are illustrated to show the efficacy of the theoretical results.

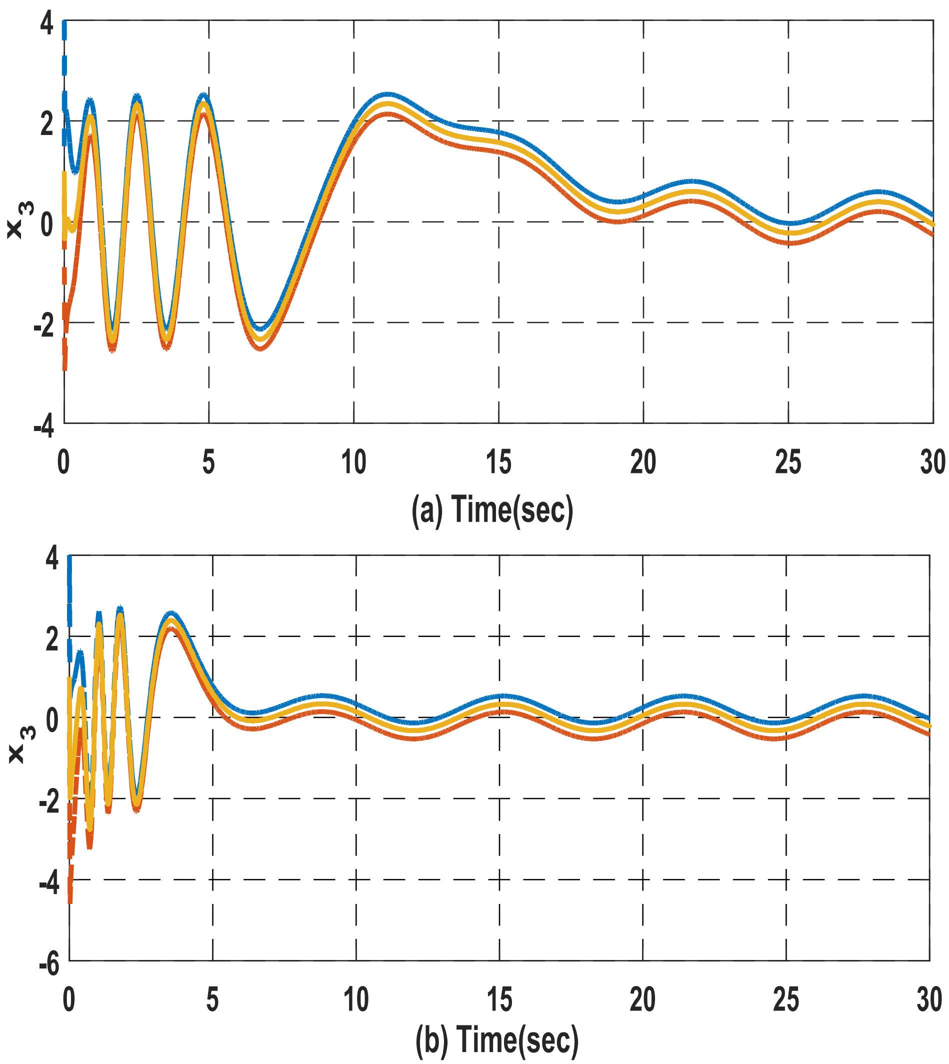

Figure 5.5: Interval observer for system (5.23) using (a) method in [57] and (b) proposed method