Chapter 3

Vector Control Lyapunov function based stabilization of nonlinear systems in arbitrary time

3.1 Introduction

Vector Lyapunov functions were first introduced in [44] to relax certain strict conditions of scalar Lyapunov functions [4–6]. In particular, it is worth observing that the components of vector Lyapunov functions need not be all positive definite and that the derivative of a vector Lyapunov function does not have to be necessarily negative or negative semi-definite to guarantee the stability of the studied systems. Hence, these functions enlarge the class of Lyapunov functions to analyze system stability.

In this chapter, a general framework is developed to analyze the arbitrary time stability of the equilibrium point of nonlinear systems using vector Lyapunov functions. Specifically, we formulate a vector comparison system in such a way that it is arbitrary time stable and after that we relate these stability features with the stability features of the original system using differential inequalities and comparison principles. Besides, we design robust universal arbitrary time convergent controllers for the large-scale systems that are robust to bounded disturbances. Moreover, in order to reduce the dimension of the comparison systems, we discuss the aggregation procedure of comparison systems which provides a simple and efficient way to derive control for the class of underactuated systems. The control of these underactuated systems is a very challenging problem as they represent a surface vessel or underactuated ships for which there is no actuation provided in the sway axis. In the end, the efficacy of the theoretic approach is verified using as example an underactuated system.

Now, we consider a nonlinear time-varying scalar differential system

$$\dot{x} = \; -\phi(t,x) := \begin{cases} \frac{-\gamma(e^{x}-1)}{e^{x}(t_a - t)}, & \text{if } \; t_0 \le t < t_a \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}$, $\gamma \in \mathbb{R}_{>1}$, t_0 is the initial time and $t_a > t_0$ is a prescribed time. It is easy to prove existence and uniqueness of the solutions of this system and to see that $\dot{x}(t) = 0$ and $x(t) = 0$ for all $t \ge t_a$ [42]. This system is the building block to establish the main results of this chapter.

Remark 3.1 *Note that t_a does not depend on any system parameter. In fact, t_a itself is an independent parameter, which is explicitly prescribed in advance. Theoretically, one can choose any arbitrarily small value of t_a. However, we recall that due to the inherent dynamics of practical systems (in particular, the actuator dynamics), these systems usually impose restrictions on assuming arbitrarily small values of t_a.*

The further part of the chapter is organized as follows. Section 3.2 provides the results of the arbitrary time stability of nonlinear systems by the exploitation of vector Lyapunov functions. Robust universal arbitrary time controllers that are robust to bounded disturbances are designed for large-scale systems in Section 3.3. In addition, we discuss the aggregation procedure of comparison systems in order to apply the derived results effectively on underactuated systems. An illustrative example with the simulation results is given in Section 3.4. Finally, conclusions are drawn in Section 3.5.

3.2 Arbitrary time stability analyzed via vector Lyapunov function

In this section, we derive results by using vector Lyapunov functions to analyze the arbitrary time stability of nonlinear systems.

Theorem 3.2 *Consider the system (2.2). Suppose that there exist a smooth vector function $V = [V_1, V_2, \ldots, V_p]^\top : D \to S$, where $p \le n$, $S \subset \mathbb{R}^p_{\ge 0}, 0 \in S$ and a vector $r \in \mathbb{R}^p_{\ge 0}$*

such that $r^\top V(x)$ *is a positive definite function, and there exists a control input* $\tau(t,x)$ *such that*

$$V'(x)f(x,\tau(t,x)) \leq\leq M\Phi_{fr}(t,V(x)), \quad x \in D \tag{3.2}$$

where $\Phi_{fr}(t,V(x)) := [\phi(t,V_1(x)),\cdots,\phi(t,V_p(x))]^\top$*,* ϕ *is the function defined in (3.1),* $M \in \mathbb{R}^{p\times p}$ *is Metzler and Hurwitz, and such that* $y^\top M \leq\leq -y^\top$ *for all non-negative vector* $y \in \mathbb{R}^p$*. Besides, suppose the following vector comparison system*

$$\dot{\eta}(t) = M\Phi_{fr}(t,\eta(t)), \qquad \eta(t_0) = \eta_0, \quad \textit{for all } t \geq t_0, \tag{3.3}$$

admits a unique solution $\eta(t) \in \mathbb{R}^p_{\geq 0}$*. Let* $x(t)$ *be any solution of (2.2) with* $\tau(t,x)$ *which satisfies (3.2), such that* $V(x_0) \leq\leq \eta_0$*. Then, the solution* $x(t) = 0$ *is arbitrary time stable for* $\gamma > p$*.*

Proof: Let us consider the comparison system (3.3). Observe that $M\Phi_{fr}(t,\eta)$ is a quasi-monotone non-decreasing function of η uniformly in t. As a consequence, the solutions to (3.3) are non-negative when $\eta_0 \in \mathbb{R}^p_{\geq 0}$ [102]. Now, let us consider the Lyapunov function $v = \eta^\top\eta$, $\eta \in \mathbb{R}^p_{\geq 0}$. Its time derivative along the trajectories of (3.3) is given by $\dot{v} = 2\eta^\top(t)M\Phi_{fr}(t,\eta(t))$. Then, from the definition of $\phi(\cdot)$ in (3.1), it follows that $\dot{v} = 0$ for all $t \geq t_a$. Now let us perform an analysis in the time interval $[t_0,t_a)$. Since $\eta^\top M \leq\leq -\eta^\top$, it follows that

$$\dot{v} \leq -2\eta^\top(t)\Phi_{fr}(t,\eta(t)), \quad \text{for all } t \in [t_0,t_a). \tag{3.4}$$

Let us introduce the function $\mathcal{M}ax$ defined by $\mathcal{M}ax(y) = \max\limits_{i\in\{1,\ldots,p\}} y_i$. Observe that the inequality (3.4) implies that

$$\dot{v} \leq -2\eta_1(t)\phi(t,\eta_1(t)),\cdots,\dot{v} \leq -2\eta_p(t)\phi(t,\eta_p(t)) \tag{3.5}$$

because $y\phi(t,y) \geq 0$ for all $y \in \mathbb{R}$ and $t \in [t_0,t_a)$. Suppose that at any particular instant $t \in [t_0,t_a)$, $\mathcal{M}ax(\eta(t)) = \eta_1(t)$. Then

$$||\eta(t)||^2 \leq p\eta_1(t)^2 \implies v(t) \leq p\eta_1(t)^2 \implies \sqrt{\frac{v(t)}{p}} \leq \eta_1(t). \tag{3.6}$$

Now using (3.5)-(3.6) and noting the fact that $-y_1\phi(y_1) \leq -y_2\phi(y_2)$ when $y_2 \leq y_1$, it is easy to obtain $\dot{v} \leq -2\sqrt{\frac{v}{p}}\phi\left(t,\sqrt{\frac{v}{p}}\right)$. Let us introduce the function: $w = \sqrt{\frac{v}{p}}$. Then, when $v(\eta_0) > 0$, the inequality $v(\eta(t)) > 0$, is satisfied for all $t \in [t_0,t_a)$. We deduce

$\dot{w} = \frac{1}{2\sqrt{vp}}\dot{v} \leq \frac{-w\phi(t,w)}{\sqrt{vp}} \leq \frac{-\phi(t,w)}{p}$, for all $t \in [t_0, t_a)$. From the definition of $\phi(\cdot)$, it follows that $\dot{w} \leq \frac{-\gamma'(e^w-1)}{e^w(t_a-t)}$, $\gamma' = \gamma/p$, for all $t \in [t_0, t_a)$. Using the fact that $\dot{v} = 0$ for all $t \geq t_a$, we deduce that $\dot{w} = 0$, for all $t \geq t_a$. Note that for $\gamma' > 1$ (i.e., $\gamma > p$), the dynamics of w is arbitrary time stable [42]. Consequently, the dynamics of v is also arbitrary time stable which implies that the solution $\eta(t) = 0$ is arbitrary time stable. Then from the results of Lemma 2.12, we conclude that the solution $x(t) = 0$ is arbitrary time stable for $\gamma > p$. Note that a similar analysis can be carried out to show the arbitrary time of convergence in the cases when $\mathcal{M}ax$ returns variables other than η_1. Let us observe that in the scalar case, i.e., $p = 1$, V reduces to V_1 and M is a constant m such that $m \leq -1$. Then, the condition in (3.2) reduces to $V_1'(x)f(x, \tau(t,x)) \leq m\phi(t, V_1(x)),\ x \in D$, which directly ensures that for $\gamma > 1$, the dynamics is arbitrary time stable. This completes the proof. □

Remark 3.3 *It is important to discuss about the matrix M that satisfies $y^\top M \leq\leq -y^\top$ for $y \in \mathbb{R}^p_{\geq 0}$. This property implies that M is Metzler and Hurwitz. Let us see some examples of M. The given condition leads to $y^\top(M + I) \leq\leq 0$, which can be alternatively written as $(M^\top + I)y \leq\leq 0$. One obtains, by selecting $M = \lambda I$, $(\lambda + I)y \leq\leq 0$ which holds for all $\lambda \leq -1$. Although several other possibilities exist for M, above ones are the simplest.*

Theorem 3.2 is generalized as follows:

Theorem 3.4 *Consider the system (2.2). Let us suppose that there exist a smooth vector function $V = [V_1, V_2, \ldots, V_p]^\top : D \to S$ where $p \leq n$, $S \subset \mathbb{R}^p_{\geq 0}, 0 \in S$ and a vector $r \in \mathbb{R}^p_{\geq 0}$ such that $r^\top V(x)$ is a positive definite function, and there exists $\tau(t,x)$ such that*

$$V'(x)f(x, \tau(t,x)) \leq\leq Q(t, V(x)), \quad x \in D,\ t \geq t_0 \tag{3.7}$$

where $Q \in C[\mathbb{R}_{\geq 0} \times S, \mathbb{R}^p]$ is a quasi-monotone non-decreasing function of V uniformly in t with $Q(t, 0) = 0$ for all $t \geq t_0$. Besides, suppose the following vector comparison system

$$\dot{\eta}(t) = Q(t, \eta(t)), \quad \eta(t_0) = \eta_0. \tag{3.8}$$

admits a unique solution $\eta(t) \in \mathbb{R}^p_{\geq 0}$ and arbitrary time stable. Let $x(t)$ be any solution of (2.2) with $\tau(t,x)$ which satisfies (3.7), such that $V(x_0) \leq\leq \eta_0$. Then, $x(t) = 0$ is arbitrary time stable.

Proof: Let us assume that $U \subseteq H$ is an open and bounded set such that $0 \in U$ and $\bar{U} \subset S$. Hence, ∂U is compact. Since, the function $v(\cdot)$ is assumed to be continuous, then, from the Weierstrass result, $v(\cdot)$ has a minimum on ∂U and $\alpha = \min_{\eta \in \partial U} v(\eta) > 0$. Suppose that $0 < \beta < \alpha$ and $D_\beta = \{\eta \in U : v(\eta) \leq \beta\}$. From the classical Lyapunov stability and positive definiteness of $v(\cdot)$, one can state that if $\epsilon > 0$, there exists $\delta > 0$ such that the ball, $B_\delta \subset D_\beta \subset H$ and $\|\eta(t)\| \leq \epsilon$, $\forall t \geq t_0$, $\|\eta_0\| < \delta$. The above analysis establishes the boundedness of the solution $\eta(t)$. Now, we analyze the scalar case and the vector case one by one. First let us consider the scalar case, i.e., $p = 1$. In this case, we have $V = V_1$ and we replace $Q(t, V(x))$ by $-\phi(t, V_1(x))$ in (3.7) to obtain $V_1'(x) f(x, \tau(t, x)) \leq -\phi(t, V_1(x))$. Due to the continuity property of $V_1(\cdot)$, there exists $\delta_2 > 0$ such that $V_1(x_0) < \delta$, $\forall \|x_0\| < \delta_2$. Next, we replace $Q(t, \eta)$ by $-\phi(t, \eta)$ in (3.8) to obtain $\dot{\eta} = -\phi(t, \eta)$, whose solution is denoted by $\eta(t) = \eta(t, \eta_0)$. Let us choose the initial condition:

$$\eta_0 = V_1(x_0) \in B_\delta, \qquad \|x_0\| < \delta_2. \tag{3.9}$$

Let us consider a scalar Lyapunov candidate function $v(\eta) = \eta^2$ whose time derivative along the trajectories of (3.8) is $\dot{v} = 2\eta\dot{\eta} = -2\eta\phi(t, \eta)$. This implies that $\dot{v} = 0$ for all $t \geq t_a$ and $\dot{v} \leq -2|\eta|\phi(t, |\eta|)$ for all $t \in [t_0, t_a)$. Noting the fact that $\sqrt{v(\eta)} = |\eta|$, we can write $\dot{v}(\eta) \leq -2\sqrt{v(\eta)}\phi(t, \sqrt{v(\eta)})$. Let us consider $w = \sqrt{v(\eta)}$, then, when $v(\eta_0) > 0$, the inequality $v(\eta(t)) > 0$ is satisfied for all $t \in [t_0, t_a)$. We deduce that $\dot{w} = \frac{1}{2\sqrt{v(\eta)}}\dot{v}(\eta) \leq -\phi(t, w)$ for all $t \in [t_0, t_a)$. We also see that $\dot{w} = 0$ for all $t \geq t_a$ leading to $w = 0$ for all $t \geq t_a$. Consequently, $v(\eta(t)) = 0$ for all $t \geq t_a$, from which it follows that

$$\eta(t) = 0, \text{ for all } t \geq t_a, \quad \eta_0 \in B_\delta. \tag{3.10}$$

Note that the conclusion (3.10) can be reached directly by observing that $\dot{\eta} = -\phi(t, \eta)$ converges to the origin in arbitrary time t_a. Now, by using the comparison principle [97], for the considered initial condition (3.9) we have:

$$V_1(x(t)) \leq \eta(t), \quad \eta_0 \in B_\delta, \quad t \in [0, \infty). \tag{3.11}$$

From (3.10)-(3.11), it follows that $V_1(x(t)) = 0$ for all $t \geq t_a$, $\|x_0\| < \delta_2$. Consequently, $x(t) = 0$ for all $t \geq t_a$. Thus the solution $x(t) = 0$ is arbitrary time stable. Now we consider the vector case, i.e., $p > 1$. Note that the vector comparison system (3.8) is assumed to be arbitrary time stable, then it guarantees that the equality in (3.10) is also valid in the vector case of (3.8). Further, we notice that $r^\top V(x)$ is positive definite. Now, since $r^\top V(x) \le \max_{i=1,\dots,p}\{r_i\} d^\top V(x), x \in D$, where d is a column vector: $[1, 1, \cdots, 1]^\top \in \mathbb{R}^p$, we deduce that $d^\top V(x)$ is also positive definite on $x \in D$. Recalling the continuity property of $V(\cdot)$, there exists $\delta_2 > 0$ such that $\|V(x_0)\| < \delta, \ \forall \|x_0\| < \delta_2$. Let us choose $\eta_0 = V(x_0) \in B_\delta$, for all $\|x_0\| < \delta_2$. Then from Lemma 2.12, it follows that $V(x(t)) \le\le \eta(t)$. Utilizing (3.10), $d^\top V(x(t)) \le d^\top \eta(t) = 0, \ \forall t \ \ge t_a$ and since $d^\top V(x(t))$ is non-negative, it follows that $d^\top V(x(t)) = 0, \ \forall \ t \ge t_a$. Since $d^\top V(\cdot)$ is positive definite, we conclude that $x(t) = 0, \ \forall t \ \ge t_a, \ \forall \ \|x_0\| < \delta_2$. Therefore, $x(t) = 0$ is arbitrary time stable. This completes the proof. □

3.3 Robust arbitrary time stabilization of large scale nonlinear systems

Let us consider the following nonlinear dynamical system consisting of p subsystems interconnected to each other with bounded non-vanishing disturbances:

$$\dot{x}_i(t) = F_i(x(t)) + H_i(x(t))(u_i(t) + D_i(t)), \tag{3.12}$$

where $F_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ with $F_i(0) = 0$ and $H_i : \mathbb{R}^n \to \mathbb{R}^{n_i \times m_i}$ are the continuous functions, for $i = \ 1, \dots, p$, $u_i \in \mathbb{R}^{m_i}$ is the control input, $x = [x_1, x_2, \cdots, x_p]^\top \in D \subseteq \mathbb{R}^n$ with $n = n_1 + n_2 + \cdots + n_p$, is the state and $\|D_i(t)\| \le D_{0i}$ is the disturbance which persists even when x has converged to zero. Furthermore, $u(t) \in \mathbb{R}^p$, where $p = m_1 + m_2 + \cdots + m_p$.

Theorem 3.5 *Consider the system (3.12). Suppose that $V = [V_1, \dots, V_p]^\top : D \to S$ with $V_i : \mathbb{R}^{n_i} \to \mathbb{R}$ is a smooth vector Lyapunov function (VLF), where $S \subset \mathbb{R}^p_{\ge 0}, 0 \in S$ and $r \in \mathbb{R}^p_{\ge 0}$ is a vector such that $r^\top V(x)$ is positive definite and*

$$V_i^{'}(x_i) F_i(x) \le\le Q_i(t, V_i(x_i)), \quad x \in \mathcal{R}_i, \ i = 1, \cdots, p$$

where $\mathcal{R}_i = \{x \in \mathbb{R}^n, x \ne 0 : V_i^{'}(x_i) H_i(x) = 0\}$. Let, the proposed universal robust control $u(t) = \tau(t, x) = [\tau_1^\top(t, x), \tau_2^\top(t, x), \dots, \tau_p^\top(t, x)]^\top$

$$\tau_i(t, x) = \begin{cases} -\left(K_i sign(V_i^{'}(x_i)) + \frac{A + \sqrt{A^2 + b_i^\top(x) b_i(x)}}{b_i(x)} \right), & b_i(x) \ne 0 \\ 0, & b_i(x) = 0 \end{cases} \tag{3.13}$$

where $A = a_i(x) - Q_i(t, V_i(x_i))$, $a_i(x) = V_i^{'}(x_i)F_i(x)$, $b_i(x) = V_i^{'}(x_i)H_i(x)$, $i = 1, \cdots, p$, $Q \in C[\mathbb{R}_{\geq 0} \times S, \mathbb{R}^p]$ *is a quasi-monotone non-decreasing function of* V *uniformly in* t *with* $Q_i(t, 0) = 0$ *for all* $t \geq t_0$, *and* K_i, $i = 1, 2, \cdots, p$, *is a constant gain. Besides, suppose the following vector comparison system*

$$\dot{\eta}(t) = Q(t, \eta(t)), \quad \eta(t_0) = \eta_0. \tag{3.14}$$

admits a unique solution $\eta(t) \in \mathbb{R}^p_{\geq 0}$ *and arbitrary time stable. Then, the solution* $x(t) = 0$ *of system (3.12) is arbitrary time stable in the time* t_a *and this stability is robust to bounded disturbances.*

Proof: Let us consider the functions: $a_i(x) = V_i^{'}(x_i)F_i(x)$ and $b_i(x) = V_i^{'}(x_i)H_i(x)$. Simple calculations give, for $i = 1$ to p:

$$\dot{V}_i(x_i) = \ a_i(x) + b_i(x)u_i(t) + b_i(x)D_i(t) \tag{3.15}$$

First case: $b_i(x) \neq 0$**.** Using the proposed universal robust control (3.13), Equation (3.15) becomes, when $H_iK_i \geq |H_i|D_{0i}$

$$\dot{V}_i(x_i) \leq\leq Q_i(t, V_i(x_i))$$

Second case: $b_i(x) = 0$**.** Control $\tau_i(t, x) = 0$. This choice ensures that $V_i^{'}(x_i)F_i(x) \leq\leq Q_i(t, V_i(x_i))$.

Thus, the derivative of VLF along the solutions of system (3.12) with the constructed control $u(t)$ satisfies $\dot{V}_i(x_i) \leq\leq Q_i(t, V_i(x_i))$, when $H_iK_i \geq |H_i|D_{0i}$. Since, it is assumed that the comparison system (3.14) is arbitrary time stable. Then from Theorem 3.4, the solution $x(t) = 0$ of system (3.12) is arbitrary time stable in the time t_a and this stability is robust to bounded disturbances when the control (3.13) is selected. □

It should be noted that the control structure (3.13) is motivated by the Sontag's universal formula [103].

Aggregation of comparison systems

In order to make the results derived above simpler and more elegant, especially for the case of underactuated systems, we aggregate comparison systems to reduce their dimension. For this, we can apply the following aggregation procedure for the linear systems:

$$\dot{x} = \ Ax + B\tau \tag{3.16}$$

where $x \in \mathbb{R}^n$ is the state vector, $\tau \in \mathbb{R}^p$ is the control input and A, B are constant matrices with appropriate dimensions. We use the transformation as $z = \mathcal{T}x$ to convert the system (3.16) into the aggregated model: $\dot{z} = Dz + G\tau$, where $z \in \mathbb{R}^m$ is the state vector, $\mathcal{T} = [\cdot]_{m \times n}$ is a non-square matrix with $m < n$ and the matrices D and G are $D = \mathcal{T}A\mathcal{T}^{-1}(\mathcal{T}\mathcal{T}^\top)^{-1}$and $G = \mathcal{T}B$ [104] under the assumption that $\mathcal{T}$ is a full rank matrix which possesses a pseudoinverse [105]. It is also assumed that $x \in N(\mathcal{T})$ if and only if $x = 0$, where the nullspace $N(\mathcal{T})$ is defined as $N(\mathcal{T}) = \{x : \mathcal{T}x = 0\}$. In a similar way, we can aggregate the nonlinear system of the form

$$\dot{x} = f(x, \tau) \tag{3.17}$$

where $x \in \mathbb{R}^n$ represents the state, $\tau \in \mathbb{R}^p$ is the control and f is a smooth nonlinear vector field. Let us apply the transformation $z = \mathcal{T}x$, where $\mathcal{T}$ is a full rank matrix that possesses a pseudoinverse to convert the system (3.17) into $\dot{z} = F(z, \tau)$, where $z \in \mathbb{R}^m$ is the state vector with $m < n$ and $F(z, \tau) = \mathcal{T}f(\mathcal{T}^{-1}z, \tau)$.

3.4 Simulation example

In the following example, we borrow the model of the surface vessel described in [106].

Example 3.6 *Consider the underactuated model of the surface vessel which defines the relation between earth fixed and body-fixed motion*

$$\begin{bmatrix} \dot{r} \\ \dot{s} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ q \end{bmatrix} - k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \\ \phi \end{bmatrix} \tag{3.18}$$

where r, s denote the ship position and ϕ represents the orientation of the ship. u, v are the velocities of the surge and sway respectively, q is the angular velocity of the ship and k is a positive constant. Furthermore, the dynamic motion of the ship [107] considering matched bounded disturbances due to environmental forces can be described as

$$\begin{aligned} M_{11}\dot{u} - M_{22}vq + D_{11}u &= \tau_1 + d_1(t) \\ M_{22}\dot{v} + M_{11}uq + D_{22}v &= 0 \\ M_{33}\dot{q} + (M_{22} - M_{11})uv + D_{33}q &= \tau_2 + d_2(t), \end{aligned} \tag{3.19}$$

where M_{11}, M_{22}, M_{33} *denote the entries of mass inertia matrix,* D_{11}, D_{22}, D_{33} *are the entries of damping matrix and* τ_1, τ_2 *denote the controls in the surge and yaw direction. Since there is no control in the sway axis, the system is an underactuated system as there are three degrees of freedom and two controls. The disturbance* $d_i(t)$ *is supposed to be bounded:* $|d_i(t)| \leq d_{0i}, i = 1, 2$ *for all* $t \geq 0$*. The model parameters are considered those in [108]* $M_{11} = 1.956, M_{22} = 2.405, M_{33} = 0.043, D_{11} = 2.436, D_{22} = 12.992$ *and* $D_{33} = 0.0564$.

We apply the aggregation procedure as discussed in Section 3.3. Let us apply the transformation as $z = \mathcal{T}x$ *with*

$$\mathcal{T} = \begin{bmatrix} 2 & 3 & 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 & 2 & 1 \end{bmatrix}, \; x = [r, s, \phi, u, v, q]^\top, \; z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

to transform the system (3.18)-(3.19) into

$$\begin{aligned} \dot{z}_1 = & \, 0.1379 z_1 (2\cos(0.1667 z_2) + 3\sin(0.1667 z_2) - 4.98) + 0.0667 z_2 (3\cos(0.1667 z_2) \\ & - 2\sin(0.1667 z_2)) + 0.01093 z_2^2 - 0.4482 k z_1 + 2.044(\tau_1 + d_1(t)) \\ \dot{z}_2 = & \, -0.59755 z_2 - 0.1035 z_1 z_2 + 23.26(\tau_2 + d_2(t)). \end{aligned} \tag{3.20}$$

We design controls τ_1 *and* τ_2 *using vector Lyapunov theory to make the solutions of the system (3.20) converge to the origin in arbitrary time. Let us consider the vector Lyapunov function,* $V = [V_1, V_2]^\top$ *with* $V_1 = (z_1 + z_2)^2$ *and* $V_2 = (z_1 - z_2)^2$*. It is easy to check that* $r^\top V$ *is positive definite, where* $r = [1, 1]^\top$*. The derivative of* V_1 *along the trajectories of (3.20) is given by,* $\dot{V}_1 = 2(z_1 + z_2)(\dot{z}_1 + \dot{z}_2)$ *for all* $t \in [t_0, t_a)$*. Let us choose:* $2.044\tau_1 + 23.26\tau_2 = -0.1379 z_1 (2\cos(0.1667 z_2) + 3\sin(0.1667 z_2) - 4.98) - 0.0667 z_2 (3\cos(0.1667 z_2) - 2\sin(0.1667 z_2)) - 0.01093 z_2^2 + 0.4482 k z_1 + 0.59755 z_2 + 0.1035 z_1 z_2 - \phi_1(t, z_1 + z_2) - K_1 sign(z_1 + z_2)$*, where* $\phi_1(t, .)$ *is the function defined in (3.1) with* $\gamma = \gamma_1$*. Then, when* $K_1 \geq 2.044 d_{01} + 23.26 d_{02}$*,*

$$\dot{V}_1 \leq \frac{-2\gamma_1 \sqrt{V_1}(e^{\sqrt{V_1}} - 1)}{e^{\sqrt{V_1}}(t_a - t)}.$$

In a similar way, when $K_2 \geq 2.044 d_{01} - 23.26 d_{02}$*,*

$$\dot{V}_2 \leq \frac{-2\gamma_2 \sqrt{V_2}(e^{\sqrt{V_2}} - 1)}{e^{\sqrt{V_2}}(t_a - t)},$$

with

$$
\begin{aligned}
&2.044\tau_1 - 23.26\tau_2 = -0.1379z_1(2\cos(0.1667z_2) + 3\sin(0.1667z_2) - 4.98) \\
&- 0.0667z_2(3\cos(0.1667z_2) - 2\sin(0.1667z_2)) - 0.01093z_2^2 + 0.4482kz_1 - 0.59755z_2 \\
&- 0.1035z_1z_2 - \phi_2(t, z_1 - z_2) - K_2 sign(z_1 - z_2),
\end{aligned}
$$

where $\phi_2(t, .)$ *is the function defined in (3.1) with* $\gamma = \gamma_2$ *and* K_1, K_2 *are positive constants. For all* $t \geq t_a$*, the designed controls* τ_1 *and* τ_2 *will maintain the dynamics (3.20) at the origin, hence,* $\dot{V}_i = 0$ *for all* $t \geq t_a$*. Now, let* $w_i = \sqrt{V_i}, i = 1, 2,$ *then* $\dot{w}_i = \frac{\dot{V}_i}{2\sqrt{V_i}} \leq \frac{-\gamma_i(e^{w_i}-1)}{e^{w_i}(t_a-t)}$*. Thus, the comparison system constructed over* $t \in [t_0, t_a)$ *is* $\dot{w}_i = \frac{-\gamma_i(e^{w_i}-1)}{e^{w_i}(t_a-t)}$*. For all* $t \geq t_a, \dot{w}_i = 0$*, for* $i = 1, 2$*. The comparison system is quasi-monotone non-decreasing and arbitrary time stable in time* t_a *with* $\gamma_1 > 2$ *and* $\gamma_2 > 2$ *as* $p = 2$*. Hence, it follows from Theorem 3.2 that the origin of the original system (3.18)-(3.19) is arbitrary time stable. The simulation results are shown in Figure 3.1 and Figure 3.2 with* $K_i = 40, k = 10, \gamma_1 = \gamma_2 = 50$*,* $d_1(t) = \sin 10t$ *and* $d_2(t) = 0.05 \sin 10t$ *with arbitrary time* $t_a = 3$ *sec, and* $t_a = 5$ *sec, respectively.*

3.5 Conclusion

We presented the generalized control design approach to stabilize nonlinear systems in arbitrary time. We have shown that it is robust to bounded disturbances by using the framework of vector Lyapunov functions and comparison systems. We designed control so that the comparison system is arbitrary time stable. After that, we relate these stability conditions with that of the original system by employing comparison principles. Furthermore, we aggregated the comparison system to reduce its dimension in order to make the proposed approach efficient and straightforward, specifically for underactuated systems. Finally, we assessed through an example accompanied by simulations the efficacy of the mathematical results.

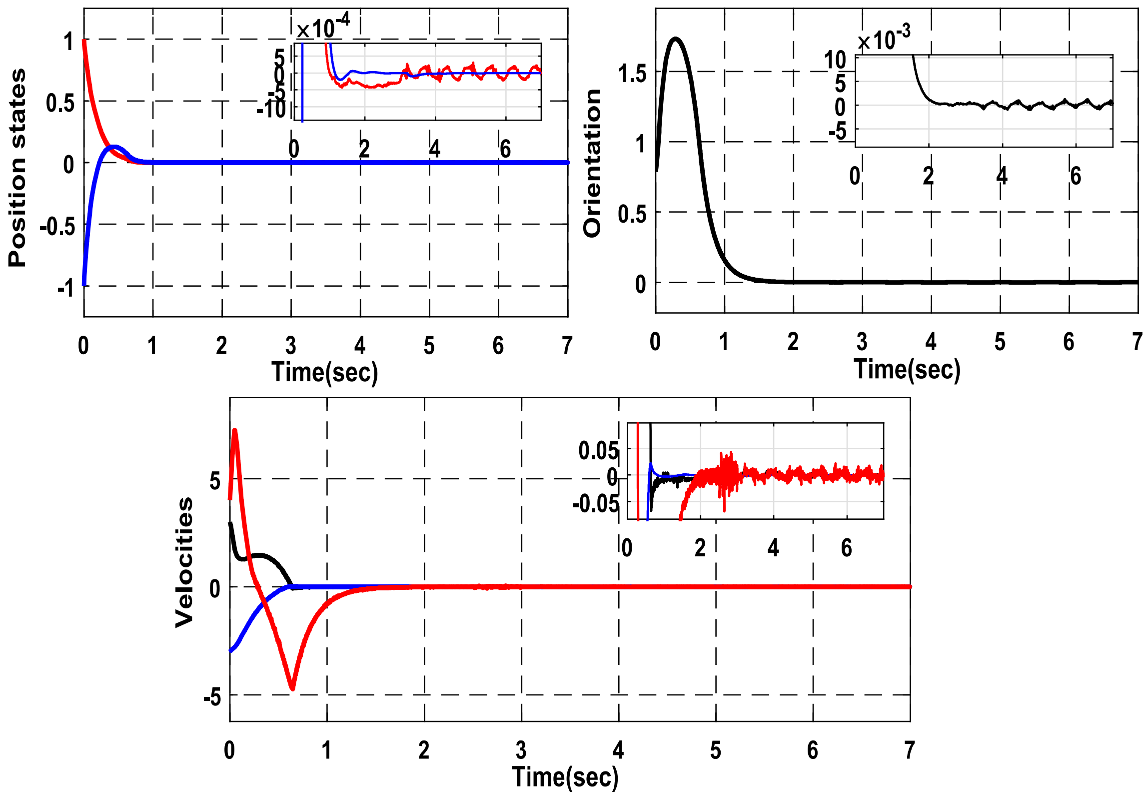

Figure 3.1: System state (3.18)-(3.19) at arbitrary time $t_a = 3$ sec

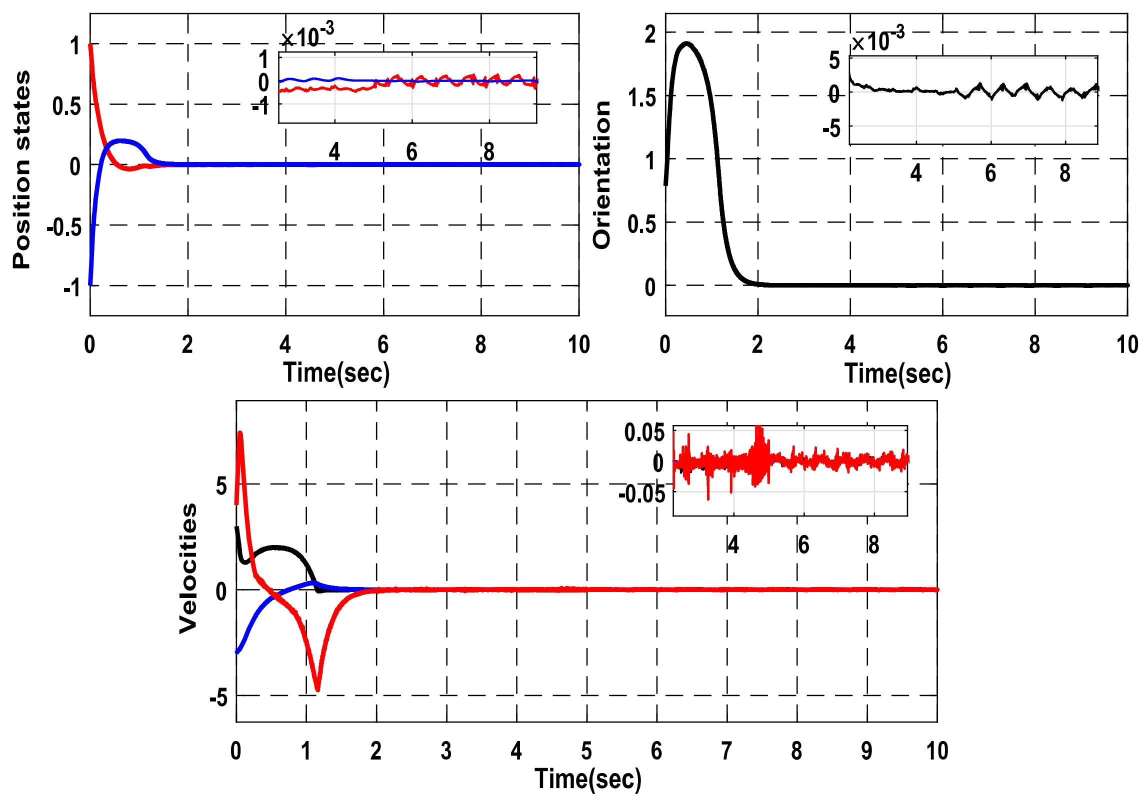

Figure 3.2: System state (3.18)-(3.19) at arbitrary time $t_a = 5$ sec