Chapter 2

Preliminaries

Throughout this thesis, we utilize the following notations and definitions. We denote $p \leq\leq q$, for $p = [p_1, p_2, \cdots, p_n]^\top$ and $q = [q_1, q_2, \cdots, q_n]^\top$, if $p_i \leq q_i$ for each $i = 1, 2, \ldots, n$. Similarly, $p << q$ is denoted if $p_i < q_i$. $[\cdot]^\top$ represents transpose. Given $x \in \mathbb{R}^n$, the Fréchet derivative of $V \in \mathbb{R}^p$ at x is denoted by $V^{'}(x)$. $C[E, F]$ denotes the set of the continuous functions from the nonempty set E to F where $E \subseteq \mathbb{R}^k$, and $F \subseteq \mathbb{R}^l$. For the set $U \in \mathbb{R}^n$, $\bar{U}$ and ∂U denote the closure and the boundary of this set, respectively. A square matrix M is known as a Metzler matrix if its off-diagonal entries are non-negative. $B \geq 0$ means that its components are non-negative (i.e., $b_{ij} \geq 0,\ i \neq j$), where B is a real matrix. The notation $\langle x, y \rangle$, for x, y in $\mathbb{R}^n$, denotes the usual inner product $x^\top y$. $\|x\|$ is the usual Euclidean norm of x in $\mathbb{R}^n$. $\|\delta x\|_v$ is a vector-valued norm of $\delta x \in \mathbb{R}^n$ as defined in the equation (4.1). For a vector $x = [x_1, x_2, \cdots, x_n]^\top \in \mathbb{R}^n$, we denote the diagonal matrix $\mathrm{diag}(x_1, x_2, \cdots, x_n)$ by $\mathrm{diag}(x)$. Let $d = [d_1, d_2, \cdots, d_n]^\top \in \mathbb{R}^n$. Corresponding to the diagonal matrix $\mathrm{diag}(d_1, d_2, \cdots, d_n)$, the vector d can be represented by $\mathrm{diag}(d)\mathbb{1}$, where $\mathbb{1}$ is the n-tuple column vector $[1, 1, \cdots, 1]^\top$. For any $x \in \mathbb{R}^n$, $\underline{x}$ and $\overline{x}$ denote the lower and upper bounds of x, respectively. The set-valued signum function $\mathrm{sign}(x) : \mathbb{R} \to \mathbb{R},\ x \in \mathbb{R}$ is defined as

$$\mathrm{sign}(x) = \begin{cases} -1, & \text{if } \ x < 0, \\ [-1, 1], & \text{if } \ x = 0 \\ 1, & \text{if } \ x > 0. \end{cases}$$

Definition 2.1 *(Quasi-monotone non-decreasing function [94, 95]). Let $E \subseteq \mathbb{R}^n$ and let $e = [e_1, e_2, \cdots, e_n]^\top$ be an element of E. A function $Q = [Q_1, Q_2, \cdots, Q_n]^\top \in C[E, \mathbb{R}^n]$*

is called quasi-monotone non-decreasing on E if for every $i \in \{1, 2, \cdots, n\}$, Q_i is non-decreasing in e_k for all $k = 1, 2, \cdots, i-1, i+1, \cdots, n$.

Definition 2.2 (Cone [96]). *A nonempty set $K \subset \mathbb{R}^n$ is called a cone if for each x in K and a non-negative scalar λ, the vector λx is in K.*

In the rest of the article, we *assume* that any cone K under consideration possesses the properties that K is a closed and convex set, $K \cap (-K) = \{0\}$, and K^o, the interior of K, is nonempty. It is to be noted that a cone K induces a partial order relation on $\mathbb{R}^n$ defined by $x \leq\leq_K y \iff y - x \in K$.

The *adjoint cone* of a cone K, denoted K^*, is defined by $K^* = \{\phi \in \mathbb{R}^n | \langle \phi, x \rangle \geq 0 \text{ for all } x \in K\}$. It is noteworthy that if ∂K denotes the boundary of the cone K, and $K_0 = K \setminus \{0\}$, then (see [96]), $x \in \partial K \iff \langle \phi, x \rangle = 0 \;\text{ for some } \phi \in K_0^*$.

2.1 Comparison functions

We provide some definitions of comparison functions that will be helpful in better understanding of the stability notions.

Definition 2.3 *[97] A function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is known as a class $\mathcal{K}$ function, if it is continuous and strictly increasing with $\psi(0) = 0$. It is known as a class $\mathcal{K}_\infty$ function, if it is a class $\mathcal{K}$ function and $\psi(r) \to \infty$ as $r \to \infty$.*

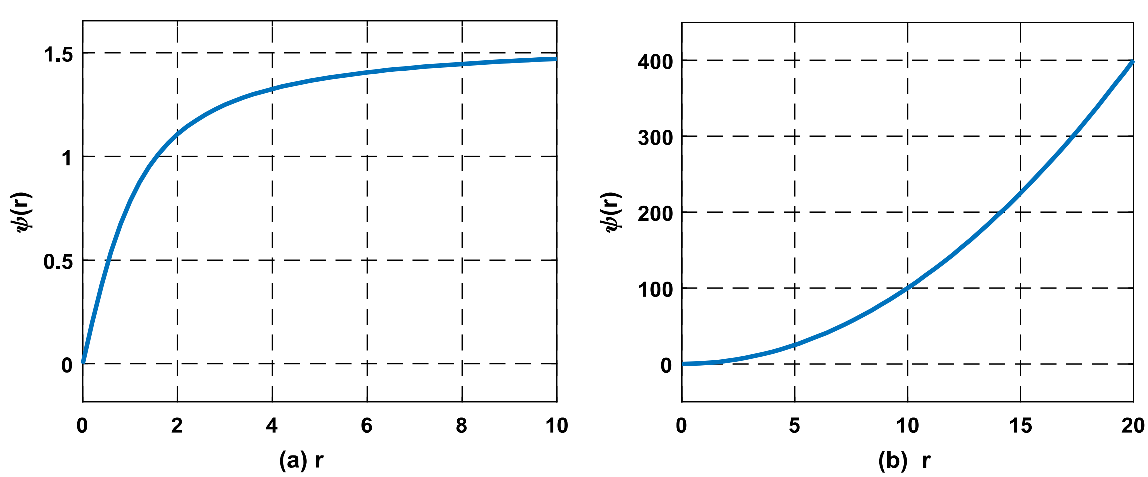

Figure 2.1: (a) Class $\mathcal{K}$ function ($\psi(r) = \tan^{-1}(r)$) (b) Class $\mathcal{K}_\infty$ function ($\psi(r) = r^2$)

To study finite-time, fixed-time and arbitrary time cases, we consider generalized functions [30].

Definition 2.4 *A function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is known as a generalized class $\mathcal{K}$ function, if it is continuous with $\varphi(0) = 0$ and satisfies*

$$\begin{cases} \varphi(r_1) > \varphi(r_2), & \text{if } \ \varphi(r_1) > 0, \ r_1 > r_2 \\ \varphi(r_1) = \varphi(r_2), & \text{if } \ \varphi(r_1) = 0, \ r_1 > r_2. \end{cases}$$

Definition 2.5 *A function $\Lambda : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a generalized class $\mathcal{KL}$ ($\mathcal{GKL}$) function, if for each fixed $t \geq t_0$, the function $\Lambda(r, t)$ with respect to r is a generalized class $\mathcal{K}$ function and the function $\Lambda(r, t)$ with respect to t is continuous and tends to zero as $t \to T$, $T < \infty$, for each fixed r. If T is some arbitrary time, then Λ is called arbitrary time $\mathcal{GKL}$ ($\mathcal{AGKL}$) function.*

2.2 Stability notions

Consider the nonlinear time-varying system

$$\dot{x} = f(t, x, \sigma), \quad x(t_0) = x_0 \tag{2.1}$$

where $x \in D \subseteq \mathbb{R}^n$ is a state vector, $\sigma \in \mathbb{R}^p$ represents constant system parameters to be tuned, $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector field such that $f(t, 0, \sigma) = 0$ for all $t \geq t_0$, that is, origin $x(t) = 0$ is an equilibrium point of system (2.1). The following definitions describe finite, fixed and arbitrary-time stability.

Definition 2.6 *[1] The system (2.1) is known as finite-time stable at the origin if it is asymptotically stable and any solution $x(t, t_0, x_0)$ of (2.1) reaches the origin at some finite time, that is, $x(t, t_0, x_0) = 0$ for all $t \geq t_0 + T(t_0, x_0)$, where $T : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the settling time.*

Definition 2.7 *[2] The system (2.1) is known as fixed-time stable at the origin if it is finite-time stable and settling time $T(t_0, x_0)$ is upper bounded by the time say, $T_m > t_0$, that is, $T(t_0, x_0) \leq T_m$*

Definition 2.8 *[42] The system (2.1) is known as arbitrary-time stable at the origin if*

- *it is fixed-time stable*
- *there exists an arbitrary desired convergence time $t_a > t_0$, which does not depend on initial conditions and system parameters and can be chosen in advance, and*
- *the inequality $t_a \geq t_{tf}$ (weak arbitrary-time stable) can be established, where t_{tf} denotes the true fixed time or actual time of convergence in which the system trajectories reach to the origin.*

We provide following definitions to differentiate among finite, fixed and arbitrary-time stability using the generalized class $\mathcal{KL}$ functions.

Definition 2.9 *The origin of the system (2.1) is called finite-time stable if there exists a class $\mathcal{GKL}$ function $\varLambda$ with $\varLambda(r,t) = 0$ when $t \geq T(r)$, where $T(r)$ is continuous with $T(0) = 0$ such that $\|x(t)\| \leq \varLambda(\|x(t_0)\|, t)$.*

Definition 2.10 *The origin of the system (2.1) is called fixed-time stable if it is finite-time stable and $sup_{r \in \mathbb{R}_{\geq 0}} T(r) < \infty$.*

Definition 2.11 *The origin of the system (2.1) is called arbitrary-time stable if there exists a class $\mathcal{AGKL}$ function $\varLambda$, and α as a class $\mathcal{K}_\infty$ function such that $\|x(t)\| \leq \varLambda(\|x(t_0)\|, t_a - t_0), \forall\ t \in [t_0, t_a),\ \forall\ \|x(t_0)\| \leq \alpha(c)$.*

Now, let us consider the forced nonlinear system

$$\dot{x} = f(x, \tau), \qquad x(t_0) = x_0, \quad t \in I \tag{2.2}$$

where state $x \in D \subseteq \mathbb{R}^n$, $\tau \in \mathbb{R}^m$ is the control, $f : D \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth nonlinear vector field such that $f(0, \tau) = 0$, that is, origin $x = 0$ is an equilibrium point of system (2.2) when control τ is applied and I is the largest interval of existence of a solution $x(t)$ of (2.2).

The following result is a fundamental comparison principle for nonlinear systems in the vector Lyapunov function framework.

Lemma 2.12 *[98] Let us consider the system (2.2). Suppose that the smooth vector function $W : D \to l \subseteq \mathbb{R}^p_{\geq 0}$ is such that, for a specific τ, $W'(x) f(x, \tau) \leq\leq Q(W(x))$, $x \in D$, where $Q : l \to \mathbb{R}^p$ is a quasi-monotone non-decreasing continuous function, such* *that* $\dot{\eta}(t) = Q(\eta(t)), \eta(t_0) = \eta_0$*, admits a unique solution* $\eta(t)$ *defined over* $[t_0, \infty)$*. If* $W(x_0) \leq\leq \eta_0$*,* $\eta_0 \in \mathbb{R}^p_{\geq 0}$*, then* $W(x(t)) \leq\leq \eta(t)$ *for all* $t \geq t_0$*, where* $x(t)$ *is the solution of the system (2.2) when control* τ *is applied.*

2.3 General conditions in interval observer design

Consider a general nonlinear system

$$\dot{x} = f(t, x, d(t)), \quad x(t_0) = x_0; \quad y = g(x) \tag{2.3}$$

where $x \in \mathbb{R}^n$ is a state vector, $y \in \mathbb{R}^p$ is the system output, $d(t) \in \mathbb{R}^l$ is a locally Lipschitz bounded disturbance, that is, $\underline{d}(t) \leq\leq d(t) \leq\leq \overline{d}(t)$, $\underline{d}(t) \in \mathbb{R}^l$, $\overline{d}(t) \in \mathbb{R}^l$, $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz nonlinear functions. The initial condition is unknown but bounded between two values, $\underline{x}(t_0) \leq\leq x(t_0) \leq\leq \overline{x}(t_0)$.

Now, if the following system

$$\dot{z} = h(t, z(t), y(t), d'(t)) \tag{2.4}$$

with initial condition $z_0 = G(t_0, \overline{x}_0, \underline{x}_0)$, $d' = (\overline{d}, \underline{d})$ and bounds for x: $\overline{x} = \overline{H}(t, z)$, $\underline{x} = \underline{H}(t, z)$ (where $h, G, \overline{H}, \underline{H}$ are Lipschitz functions with appropriate dimensions), $z \in \mathbb{R}^n$, follows two conditions:

- *Framer property*: For any set of initial conditions $x_0, \underline{x}_0$ and $\overline{x}_0$ in $\mathbb{R}^n$ satisfying $\underline{x}_0 \leq\leq x_0 \leq\leq \overline{x}_0$, the solutions of system (2.3)-(2.4) follows

$$\underline{x}(t) = \underline{H}(t, z(t)) \leq\leq x(t) \leq\leq \overline{x}(t) = \overline{H}(t, z(t)), \quad \forall\, t \geq t_0$$

  with initial conditions x_0 and $G(t_0, \overline{x}_0, \underline{x}_0)$.

- *Convergence property*: when $d(t) = 0$, then the norm of the error $\|e(t)\| = \|\overline{x}(t) - x(t)\|$ or $\|x(t) - \underline{x}(t)\|$ converges exponentially to zero.

  Then, it is called an interval observer for the system (2.3).

2.4 Graph theory

Now we define some notions of graph theory [99]. An undirected connected graph with weights is represented as $G(X, D, B)$, where $X = \{x_1, x_2, \ldots, x_n\}$ denotes the set of nodes,

$D \subseteq X \times X$ denotes the set of edges and $B = [b_{ij}]_{n\times n}$ denotes the adjacency matrix with weights consisting of positive elements b_{ij}. $d(x_i)$ indicates the number of nodes connected to the node x_i. The $n\times n$ matrix $A = \text{diag}(d(x_i))$ is known as the degree matrix. Now the Laplacian of graph G is $n \times n$ matrix which can be represented as $L = A - B$. A graph is said to be connected if there exists a path to each node or in another way there exists at least one spanning tree of the graph. A graph G generated by the switching signal $s(t)$ is said to be $\sigma-$jointly connected [100] if there exists $\sigma < \infty$, such that the graph G with set of vertices $V(G) = V(G_s)$ and the set of edges $D(G) = D(G_{s(t_i)}) \cup \cdots, D(G_{s(t_k)})$ is connected for all $t_1 \geq 0$, where $t_i, \cdots, t_k$ are the switching time instants in $[t_1, t_1 + \sigma]$.

Definition 2.13 *[99] The agents in multi-agent systems are said to reach consensus if each agent lies in the set*

$$\mathcal{A} = \{x \in \mathbb{R}^n \mid x_i = x_j, \quad i, j \in \{1, 2, \cdots, n\}\}.$$

Definition 2.14 *[101] Suppose the heterogeneous multi-agent system has n number of agents out of which m number of agents $(m < n)$ are denoted by third-order dynamics and remaining agents are denoted by second-order dynamics. The multi-agent system with heterogeneous nodes is said to reach a common value (consensus) if we have*

$$\begin{aligned}
\lim_{t\to\infty} \|r_i(t) - r_j(t)\| &= 0, \quad i, j \in I_n \\
\lim_{t\to\infty} \|s_i(t) - s_j(t)\| &= 0, \quad i, j \in I_n \\
\lim_{t\to\infty} \|q_i(t) - q_j(t)\| &= 0, \quad i, j \in I_m
\end{aligned}$$

for all initial conditions assuming that each agent communicates all the states with its neighbor. The variables r, s and q denote the position, velocity and acceleration of the agents. The symbols I_m and I_n are the sets defined as $I_m = \{1, 2\ldots, m\}$ and $I_n = \{1, 2, \ldots m, m+1, \ldots, n\}$ respectively.