Chapter 2

Continuous-time static output feedback controller design

This chapter addresses SOF controller design for CT systems. The result of [145, 146] is recalled for completeness in the presentation of the new design problems that are addressed. This development involves suitable decomposition of Lyapunov matrices and deriving LMI criterion that ensures H_∞ performance. LMI criteria for pole-placement in LMI region are also derived. Next, the SOF design condition is extended to the PID controller design problem for higher-order MIMO systems. The design approach is more suitable for easy design of PID controller gains and implementation.

Finally, extension of this concept for more realistic system representation, such as uncertain systems with parametric perturbations and actuator saturation is developed. Different from the conditions developed in [145] for designing restricted SOF controllers, where decomposition of the Lyapunov matrix has been considered, here, we present new sufficient conditions for SOF control design for a class of polytopic systems both with and without actuator saturation by introducing an auxiliary matrix variable and decomposing this variable instead of the Lyapunov matrix. Numerical examples are presented to demonstrate the effectiveness of the proposed results.

2.1 SOF controller design for LTI systems

In this section, new LMI criteria for SOF design are derived that can be used even for restricted feedback controller design. The development is in the line of [57], where a diagonal decomposition of the Lyapunov matrix is considered. However, the off-diagonal terms of the Lyapunov matrix have been neglected therein. This resulted in restrictions in the developed criterion. Appropriate off-diagonal terms in the Lyapunov matrix are considered in [146] that yields improvement over the work of [57]. Two decompositions are considered corresponding to the nonlinear matrix multiplication terms $BKCX$ and $PBKC$ (individual matrices of these two terms are defined in the upcoming section). LMI criteria are derived for the H_∞ control and the pole-placement in the LMI region with SOF controller.

2.1.1 Problem statement and preliminaries

Consider an LTI plant as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_w w(t) + Bu(t), \\ z(t) &= C_z x(t) + D_{zu} u(t) + D_{zw} w(t), \\ y(t) &= Cx(t) + D_{yw} w(t), \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input and $z(t) \in \mathbb{R}^{p_1}$ is the controlled output of the system. $w(t) \in \mathbb{R}^{m_1}$ and $y(t) \in \mathbb{R}^p$ are disturbance input and measured output, respectively. $A, B, B_w, C_z, D_{zu}, D_{zw}, C, D_{yw}$ are matrices of appropriate dimensions. For system (2.1), consider a SOF controller as:

$$u(t) = Ky(t) \tag{2.2}$$

where K is the controller gain. The generic case of centralized control is considered for the developments. The decentralized and other restricted feedback cases can be incorporated by imposing restrictions in the SOF gain matrix and corresponding LMI variables that are discussed later. The closed-loop system is given by

$$\begin{bmatrix} \dot{x}_{cl}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} x_{cl}(t) \\ w(t) \end{bmatrix}, \tag{2.3}$$

where $x_{cl}(t) = x(t)$, $A_{cl} = A + BKC, B_{cl} = B_w + BKD_{yw}$, $C_{cl} = C_z + D_{zu}KC$ and $D_{cl} = D_{zw} + D_{zu}KD_{yw}$ are closed-loop system matrices. The closed-loop transfer function matrix from $w(t)$ to $z(t)$ is

$$T_{zw}(s) = C_{cl}(sI - A_{cl})^{-1}B_{cl} + D_{cl} \tag{2.4}$$

For H_∞ performance of (2.4), the Lemma 1.5 is used. In Lemma 1.5, conditions (1.15) and (1.16) are BMIs due to the involvement of the terms $BKCX$ and $PBKC$, respectively. LMI conditions corresponding to (1.15) and (1.16) are derived here, either of which can be used to design SOF controllers.

2.1.2 Decomposition of constituent matrices

This section presents appropriate decompositions of Lyapunov matrices X and P. This is useful in deriving the LMI conditions corresponding to both (1.15) and (1.16).

2.1.2.1 Decomposition of X

Consider the matrices $Q_C \in \mathcal{N}(C)$ and $R_C = C^\dagger + Q_C L_C$ [57], where L_C is an $(n-p) \times p$ matrix, $C^\dagger = C^T(CC^T)^{-1}$ is the Moore-Penrose pseudo inverse of C and $\mathcal{N}(\cdot)$ represent the null-space. Then the below facts are straightforward.

Fact 1 *Any $X > 0$ can be decomposed as*

$$X = \begin{bmatrix} Q_C^T \\ R_C^T \end{bmatrix}^T \begin{bmatrix} X_Q & X_S \\ X_S^T & X_R \end{bmatrix} \begin{bmatrix} Q_C^T \\ R_C^T \end{bmatrix}, \textit{with} \begin{bmatrix} X_Q & X_S \\ X_S^T & X_R \end{bmatrix} > 0, \tag{2.5}$$

where $X_Q \in \mathbb{R}^{(n-p)\times(n-p)}$, $X_S \in \mathbb{R}^{(n-p)\times p}$ and $X_R \in \mathbb{R}^{p\times p}$.

In view of that $CQ_C = 0$, and R_C is so chosen that $CR_C = I$, and $X = Q_C X_Q Q_C^T + R_C X_S^T Q_C^T + Q_C X_S R_C^T + R_C X_R R_C^T$, one obtains

$$CX = X_R R_C^T + X_S^T Q_C^T. \tag{2.6}$$

Fact 2 *There exist matrices $Y_C \in \mathbb{R}^{m\times p}$ and invertible matrix $X_R \in \mathbb{R}^{p\times p}$, such that the following decomposition holds for all K and X.*

$$KCX = Y_C R_C^T + Y_C X_R^{-1} X_S^T Q_C^T \tag{2.7}$$

with

$$KX_R = Y_C \tag{2.8}$$

Remark 2.1 *The matrix Q_C is one whose columns form the basis of the null(C), i.e., $CQ_C = 0$. Further, R_C is a given matrix such that $CR_C = I$ (R_C is the right inverse* *of C). Notice that R_C can be parametrized as $R_C = C^\dagger + Q_C L_C$ [57], where L_C is an $(n-p)\times p$ matrix and $C^\dagger = C^T(CC^T)^{-1}$ is the Moore-Penrose pseudo inverse of C. This can be seen since $C(C^\dagger + Q_C L_C) = I$ and, conversely, if $CR_C = I$ then $C(R_C - C^\dagger) = 0$. Thus, the choice of R_C is not unique.*

Note that, a restricted decomposition of the above has been considered in [57], particularly with $X_S = 0$. Such a restriction and corresponding development introduced conservatism in the resulting SOF design criterion. Here, X_S is retained and thereby develop criteria that are less conservative.

2.1.2.2 Decomposition of P

Similar to the above decomposition of X, decomposition of the Lyapunov matrix P is presented next. Consider the matrices $Q_B \in \mathcal{N}(B^T)$ and R_B so chosen that $R_B^T B = I$. Then, we have the following facts.

Fact 3 *Any $P > 0$ can be decomposed as:*

$$P = \begin{bmatrix} Q_B^T \\ R_B^T \end{bmatrix}^T \begin{bmatrix} P_Q & P_S \\ P_S^T & P_R \end{bmatrix} \begin{bmatrix} Q_B^T \\ R_B^T \end{bmatrix}, \textit{with} \begin{bmatrix} P_Q & P_S \\ P_S^T & P_R \end{bmatrix} > 0, \tag{2.9}$$

where $P_Q \in \mathbb{R}^{(n-m)\times(n-m)}$, $P_S \in \mathbb{R}^{(n-m)\times m}$ and $P_R \in \mathbb{R}^{m\times m}$.

In view of that $Q_B^T B = 0$, and $R_B^T B = I$ and $P = Q_B P_Q Q_B^T + R_B P_S^T Q_B^T + Q_B P_S R_B^T + R_B P_R R_B^T$, one obtains

$$PB = R_B P_R + Q_B P_S. \tag{2.10}$$

Fact 4 *There exist matrices $Y_B \in \mathbb{R}^{m\times p}$ and invertible matrix $P_R \in \mathbb{R}^{m\times m}$, such that the following decomposition holds for all K and P.*

$$PBK = R_B Y_B + Q_B P_S P_R^{-1} Y_B \tag{2.11}$$

with

$$P_R K = Y_B \tag{2.12}$$

2.1.3 H_∞ controller design

Two LMI criteria are derived in this section based on the decompositions made in the previous section. Either of these two criteria can be used for controller design. The below result corresponds to the criterion (1.15).

Theorem 2.2 *[145] The system (2.1) is stable with the SOF controller (2.2) and performance $\|T_{zw}(s)\| < \gamma$ is guaranteed if, for scalars ρ_1 and ρ_2, there exist $X_Q = X_Q^T$, $X_R = X_R^T$, $Z_R = Z_R^T$, X_S and Y_C satisfying the below LMI.*

$$\begin{bmatrix} \Psi_1 & * & * & * & * \\ \Psi_2 & -\gamma^2 I & * & * & * \\ \Psi_3 & D_{zw} & -I & * & * \\ \Psi_4^T & 0 & Y_C^T D_{zu}^T & -Z_R & * \\ \rho_1 X_S^T Q_C^T & \rho_1 D_{yw} & 0 & 0 & -\Psi_5 \end{bmatrix} < 0, \tag{2.13}$$

$$\begin{bmatrix} X_Q & X_S \\ X_S^T & X_R \end{bmatrix} > 0, \quad Z_R > 0 \tag{2.14}$$

where $\Psi_1 = Sym\{\Xi_1 + \Xi_2\}$, $\Xi_1 = AQ_CX_QQ_C^T + AR_CX_S^TQ_C^T + AQ_CX_SR_C^T + AR_CX_RR_C^T$, $\Xi_2 = BY_CR_C^T - \rho_2R_CX_S^TQ_C^T$, $\Psi_2 = B_w^T - \rho_2D_{yw}^TR_C^T$, $\Psi_3 = C_zX + D_{zu}Y_CR_C^T$, $\Psi_4 = BY_C + \rho_2R_CX_R$, $\Psi_5 = Sym\{\rho_1X_R\} - Z_R$ The SOF controller gain can be computed as $K = Y_CX_R^{-1}$.

Note that, with $X_S = 0$ in Theorem 2.2, we get back the result of [57]. On the other hand, retaining X_S yields full matrix X leading to less conservative result. Similar to the decomposition of X in (1.15) one may use (1.16) along with appropriate decomposition of P. Next, the result corresponding to the decomposition of P is presented.

Theorem 2.3 *[145] The system (2.1) is stable with the SOF controller (2.2) and a performance $\|T_{zw}(s)\| < \gamma$ is ensured if, for known scalars $\bar{\rho}_1$ and $\bar{\rho}_2$, there exist $P_Q = P_Q^T$, $P_R = P_R^T$, $H_R = H_R^T$, P_S and Y_B such that the below convex optimization problem is* *solvable.*

$$\begin{bmatrix} \bar{\Psi}_1 & * & * & * & * \\ \bar{\Psi}_2 & -\gamma^2 I & * & * & * \\ \bar{\Psi}_3 & D_{zw} & -I & * & * \\ P_S^T Q_B^T & 0 & D_{zu}^T & -H_R & * \\ \bar{\rho}_1\bar{\Psi}_4 & \bar{\rho}_1 Y_B D_{yw} & 0 & 0 & -\bar{\Psi}_5 \end{bmatrix} < 0, \tag{2.15}$$

$$\begin{bmatrix} P_Q & P_S \\ P_S^T & P_R \end{bmatrix} > 0, H_R > 0 \tag{2.16}$$

where $\bar{\Psi}_1 = Sym\{\bar{\Xi}_1+\bar{\Xi}_2\}$, $\bar{\Xi}_1 = Q_B P_Q Q_B^T A + R_B P_S^T Q_B^T A + Q_B P_S R_B^T A + R_B P_R R_B^T A$, $\bar{\Xi}_2 = R_B Y_B C - \bar{\rho}_2 Q_B P_S R_B$, $\bar{\Psi}_2 = B_w^T P + D_{yw}^T Y_B^T R_B^T$, $\bar{\Psi}_3 = C_z - \bar{\rho}_2 D_{zu} R_B$, $\bar{\Psi}_4 = Y_B C + \bar{\rho}_2 P_R R_B$, $\bar{\Psi}_5 = Sym\{\bar{\rho}_1 P_R\} - H_R$. *The controller gain can be computed as* $K = P_R^{-1} Y_B$.

It can be seen that the expression of gain matrix $K = Y_C X_R^{-1}$ in Theorem 2.2 and $K = P_R^{-1} Y_B$ in Theorem 2.3 have the flexibility in choosing different structure of K by imposing structural constraints in X_R, Y_C, P_R and Y_B matrices. Commonly, for restricted feedback control case, P_R can be block-diagonal and Y_B takes same structure as the desired K. The same structural consideration can be made while choosing X_R and Y_C to get back the desired structure of K. It is worth noting that P_Q is a full matrix, and the blocks of P_R and Y_B might contain large number of free entries for large-scale systems.

2.1.4 Pole-placement in LMI region

Next performance criterion considered is the pole-placement in LMI regions [10] through SOF control. It is well known that, though the H_∞ control in the previous section yields good robust performance, it lacks in ensuring transient performance. The transient performance of the closed-loop system can be improved through locating the closed-loop poles in specified regions for which transient performance guaranteeing minimum performance criteria, such as damping ratio, decay rate, are known. The SOF design solution in the previous section is extended to the pole-placement in LMI region problem in this section. To proceed further, consider the nominal closed-loop system of (2.3) as:

$$\dot{x}(t) = A_{cl} x(t) \tag{2.17}$$

with $A_{cl} = A + BKC$ following the notations as defined in (2.1). The objective is to design the SOF gain K so that the closed-loop poles corresponding to (2.17) are placed in specified LMI region.

Consider an LMI region $\mathcal{D}_C(\alpha,\theta)$ consists of a set of complex numbers $p+jq$ such that $p<-\alpha<0$ and $p\tan\theta<-|q|$ as shown in Fig.1.5. Conditions (1.23) and (1.24) represent the characteristic function and matrix inequality criterion for placing the closed-loop poles of (2.17) in $\mathcal{D}_C(\alpha,\theta)$. Now, the following result can be obtained corresponding to (1.24).

Theorem 2.4 *[145] The closed-loop poles of system (2.17) is placed in the region $\mathcal{D}_C(\alpha,\theta)$ if, for scalars α_1 and α_2, there exist $X_Q=X_Q^T$, $X_R=X_R^T$, $Z_{R1}=Z_{R1}^T>0$, $Z_{R2}=Z_{R2}^T>0$, X_S and Y_C satisfying the below LMI.*

$$\begin{bmatrix} s_\theta(\hat{\Psi}_1) & * & * & * & * & * \\ c_\theta\Psi_6 & s_\theta(\hat{\Psi}_1) & * & * & * & * \\ s_\theta\Psi_4^T & -c_\theta\Psi_4^T & -Z_{R1} & * & * & * \\ c_\theta\Psi_4^T & s_\theta\Psi_4^T & 0 & -Z_{R2} & * & * \\ \alpha_1\Pi_1^T & 0 & 0 & 0 & -\Pi_2 & * \\ 0 & \alpha_2\Pi_1^T & 0 & 0 & 0 & -\Pi_3 \end{bmatrix} < 0 \tag{2.18}$$

where $\hat{\Psi}_1=\Psi_1+2\alpha X$, $\Psi_1=Sym\{\Xi_1+\Xi_2\}$, $\Psi_6=\Xi_1^T-\Xi_1+\Xi_2^T-\Xi_2$, $\Pi_1=Q_cX_S$, $\Pi_2=Sym\{\alpha_1X_R\}-Z_{R1}$, $\Pi_3=Sym\{\alpha_2X_R\}-Z_{R2}$, Ψ_4 defined earlier and α_1, α_2 are known scalars.

Next, criterion for pole-placement in LMI region corresponding to the decomposition of P is presented. For the purpose, it is easy to obtain an alternative matrix inequality by pre- and post-multiplying (1.24) with $diag\{P,P\}$, as:

$$\begin{bmatrix} s_\theta(Sym\{PA_{cl}\}+2\alpha P) & * \\ c_\theta(A_{cl}^TP-PA_{cl}) & s_\theta(Sym\{PA_{cl}\}+2\alpha P) \end{bmatrix} < 0 \tag{2.19}$$

The following result is obtained corresponding to (2.19) for SOF controller design.

Theorem 2.5 *[145] The closed-loop poles of system (2.1) is placed in the region $\mathcal{D}_C(\alpha,\theta)$ if, for scalars $\bar{\alpha}_1$ and $\bar{\alpha}_2$, there exist $P_Q=P_Q^T$, $P_R=P_R^T$, $H_{R1}=H_{R1}^T>0$, $H_{R2}=H_{R2}^T>0$,*

H_S and Y_B such that the below convex optimization problem is solvable.

$$\begin{bmatrix} s_\theta(\hat{\bar{\Psi}}_1) & * & * & * & * & * \\ c_\theta\bar{\Psi}_6 & s_\theta(\hat{\bar{\Psi}}_1) & * & * & * & * \\ s_\theta\bar{\Pi}_1^T & -c_\theta\bar{\Pi}_1^T & -H_{R1} & * & * & * \\ c_\theta\bar{\Pi}_1^T & s_\theta\bar{\Pi}_1^T & 0 & -H_{R2} & * & * \\ \bar{\alpha}_1\bar{\Psi}_4 & 0 & 0 & 0 & -\bar{\Pi}_2 & * \\ 0 & \bar{\alpha}_2\bar{\Psi}_4 & 0 & 0 & 0 & -\bar{\Pi}_3 \end{bmatrix} < 0 \tag{2.20}$$

where $\hat{\bar{\Psi}}_1 = \bar{\Psi}_1 + 2\alpha P$, $\bar{\Psi}_1 = Sym\{\bar{\Xi}_1 + \bar{\Xi}_2\}, \bar{\Psi}_6 = \bar{\Xi}_1^T - \bar{\Xi}_1 + \bar{\Xi}_2^T - \bar{\Xi}_2$, $\bar{\Pi}_1 = Q_B P_S$, $\bar{\Pi}_2 = Sym\{\bar{\alpha}_1 P_R\} - H_{R1}$, $\bar{\Pi}_3 = Sym\{\bar{\alpha}_2 P_R\} - H_{R2}$, $s_\theta = \sin\theta$, $c_\theta = \cos\theta$, $\bar{\Psi}_4$ *defined earlier and* $\bar{\alpha}_1, \bar{\alpha}_2$ *are known scalars.*

2.1.5 H_∞ controller design with norm bounded uncertainty

In this section, an extension of the proposed methodology for systems with norm bounded uncertainties in the input matrix is presented based on the decomposition of X made in the previous section. The uncertain system is represented as:

$$\dot{x}(t) = Ax(t) + B_w w(t) + (B + \Delta B(t))u(t) \tag{2.21}$$

with $x(t)$, $u(t)$, $w(t)$, A, B, B_w as defined in (2.1). The controlled output $z(t)$ and the measured output $y(t)$ equations are the same as in (2.1). $\Delta B(t)$ captures time-varying parameter uncertainties in B and it can be decomposed as the following:

$$\Delta B(t) = DF(t)E, \ \ \|F(t)\| \leq 1 \tag{2.22}$$

where D and E are constant matrices of appropriate dimensions with $F(t)$ takes care of the time variations in $\Delta B(t)$. The Lemma 1.8 defined in chapter 1 ensures H_∞ performance of (2.21). Next, the following result is derived corresponding to (1.27) for the SOF design.

Theorem 2.6 *[145] The system (2.21) is stable with the SOF controller (2.2) and performance $\|T_{zw}(s)\| < \gamma$ is guaranteed if, for scalars β_1 and β_2, there exist $X_Q = X_Q^T$,*

$X_R = X_R^T$, $S_R = S_R^T$, X_S and Y_C satisfying the below LMI:

$$\begin{bmatrix} \Psi_{11} & * & * & * & * & * & * \\ \Psi_{22} & -\gamma^2 I & * & * & * & * & * \\ \Psi_{33} & D_{zw} & -I & * & * & * & \\ EY_C R_C^T & 0 & 0 & -\epsilon I & * & * & \\ \Psi_{44}^T & 0 & Y_C^T D_{zu}^T & Y_C^T E^T & -S_R & * & \\ \beta_1 X_S^T Q_C^T & \beta_1 D_{yw} & 0 & 0 & 0 & -\Psi_{55} & \end{bmatrix} < 0, \tag{2.23}$$

$$\begin{bmatrix} X_Q & X_S \\ X_S^T & X_R \end{bmatrix} > 0, \quad S_R > 0 \tag{2.24}$$

where $\Psi_{11} = Sym\{\Xi_{11} + \Xi_{22}\} + \epsilon DD^T$, $\Xi_{11} = AQ_C X_Q Q_C^T + AR_C X_S^T Q_C^T + AQ_C X_S R_C^T + AR_C X_R R_C^T$, $\Xi_{22} = BY_C R_C^T - \beta_2 R_C X_S^T Q_C^T$, $\Psi_{22} = B_w^T - \beta_2 D_{yw}^T R_C^T$, $\Psi_{33} = C_z X + D_{zu} Y_C R_C^T$, $\Psi_{44} = BY_C + \beta_2 R_C X_R$, $\Psi_{55} = \beta_1 X_R + \beta_1 X_R^T - S_R$. The SOF controller gain can be computed as $K = Y_C X_R^{-1}$.

Similar method can be followed for deriving condition based on the decomposition of P. Also, it can be extended to the systems with uncertainties only in the output matrix C. However, for systems with uncertainties in A, B, C matrices obtaining an LMI condition will be complex.

In this section, the problem of designing SOF controller for CT linear system is studied. Sufficient LMI conditions are derived for designing SOF controller guaranteeing H_∞ performance and pole-placement in damping region. The development involves suitable decomposition of the X or P matrices that are effectively used to obtain LMI criteria.

2.1.6 Dynamic controller design

Consider the following DOF controller for system (2.1).

$$\begin{aligned} \dot{x}(t) &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) + D_c y(t) \end{aligned} \tag{2.25}$$

where $x_c(t) \in \mathbb{R}^{n_c}$ is the controller state with n_c being the chosen controller order, A_c, B_c, C_c and D_c are appropriate dimensional controller matrices to be designed. Let us define a new state variable as the augmented plant state and a fictitious error state as:

$$\widetilde{x}(t) = \begin{bmatrix} x(t) \\ Tx(t) - x_c(t) \end{bmatrix}, \tag{2.26}$$

where $T \in \mathbb{R}^{n_c \times n}$ is a full row-rank matrix. For full-order DOF controller, ($n_c = n$), a straightforward choice is $T = I$. Then, the closed-loop system dynamics corresponding to the state definition (2.26) can be written as:

$$\begin{aligned} \dot{\widetilde{x}}(t) &= \widetilde{A}\widetilde{x}(t) + \widetilde{B}_w w(t) + \widetilde{B}\widetilde{u}(t) \\ \widetilde{z}(t) &= \widetilde{C}_z\widetilde{x}(t) + \widetilde{D}_{zu}\widetilde{u}(t) + \widetilde{D}_{zw}w(t) \\ \widetilde{y}(t) &= \widetilde{C}\widetilde{x}(t) + \widetilde{D}_{yw}w(t) \end{aligned} \tag{2.27}$$

with $\widetilde{u}(t) = \widetilde{K}\widetilde{y}(t)$ and

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ TA & 0 \end{bmatrix}, \ \widetilde{B} = \begin{bmatrix} B & 0 \\ TB & -I \end{bmatrix}, \ \widetilde{C} = \begin{bmatrix} C & 0 \\ T & -I \end{bmatrix}, \ \widetilde{B}_w = \begin{bmatrix} B_w \\ TB_w \end{bmatrix}, \ \widetilde{C}_z = \begin{bmatrix} C_z & 0 \end{bmatrix},$$

$$\widetilde{D}_{zu} = \begin{bmatrix} D_{zu} & 0 \end{bmatrix}, \ \widetilde{K} = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}, \ \widetilde{D}_{yw} = \begin{bmatrix} D_{yw} \\ 0 \end{bmatrix}, \ \widetilde{D}_{zw} = D_{zw}$$

The system (2.27) is in the form of (2.1) with the $\widetilde{K}$ is the SOF controller as in (2.2). Hence, the DOF controller parameters with the parameters embedded in $\widetilde{K}$ can be designed using the same SOF design criteria developed previously.

2.2 Robust PID controller design in SOF framework

This section considers the design of robust PID controller for higher order MIMO plants. The design problem is first recast into a SOF design problem and then the transformed SOF problem is solved within the framework of LMIs through a decomposition of the Lyapunov matrix variable. Sufficient LMI criteria are derived that ensure H_∞ performance of the underlying system. This work extends the same approach of SOF controller design discussed in the previous section to the PID controller design problem. By means of a numerical example, it is shown that the designed controller yields less conservative results. Also, a comparative study is done with the existing techniques to demonstrate the efficacy of the proposed method.

2.2.1 H_∞ controller design

The following result is in the line of work [145]. The derived condition is based on the decomposition of Lyapunov matrix X as presented in the previous section. Similar method can be followed for deriving condition based on the decomposition of P. In [145], sufficient LMI conditions are derived for SOF controller design using Lemma 1.15 but here, we use Lemma 1.16 to obtain the LMI condition which yields less conservative results. This has been illustrated using the numerical examples later.

Theorem 2.7 *Given a CT system described by (2.1) along with SOF controller* (2.2), *the former is stable and a performance* $\|T_{wz}(s)\| < \gamma$ *is guaranteed if, there exist matrices* $X = X_Q^T$, $X_R = X_R^T$, X_S, Y_C *and scalars* α *and* β *such that the below LMI conditions are satisfied.*

$$\begin{bmatrix} \Phi_1 & * & * & * \\ \Phi_2 & -\gamma^2 I & * & * \\ \Phi_3 & D_{zw} & -I & * \\ \Phi_4 & D_{yw} & \alpha Y_C^T D_{zu}^T & -\Phi_5 \end{bmatrix} < 0, \tag{2.28}$$

$$\begin{bmatrix} X_Q & X_S \\ X_S^T & X_R \end{bmatrix} > 0 \tag{2.29}$$

where $\Psi = AQ_CX_QQ_C^T + AR_CX_S^TQ_C^T + AQ_CX_SR_C^T + AR_CX_RR_C^T$, $\Phi_1 = Sym\{\Psi\} + Sym\{\Xi\}$, $\Phi_2 = B_w^T - \beta D_{yw}^T R_C^T$, $\Phi_3 = C_zX + D_{zu}Y_CR_C^T$, $\Phi_4 = \alpha\Gamma^T + X_S^TQ_C^T$, $\Phi_5 = Sym\{\alpha X_R\}$, $\Xi = BY_CR_C^T - \beta R_CX_S^TQ_C^T$, $\Gamma = BY_C + \beta R_CX_R$. *The feedback controller gain can be computed as* $K = Y_CX_R^{-1}$.

Proof: Given $\begin{bmatrix} Q_C & R_C \end{bmatrix}$ is full rank and from (2.5) and (2.29), it is clear that $X > 0$. Then it remains to show that (2.28) is sufficient for (1.15). Replacing (2.7) in (1.15), one can rewrite a sufficient criterion of (1.15) as

$$\begin{bmatrix} \Phi_1 & * & * \\ \Phi_2 & -\gamma^2 I & * \\ \Phi_3 & D_{zw} & -I \end{bmatrix} + \begin{bmatrix} Sym\{\Gamma X_R^{-1} X_S^T Q_C^T\} & * & * \\ D_{yw}^T X_R^{-1} \Gamma^T & 0 & * \\ D_{zu} Y_C X_R^{-1} X_S^T Q_C^T & D_{zu} Y_C X_R^{-1} D_{yw} & 0 \end{bmatrix} < 0 \tag{2.30}$$

The above equation (2.30) can be rewritten as

$$\begin{bmatrix} \Phi_1 & * & * \\ \Phi_2 & -\gamma^2 I & * \\ \Phi_3 & D_{zw} & -I \end{bmatrix} + \begin{bmatrix} \Gamma \\ 0 \\ D_{zu}Y_C \end{bmatrix} X_R^{-1} \begin{bmatrix} Q_CX_S \\ D_{yw}^T \\ 0 \end{bmatrix}^T + \begin{bmatrix} Q_CX_S \\ D_{yw}^T \\ 0 \end{bmatrix} X_R^{-1} \begin{bmatrix} \Gamma \\ 0 \\ D_{zu}Y_C \end{bmatrix}^T < 0 \tag{2.31}$$

Finally, applying Lemma 1.16 and substituting $\mathscr{L} = X_R$, one obtains (2.28). $\square$

In the upcoming section, we discuss the method of PID controller design by appropriately transforming it into SOF design problem.

2.2.2 PID controller design

The schematic diagram of the overall closed-loop system is shown in Figure 2.1. The figure clearly demonstrates that the PID controller undergoes a transformation using the $T(\cdot)$ function block to SOF controller design which is inverse transformed using the $T^{-1}(\cdot)$ function block to compute the original gains of the PID controller. The appropriate control signal is then generated using these recovered gains and is fed back into the plant.

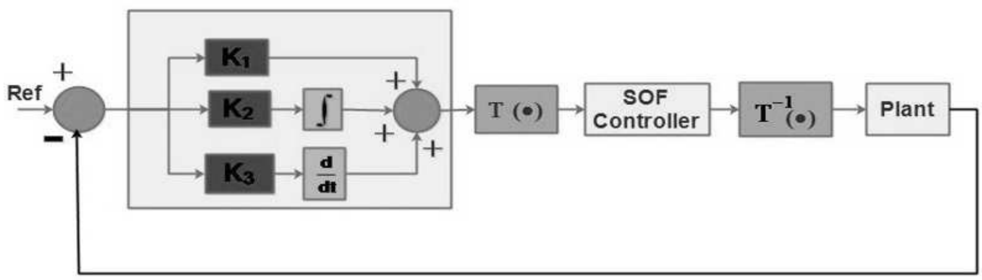

Figure 2.1: Closed-loop system block diagram

2.2.2.1 Transformation from PID to SOF Controller Design Problem

Consider the nominal CT LTI system with $w(t) = 0$ in (2.1), given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{2.32}$$

with the PID controller in the form as

$$u(t) = K_1 y(t) + K_2 \int_0^t y(\tau)d\tau + K_3 \frac{dy(t)}{dt} \tag{2.33}$$

where $K_1, K_2, K_3 \in \mathbb{R}^{m \times n_y}$ represent the proportional, integral and derivative gains, respectively which are to be designed. Here, our objective is to transform the PID controller design problem into an SOF design problem. In order to achieve our goal, we consider the coordinate transformation variable defined by [35] as $\nu_1(t) = x(t)$, $\nu_2(t) = \int_0^t y(\tau)d\tau$ to incorporate the states to be tracked for incorporating the integral control terms. Let us define $\nu(t) = \begin{bmatrix} \nu_1^T(t) & \nu_2^T(t) \end{bmatrix}^T$, where $\nu(t)$ can be seen as the new state vector of the transformed system, whose dynamics is given by

$$\begin{cases} \dot{\nu}_1(t) = A\nu_1(t) + Bu(t), \\ \dot{\nu}_2(t) = C\nu_1(t). \end{cases} \tag{2.34}$$

Rewriting (2.34), we get,

$$\begin{cases} \dot{\nu}(t) = \bar{A}\nu(t) + \bar{B}u(t), \\ \bar{y}(t) = \bar{C}\nu(t), \end{cases} \tag{2.35}$$

where $\bar{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\bar{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$, $\bar{y}(t) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $C_1 = \begin{bmatrix} C & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & I \end{bmatrix}$, $C_3 = \begin{bmatrix} CA & 0 \end{bmatrix}$. Assuming that the matrix $(I - K_3CB)$ is invertible and using above matrix definitions, the system (2.35) and controller (2.33) reduce to an SOF controller as

$$u(t) = \bar{K}\bar{y}(t) \tag{2.36}$$

where, $\bar{K} = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 & \bar{K}_3 \end{bmatrix}$, $\bar{K}_1 = (I - K_3CB)^{-1}K_1$, $\bar{K}_2 = (I - K_3CB)^{-1}K_2$, $\bar{K}_3 = (I - K_3CB)^{-1}K_3$. Once the controller gain matrix $\bar{K} = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 & \bar{K}_3 \end{bmatrix}$ is computed, the original PID controller gains can be recalculated as

$$K_3 = \bar{K}_3(I + CB\bar{K}_3)^{-1}, \; K_2 = (I - K_3CB)\bar{K}_2, \; K_1 = (I - K_3CB)\bar{K}_1.$$

The following lemma guarantees the existence and invertibility of the matrix $(I + CB\bar{K}_3)$.

Lemma 2.8 ([76]) *The matrix $(I + CB\bar{K}_3)$ is always invertible if and only if the matrix $(I - K_3CB)$ is invertible, where K_3 and $\bar{K}_3$ are related to each other as $\bar{K}_3 = (I - K_3CB)^{-1}K_3$, or equivalently $K_3 = \bar{K}_3(I + CB\bar{K}_3)^{-1}$*

2.2.3 H_∞ based robust PID controller

This section focuses on the design of PID controllers with the underlying performance criteria chosen as H_∞ performance (γ). Consider the system (2.1) and PID controller (2.33). Under the assumption that the matrix $(I - K_3CB)$ is invertible and using the transformation discussed in the earlier section to change the PID controller design problem to SOF design problem, the augmented system dynamics are as follows

$$\begin{cases} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t) + \tilde{B}_w w(t) \\ \tilde{z}(t) &= \tilde{C}_z\tilde{x}(t) + \tilde{D}_{zu}\tilde{u}(t) + \tilde{D}_{zw}w(t) \\ \tilde{y}(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}_{yw}w(t) \\ \tilde{u}(t) &= \tilde{K}\tilde{y}(t) \end{cases} \tag{2.37}$$

where $\tilde{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\tilde{B}_w = \begin{bmatrix} B_w \\ 0 \end{bmatrix}$, $\tilde{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$, $C_1 = \begin{bmatrix} C & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & I \end{bmatrix}$, $C_3 = \begin{bmatrix} CA & 0 \end{bmatrix}$, $\tilde{C}_z = \begin{bmatrix} C_z & 0 \end{bmatrix}$, $\tilde{D}_{zu} = D_{zu}$, $\tilde{D}_{zw} = D_{zw}$, $\tilde{D}_{yw} = D_{yw}$. Thus, the composite feedback controller gain matrices $\tilde{K} = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 & \bar{K}_3 \end{bmatrix}$ can be obtained by applying Theorem 2.7 to system (2.37).

2.2.4 Numerical example

A numerical example is considered in this section to demonstrate the efficacy of the proposed design. Note that the scalar parameters α, β in Theorem 2.7 are obtained using linear search algorithm of *fminsearch* [147] in MATLAB.

Example 2.9 *Consider the state space linearized model of an aircraft system given by [35] where only H_∞ output feedback optimization problem is considered with the given parameters*

$$A = \begin{bmatrix} -0.0266 & -36.6170 & -18.8970 & -32.0900 & 3.2509 & -0.7626 \\ 0.0001 & -1.8997 & 0.9831 & -0.0007 & -0.1708 & -0.0050 \\ 0.0123 & 11.7200 & -2.6316 & 0.0009 & -31.6040 & 22.3960 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 30 & 0 \\ 0 & 30 \end{bmatrix}, B_w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 30 \\ 0 \end{bmatrix}, C_z = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, D_{zu} = \begin{bmatrix} 1 & 1 \end{bmatrix},\ D_{zw} = 0,\ D_{yw} = [0 \ \ 0]^T.$$

The results computed using Theorem 2.7 are listed in Table 1. These are further compared

Table 2.1: SOF and PID Controllers and their respective performances

Design Approach	Problem Type	Feedback Gains	Closed-loop Poles	Performance	Parameters
[35], [9]	SOF stabilization	$K = \begin{bmatrix} 7.0158 & -4.3414 \\ 2.1396 & -4.4660 \end{bmatrix}$	$-0.0475 \pm j0.0853$, $-0.7576 \pm j0.7543$, $-29.2613, -33.6825$	Stable	α = -1.9 $\times 10^{-4}$
[76]	SOF stabilization	$K = \begin{bmatrix} 0.6828 & 0.2729 \\ -0.1024 & -0.0348 \end{bmatrix}$	$-1.3274 \pm j4.6317$, $-0.7735, -0.0665$, $-30.6841, -30.0117$	Stable	$\alpha = -0.1330$
Theorem 2.7 (Proposed)	SOF stabilization	$K = \begin{bmatrix} 0.4099 & 0.1906 \\ -0.1504 & 0.0610 \end{bmatrix}$	$-1.4831 \pm j3.0245$, $-0.7875, -0.1083$, $-31.0626, -30.0006$	Stable	$\alpha = 77.6023$, $\beta = 0.4403$
[35]	PID stabilization	$K_1 = \begin{bmatrix} 10.1359 & -1.7947 \\ 6.9912 & -9.4140 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0.3817 & -0.6939 \\ 0.6528 & -1.1978 \end{bmatrix}$, $K_3 = \begin{bmatrix} 2.6162 & -1.4722 \\ 0.8212 & -1.6284 \end{bmatrix}$	$-17.38 \pm j34.73$, $-3.07 \pm j0.46$, $-0.0003, -0.0243$, $-0.0774, -37.0751$	Stable	α =-4.4 $\times 10^{-4}$
[76]	PID stabilization	$K_1 = \begin{bmatrix} 13.9831 & 2.0458 \\ 0.8758 & -1.3309 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0.1710 & -0.3188 \\ 0.1550 & -0.2891 \end{bmatrix}$, $K_3 = \begin{bmatrix} 5.7954 & -4.5855 \\ 1.1572 & -1.9783 \end{bmatrix}$	$-19.4227 \pm j31.7602$, $-49.8699, -4.9964$, $-0.6490, -0.0275$, $-0.0388, -1.3296 \times 10^{-5}$	Stable	α = -4.4 $\times 10^{-4}$
Theorem 2.7 (Proposed)	PID stabilization	$K_1 = \begin{bmatrix} -0.0798 & 0.6675 \\ -7.6582 & -0.7689 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0.1291 & -0.1357 \\ 0.1152 & -0.1466 \end{bmatrix}$, $K_3 = \begin{bmatrix} -0.6425 & 1.2466 \\ -3.6122 & 3.2701 \end{bmatrix}$	$-19.9208 \pm j22.2367$, $-15.2952, -4.8236$, $-0.6799, -0.0043$, $-0.0468, -0.0326$	Stable	$\alpha = 17.2488$, $\beta = 0.4327$

with the existing results given by [35] and [9] for two different cases. The first is that of an SOF stabilization problem and the second considers SOF design with H_∞ performance measure (γ). Note that smaller the value of the performance measure, better is the dis-

[35], [9]	SOF (H_∞)	$K=\begin{bmatrix}0.2838 & 0.0313\\ -0.8725 & -0.0289\end{bmatrix}$	$\begin{cases}-0.1042\pm j0.1536\\ -1.6525\pm j4.3864\\ -30.0027, -31.0378\end{cases}$	$\gamma=0.863$	α =-4.5 $\times10^{-2}$
[76]	SOF (H_∞)	$K=\begin{bmatrix}2.3982 & 0.2302\\ -3.3982 & -0.2302\end{bmatrix}$	$\begin{cases}-0.0010\pm j11.6634\\ -0.0547, -0.1260\\ -34.3691, -30.0061\end{cases}$	$\gamma=0.323$	$\alpha=-0.0020$
Theorem 2.7 (Proposed)	SOF (H_∞)	$K=\begin{bmatrix}0.1892 & 0.0438\\ -1.1895 & -0.0438\end{bmatrix}$	$\begin{cases}-1.5557\pm j4.8905\\ -0.1145\pm j0.1284\\ -31.2124, -30.0052\end{cases}$	$\gamma=0.0870$	α =1.4500 $\times10^3$, β = -0.0001 $\times10^3$
[35]	PID (H_∞)	$K_1=\begin{bmatrix}442.17 & 221.64\\ -188.84 & -104.44\end{bmatrix}$ $K_2=\begin{bmatrix}0.3845 & -0.5019\\ 0.1068 & -0.3373\end{bmatrix}$ $K_3=\begin{bmatrix}48.03 & -3.85\\ -19.31 & -8.80\end{bmatrix}$	$\begin{cases}-0.0001, -0.0020\\ -0.0050, -0.7200\\ -19.9100, -191.000\\ -48.02\pm j77.79\end{cases}$	$\gamma=1.003$	α =-4.9 $\times10^{-4}$
[76]	PID (H_∞)	$K_1=\begin{bmatrix}22.5780 & 5.9056\\ -15.3968 & -4.9824\end{bmatrix}$ $K_2=\begin{bmatrix}39.0276 & 20.4624\\ -24.1171 & -13.3738\end{bmatrix}$ $K_3=\begin{bmatrix}6.2065 & -4.6857\\ -4.1822 & 2.9566\end{bmatrix}$	$\begin{cases}-22.9441\pm j32.1154\\ -35.1083, -9.1967\\ -4.7860, -0.0249\\ -0.0637, -0.6649\end{cases}$	$\gamma=1.001$	α =-1.3142 $\times10^{-4}$
Theorem 2.7 (Proposed)	PID (H_∞)	$K_1=\begin{bmatrix}26.3361 & 6.4134\\ -27.2839 & -6.4000\end{bmatrix}$ $K_2=\begin{bmatrix}-0.0013 & 0.0361\\ 0.0013 & -0.0360\end{bmatrix}$ $K_3=\begin{bmatrix}5.4216 & -2.3951\\ -5.4274 & 2.3991\end{bmatrix}$	$\begin{cases}-25.7949\pm j62.4829\\ -29.8858, -9.5803\\ -0.4392, -0.0223\\ -0.0069, -0.0000\end{cases}$	$\gamma=1.000$	$\alpha=69.2078$, $\beta=-0.0132$

turbance rejection capability of the controller. It is clearly seen from the Table 2.1 that the proposed results give improved results over existing designs.

2.3 SOF controller design for a class of polytopic systems with actuator saturation

This section presents a robust $\mathcal{L}_2$ based SOF controller design for a class of linear CT polytopic systems subject to actuator saturation, which is a feature common to vast industrial processes. New sufficient LMI conditions are derived for the design of SOF controllers using parameter dependent Lyapunov function. The development incorporates decomposition of an auxiliary matrix variable so that approximation in deducing the LMIs involve reduced-size matrices. In addition, $\mathcal{L}_2$ performance under the constraints of actuator saturation is considered to develop a framework of a considerably complete problem. Such a problem in its entirety has not been considered so far in literature.

Different from the conditions developed in [145] for designing restricted SOF controllers, where decomposition of the Lyapunov matrix has been considered, here, we present new sufficient conditions for SOF control design for systems with matched output containing polytopic uncertainties both with and without actuator saturation by introducing an auxiliary matrix variable and decomposing this variable instead of the Lyapunov matrix. Several design cases are taken to demonstrate the efficacy of the proposed method and a comparison of the obtained results with those existing in literature is presented .

2.3.1 Problem statement and preliminaries

Consider a linear polytopic system described as:

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B(\theta)\text{sat}\{u(t)\} + B_w(\theta)w(t) \\ z(t) &= C_z(\theta)x(t) + D_{zu}(\theta)\text{sat}\{u(t)\} + D_{zw}(\theta)w(t) \\ y(t) &= Cx(t) + D_{yw}(\theta)w(t) \end{aligned} \tag{2.38}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $z(t) \in \mathbb{R}^{p_1}$ is the controlled output of the system, and $w(t) \in \mathbb{R}^{m_2}$ and $y(t) \in \mathbb{R}^p$ are disturbance input and measured output, respectively. The parameter θ=$[\theta_1, \theta_2, \cdots, \theta_r]^T \in \mathbb{R}^r$ is uncertain. The saturation function $sat\{\cdot\} : \mathbb{R}^m \to \mathbb{R}^m$ is defined as $sat\{u(t)\} := [sat\{u_1(t)\}, \cdots, sat\{u_m(t)\}]^T$, where $sat\{u_i(t)\} = sign(u_i(t)) \min\{\bar{u}_i, |u_i(t)|\}$, $\bar{u}_i > 0$ is the saturation amplitude of the i^{th} input. Note that the notation $sat\{\cdot\}$ is used for both the scalar and vector valued functions.

Assumption 1 *Following assumptions are considered for system* (2.38)*.*

1. *The parameter dependent matrices $A(\theta)$, $B(\theta)$, $B_w(\theta)$, $C_z(\theta)$, $D_{zu}(\theta)$, $D_{zw}(\theta)$, $D_{yw}(\theta)$ are assumed to be continuous in their arguments and bounded, having an affine dependence on θ.*

2. *The parameter θ ranges in a polytope Λ as:*

$$\theta \in \Lambda := Co\{\Omega_1, \Omega_2, \cdots, \Omega_N\} = \left\{ \sum_{i=1}^{N=2^r} \zeta_i \Omega_i : \zeta_i \geq 0, \ \sum_{i=1}^{N} \zeta_i = 1 \right\}$$

where N represents the cardinality of the vertices of the polytope.

Remark 2.10 *The matched assumption made in the measured output matrix of the system* (2.38)*, i.e., C is constant, may restrict the application of the presented result to general class of polytopic systems. However, many systems that are modeled in polytopic form possess this feature such as autonomous vehicles (land robots [84, 148, 149], airborne vehicles [85, 150], underwater vehicles [86]), electric motors [151–154] and DC-DC converters [34, 87, 155]. Hence, the class of polytopic systems considered here has wide applications.*

For designing SOF H_∞ controller in LMI framework, previous works in [43], [26], [82], require $D_{yw} = 0$. The work by [79] requires $D_{yw} = 0$, $D_{zw} = 0$ and $C_z^T D_{zu} = 0$. In contrast, such constraints on system matrices are not imposed here. However, it is considered that the feedback control system is strictly proper, which is more clear from the following polytopic representation of the system matrices of (2.38) as :

$$\begin{bmatrix} A(\theta) & B(\theta) & B_w(\theta) \\ C_z(\theta) & D_{zu}(\theta) & D_{zw}(\theta) \\ C & 0 & D_{yw}(\theta) \end{bmatrix} = \sum_{i=1}^{N} \zeta_i \begin{bmatrix} A(\Omega_i) & B(\Omega_i) & B_w(\Omega_i) \\ C_z(\Omega_i) & D_{zu}(\Omega_i) & D_{zw}(\Omega_i) \\ C & 0 & D_{yw}(\Omega_i) \end{bmatrix}. \tag{2.39}$$

The following is the definition of parameter dependent Lyapunov function that will form the basis of main results in this paper.

Definition 2.11 *[156] The function $V(t) = x^T(t)P(\theta)x(t)$ is said to be a PDLF if*

$$P(\theta) = \sum_{i=1}^{N} \zeta_i P_i, \zeta_i \in [0, 1], \sum_{i=1}^{N} \zeta_i = 1, \tag{2.40}$$

where P_is are symmetric positive definite matrices of appropriate dimension.

Next, recall the following preliminaries pertaining to defining the region of attraction for systems with actuator saturation (ROA).

Definition 2.12 *[157] A set $\mathscr{S} \in \mathbb{R}^n$ is said to be invariant if all trajectories of* (2.38) *originating from it stay within.*

Definition 2.13 *[18] A compact ellipsoidal convex set $\bar{\Theta}(P(\theta), \xi) \subset \mathscr{S}$ for $P(\theta)$ and $\xi > 0$, is defined as*

$$\bar{\Theta}(P(\theta), \xi) = \{x(t) \in \mathbb{R}^n : x^T(t)P(\theta)x(t) \leq \xi\}$$

Definition 2.14 *[18] Let* $\phi(t, x_0)$ *be the state trajectories of* (2.38) *with* $w(t) = 0$ *and initial state* x_0*. The region of attraction of the origin denoted by* $\mathcal{S}$ *is defined as*

$$\mathcal{S} = \{x_0 \in \mathbb{R}^n : \phi(t, x_0) \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

From Definition 2.13 and (1.33)*, the compact ellipsoid lies inside* $\sigma(H, u_j)$*, i.e.,* $\bar{\Theta}(P(\theta), \xi) \subset \sigma(H, u_j)$*, if* $1 \geq \frac{x^T(t)P(\theta)x(t)}{\xi} \geq \frac{x^T(t)C^T h_j^T h_j C x(t)}{\bar{u}_j^2}$*,* $\forall j$*. It can be shown following [14, page no. 407] that it is equivalent to*

$$\xi h_j C P(\theta)^{-1} C^T h_j^T \leq \bar{u}_j^2, \; j = 1, \cdots, m. \tag{2.41}$$

Now, with the SOF controller given as

$$u(t) = Ky(t), \tag{2.42}$$

where K *is the SOF gain matrix to be designed, the overall closed-loop system can be formed using* (2.38) *as:*

$$\begin{aligned} \dot{x}(t) &= A_o(\theta)x(t) + B_o(\theta)w(t) \\ z(t) &= C_o(\theta)x(t) + D_o(\theta)w(t), \end{aligned} \tag{2.43}$$

with

Case 1 (Without actuator saturation)

$$\begin{aligned} A_o(\theta) &= A(\theta) + B(\theta)KC, \quad B_o(\theta) = B_w(\theta) + B(\theta)KD_{yw}(\theta), \\ C_o(\theta) &= C_z(\theta) + D_{zu}(\theta)KC, \quad D_o(\theta) = D_{zw}(\theta) + D_{zu}(\theta)KD_{yw}(\theta). \end{aligned}$$

Case 2 (With actuator saturation)

$$\begin{aligned} A_o(\theta) &= A(\theta) + B(\theta)\left\{\sum_{i=1}^{2^m} \eta_i \left(\prod\nolimits_i K + \bar{\prod}\nolimits_i H\right)\right\} C, \\ B_o(\theta) &= B_w(\theta) + B(\theta)\left\{\sum_{i=1}^{2^m} \eta_i \left(\prod\nolimits_i K + \bar{\prod}\nolimits_i H\right)\right\} D_{yw}(\theta), \\ C_o(\theta) &= C_z(\theta) + D_{zu}(\theta)\left\{\sum_{i=1}^{2^m} \eta_i \left(\prod\nolimits_i K + \bar{\prod}\nolimits_i H\right)\right\} C, \\ D_o(\theta) &= D_{zw}(\theta) + D_{zu}(\theta)\left\{\sum_{i=1}^{2^m} \eta_i \left(\prod\nolimits_i K + \bar{\prod}\nolimits_i H\right)\right\} D_{yw}(\theta). \end{aligned}$$

The objective of this work is to design the SOF controller (2.42) *such that the system* (2.43) *attains* $\mathcal{L}_2$ *performance as given in Definition 1.3. Note that, the condition* (1.30) *corresponds to a PDLF with the Lyapunov function considered as* $V(x(t)) = x^T(t)P(\theta)x(t)$ *with* $P(\theta)^{-1} = X(\theta)$.

Remark 2.15 *The PDLF is quadratic with respect to the states of the system and has an affine dependence on the uncertain parameter. For* $P(\theta) = P$ *and* $X(\theta) = X$, *the condition* (1.30) *corresponds to the constant Lyapunov function based quadratic analysis. The constant Lyapunov function is known to yield conservative results for slowly varying parameters or constant ones. In contrast, the affine structure of the PDLF yields less conservative results.*

The main objective is to derive LMI conditions corresponding to the BMI (1.30) (due to the involvement of the terms $B(\theta)KCX(\theta)$*) for designing the SOF controller* (2.42).

2.3.2 Decomposition of auxiliary variable G

In this section, an alternate LMI condition corresponding to (1.30) *is presented. An auxiliary variable* G *is introduced to separate out the system matrix from the Lyapunov one so that the latter remains free and enables not imposing restriction on* $X(\theta)$ *to obtain a sufficient condition corresponding to* (1.30).

Lemma 2.16 *Consider the closed-loop system* (2.43) *with the SOF controller* (2.42). *For scalars* α, $\gamma > 0$, *if there exist matrices* $X(\theta) = X^T(\theta) > 0$ *and* G *satisfying the following matrix inequality:*

$$\begin{bmatrix} -2\alpha X(\theta) & * & * & * \\ B_o^T(\theta) & -\gamma^2 I & * & * \\ 0 & D_o(\theta) & -I & * \\ X(\theta) + G^T A_o^T(\theta) + \alpha G^T & 0 & G^T C_o^T(\theta) & -Sym\{G\} \end{bmatrix} < 0, \tag{2.44}$$

then the system satisfies the $\mathcal{L}_2$ *performance* (1.12).

Proof : *Adding and subtracting* $2\alpha X(\theta)$ *to the term* $Sym\{A_o(\theta)\ X(\theta)\}$ *in* (1.30), *one obtains:*

$$\begin{bmatrix} Sym\{(A_o(\theta) + \alpha I)X(\theta)\} - 2\alpha X(\theta) & * & * \\ B_o^T(\theta) & -\gamma^2 I & * \\ C_o(\theta)X(\theta) & D_o(\theta) & -I \end{bmatrix} < 0.$$

Separating the nonlinear terms involving θ, the above can be written as:

$$\begin{bmatrix} -2\alpha X(\theta) & * & * \\ B_o^T(\theta) & -\gamma^2 I & * \\ 0 & D_o(\theta) & -I \end{bmatrix} + \begin{bmatrix} (A_o(\theta)+\alpha I) \\ 0 \\ C_o(\theta) \end{bmatrix} \begin{bmatrix} X(\theta) \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} X(\theta) \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (A_o(\theta)+\alpha I) \\ 0 \\ C_o(\theta) \end{bmatrix}^T < 0. \tag{2.45}$$

Finally, applying Lemma 1.16 to (2.45) *with $L = G^T$ and $\beta = 1$, one obtains* (2.44). □

Remark 2.17 *Performance conditions* (2.44) *involving the auxiliary variable G has not been used in literature so far, though such a variable has been used in [26] for DT systems to tackle the nonlinear terms that arise because of the coupling between controller gain and system matrices.*

A decomposition of the auxiliary variable G is used in this work to obtain the main results. In [57], a diagonal decomposition of Lyapunov matrix is considered. However, the off-diagonal elements of the Lyapunov matrix have been omitted therein. This imposed restriction in the developed condition. In the recent work of [145], the off-diagonal terms considered for obtaining LMI criteria for SOF design. In the present work, sufficient conditions are developed for designing $\mathcal{L}_2$ based SOF controllers with and without actuator saturation through decomposition of the auxiliary matrix G instead of the Lyapunov matrix. Note that since PDLF is employed in this work, we will have to decompose each X_i if we proceed with the decomposition of Lyapunov matrix as in [145], which will consequently lead to increase the number of LMI variables. The free variable G is decomposed in this work instead of X_i as follows.

A decomposition of the auxiliary matrix variable in the similar lines as in [145] is presented next. Let the matrices $Q \in \mathcal{N}(C)$ and $R = C^\dagger + QL_C$ [57], where L_C is an $(n-p)\times p$ matrix, $C^\dagger = C^T(CC^T)^{-1}$ is the Moore-Penrose pseudo inverse of C where $\mathcal{N}(C)$ represents the null space of C. Then the below result is straightforward.

Fact 5 *A matrix G can be decomposed as*

$$G = \begin{bmatrix} Q^T \\ R^T \end{bmatrix}^T \begin{bmatrix} G_Q & G_S \\ G_T & G_R \end{bmatrix} \begin{bmatrix} Q^T \\ R^T \end{bmatrix},$$

where $G_Q \in \mathbb{R}^{(n-p)\times(n-p)}$, $G_S \in \mathbb{R}^{(n-p)\times p}$, $G_T \in \mathbb{R}^{p\times(n-p)}$ *and* $G_R \in \mathbb{R}^{p\times p}$ *and* $CQ = 0$. *If* R *is so chosen that* $CR = I$, *then one obtains* $CG = G_R R^T + G_T Q^T$.

The matrix Q *is one whose columns form the basis of the* $null(C)$, *i.e.,* $CQ = 0$. *Further,* R *is a given matrix such that* $CR = I$ (R_C *is the right inverse of* C). *Notice that* R *can be parametrized as* $R = C^\dagger + QL_C$ *[57], where* L_C *is an* $(n-p)\times p$ *matrix and* $C^\dagger = C^T(CC^T)^{-1}$ *is the Moore-Penrose pseudo inverse of* C. *This can be seen since* $C(C^\dagger + QL_C) = I$ *and, conversely, if* $CR = I$ *then* $C(R - C^\dagger) = 0$. *Thus, the choice of* R *is not unique. The dependence of solution of LMIs on the choice of* Q *and* R *is further studied numerically in Example 2.24 in design section later.*

2.3.3 $\mathcal{L}_2$ based SOF controller design

This section presents sufficient LMI conditions for $\mathcal{L}_2$ based SOF controller design with and without saturation of the control input based on the idea of decomposition of G discussed in the previous section.

2.3.3.1 Without actuator saturation

Using Lemmas 1.16 and 2.16, we now state Theorem 2.18 below, which provides a new LMI criterion for designing $\mathcal{L}_2$ based SOF controller (2.42), with $sat\{u(t)\}$ in (2.38) replaced by $u(t)$ because of the unsaturated nature of the control input.

Theorem 2.18 *The CT system (2.38) with unsaturated control input is stabilizable by SOF controller gain* K *with* $\mathcal{L}_2$ *performance* γ *if, for scalars* α, ρ, *there exist* $X_i = X_i^T > 0$ G_Q, G_R, G_T, G_S *(therefore* G*) and* Y_C *such that*

$$\begin{bmatrix} -2\alpha X_i & * & * & * & * \\ B_{w_i}^T - \rho D_{yw_j}^T R^T & -\gamma^2 I & * & * & * \\ 0 & D_{zw_i} & -I & * & * \\ \Psi_{3_i} & 0 & G^T C_{z_i}^T + RY_C^T D_{zu_i}^T & -Sym\{G\} & * \\ (\rho B_i Y_C + \rho^2 R G_R)^T & D_{yw_j} & \rho Y_C^T D_{zu_i}^T & G_T Q^T & -Sym\{\rho G_R\} \end{bmatrix} < 0,$$

$$i, j = 1, \cdots, N, \tag{2.46}$$

where $\Psi_{3_i} = X_i + \alpha G^T + G^T A_i^T + RY_C^T B_i^T - \rho Q G_T^T R^T$. *The feedback gain can be computed as* $K = Y_C G_R^{-1}$.

Proof : Multiplying (2.46) with $\zeta_i\zeta_j$ and summing up all the conditions, and in view of (2.39) and (2.40), one can write the equivalent condition of (2.46) as:

$$\begin{bmatrix} -2\alpha X(\theta) & * & * & * & * \\ B_w^T(\theta) - \rho D_{yw}^T(\theta)R^T & -\gamma^2 I & * & * & * \\ 0 & D_{zw}(\theta) & -I & * & * \\ \Psi_3(\theta) & 0 & \Psi_4(\theta) & -Sym\{G\} & * \\ (\rho B(\theta)Y_C + \rho^2 RG_R)^T & D_{yw}(\theta) & \rho Y_C^T D_{zu}^T(\theta) & G_T Q^T & -Sym\{\rho G_R\} \end{bmatrix} < 0, \quad (2.47)$$

where $X(\theta) = P(\theta)^{-1}$, $\Psi_3(\theta) = X(\theta) + \alpha G^T + G^T A^T(\theta) + RY_C^T B^T(\theta) - \rho Q G_T^T R^T$ and $\Psi_4(\theta) = G^T C_z^T(\theta) + RY_C^T D_{zu}^T(\theta)$. Applying Lemma 1.16 to (2.47) with $L = G_R$, we get

$$\begin{bmatrix} -2\alpha X(\theta) & * & * & * \\ B_w^T(\theta) - \rho D_{yw}^T(\theta)R^T & -\gamma^2 I & * & * \\ 0 & D_{zw}(\theta) & -I & * \\ \Psi_3(\theta) & 0 & \Psi_4(\theta) & -Sym\{G\} \end{bmatrix} + Sym\left\{ \begin{bmatrix} 0 \\ D_{yw}^T(\theta) \\ 0 \\ QG_T^T \end{bmatrix} G_R^{-T} \begin{bmatrix} B(\theta)Y_C + \rho RG_R \\ 0 \\ D_{zu}(\theta)Y_C \\ 0 \end{bmatrix}^T \right\} < 0. \quad (2.48)$$

Now, from Fact 5, we have $KCG = Y_C R^T + Y_C G_R^{-1} G_T Q^T$, where $Y_C \in \mathbb{R}^{m\times p}$, $Y_C = KG_R$ and G_R is invertible. Performing simple manipulations in (2.48) by replacing $Y_C R^T + Y_C G_R^{-1} G_T Q^T$ with KCG, we get (2.44). Thereby, Lemma 2.16 is satisfied and the proof is complete. □

2.3.3.2 With actuator saturation

Next, sufficient LMI conditions for designing $\mathcal{L}_2$ based SOF controller for (2.38) is developed in the below Theorem.

Theorem 2.19 *The CT system (2.38) is stabilizable by SOF controller gain K with $\mathcal{L}_2$ performance γ if, for scalars $\bar{\alpha}$ and $\bar{\rho}$, there exist matrices $X_i = X_i^T > 0$, G_Q, G_R, G_T,*

G_S (therefore G), Y_{R_1} and Y_{R_2} satisfying the below LMIs:

$$\begin{bmatrix} -2\bar{\alpha}X_i & * & * & * & * \\ B_{w_i}^T - \bar{\rho}D_{yw_k}^T R^T & -\gamma^2 I & * & * & * \\ 0 & D_{zw_i} & -I & * & * \\ \bar{\Psi}_{3_i} & 0 & G^T C_{z_i}^T + RY^T D_{zu_i}^T & -Sym\{G\} & * \\ \bar{\rho}(B_i Y + \bar{\rho}RG_R)^T & D_{yw_k} & \bar{\rho}Y^T D_{zu_i}^T & G_T Q^T & -Sym\{\bar{\rho}G_R\} \end{bmatrix} < 0, \tag{2.49a}$$

$$\begin{bmatrix} -\bar{u}_j^2 X_i & * & * \\ \bar{\rho}CX_k & -Sym\{\bar{\rho}G_R\} & * \\ 0 & e_j Y_{R_2} & -I \end{bmatrix} \leq 0, \ j \in [1, m], \ i, k = 1, \cdots, N. \tag{2.49b}$$

where $\bar{\Psi}_{3_i} = X_i + \bar{\alpha}G^T + G^T A_i^T + RY^T B_i^T - \bar{\rho}QG_T^T R^T$, $Y = \prod_i Y_{R_1} + \bar{\prod}_i Y_{R_2}$, $i = 1, \cdots, 2^m$. e_j represents the j^{th} row of $I_{m\times m}$. The controller gain K can be computed as $K = Y_{R_1} G_R^{-1}$.

Proof : Consider the closed-loop system (2.43) with the control input (2.42). The LMI criterion (2.49a) is obtained following similar derivation steps as in Theorem 2.18 by using $KCG = Y_{R_1}R^T + Y_{R_1}G_R^{-1}G_T Q^T$ and $HCG = Y_{R_2}R^T + Y_{R_2}G_R^{-1}G_T Q^T$ with $Y_{R_1} = KG_R, Y_{R_2} = HG_R$ from Fact 5.

Next, consider the ellipsoid ($\bar{\Theta}(P(\theta), 1)$ is inside $\sigma(H, u_j)$, i.e., $\bar{\Theta}(P(\theta), 1) \subset \sigma(H, u_j)$. It is so if the conditions $\xi h_j CP(\theta)^{-1}C^T h_j^T \leq \bar{u}_j^2$, $j = 1, \cdots, m$, hold, which can be written as :

$$-\bar{u}_j^2 P(\theta) + C^T G_R^{-T} Y_{R_2}^T e_j^T e_j Y_{R_2} G_R^{-1} C \leq 0, \ j = 1, \cdots, m. \tag{2.50}$$

Pre- and post-multiplying (2.50) by $P(\theta)^{-1}$ and its transpose, respectively, and substituting $P(\theta)^{-1} = X(\theta)$, yields

$$-\bar{u}_j^2 X(\theta) + X^T(\theta)C^T G_R^{-1} Y_{R_2}^T e_j^T e_j Y_{R_2} G_R^{-1} CX(\theta) \leq 0. \tag{2.51}$$

Now consider (2.49b). Multiply (2.49b) with $\zeta_i \zeta_k$ and twice summing over all i and k, we get

$$\begin{bmatrix} -\bar{u}_j^2 X(\theta) & * & * \\ \bar{\rho}CX(\theta) & -Sym\{\bar{\rho}G_R\} & * \\ 0 & e_j Y_{R_2} & -I \end{bmatrix} \leq 0. \tag{2.52}$$

Now, applying Lemma 1.15 on (2.52), with $\mathcal{T} = -\bar{u}_j^2 X(\theta)$, $M = G_R^{-1} C X(\theta)$, $R = Y_{R_2}^T e_j^T e_j Y_{R_2}$ and $L = \bar{\rho} G_R$, we get

$$\begin{bmatrix} -\bar{u}_j^2 X(\theta) & * \\ \bar{\rho} C X(\theta) & -Sym\{\bar{\rho} G_R\} + Y_{R_2}^T e_j^T e_j Y_{R_2} \end{bmatrix} \leq 0$$

Finally, using schur complement, a stricter inequality condition can be written as (2.51). The proof is complete. □

Next, to enlarge the region of attraction satisfying the criteria in Theorem 2.19, the following LMI condition is considered:

$$\begin{bmatrix} \lambda^{-2} \Lambda & I \\ I & X_i \end{bmatrix} \geq 0, \; i = 1, \cdots, N, \tag{2.53}$$

where λ signifies a measure of the ROA. Larger λ indicates larger ROA, which is desired.

To obtain the LMI condition (2.53), consider $\chi_\Lambda \subset \mathbb{R}^n$ as a bounded convex set containing the equilibrium point. With an aim to find the largest region inside $\bar{\Theta}(P(\theta), \xi)$, define the set

$$\lambda_\Lambda(\bar{\Theta}(P(\theta), \xi)) = sup\left\{\lambda > 0 | (\lambda \chi_\Lambda) \subset \bar{\Theta}(P(\theta), \xi)\right\}$$

where λ is a scalar and χ_Λ defined as $\chi_\Lambda = \{x \in \mathbb{R}^n : x^T \Lambda x \leq 1\}$. Now, in order to ensure $\lambda \chi_\Lambda \subset \bar{\Theta}(P(\theta), \xi)$, we have the equivalent LMI condition such that $\frac{\Lambda}{\lambda^2} \geq \frac{P(\theta)}{\xi}$ where Λ is a positive definite matrix. Using Schur complement and assuming $\xi = 1$ (without loss of generality) and substituting $P(\theta)^{-1} = X(\theta)$, one obtains a LMI condition (2.53).

Now, in order to get the largest estimate of the region of attraction along with smallest $\mathcal{L}_2$ performance γ, the following LMI optimization problem is defined.

Problem 1

$$\min(\lambda^{-2} + \gamma^2)$$

subject to (2.49a), (2.49b) *and* (2.53).

This problem is solved in the numerical examples wherever SOF design with actuator saturation is considered.

Remark 2.20 *By setting $X_i = X$, $\forall i \in [1, 2^m]$, conditions (2.49a), (2.49b) and (2.53) reduce to the one corresponding to a quadratic constant Lyapunov function (CLF). Using*

PDLF, the proposed controller provides less conservative results (here, larger region of attraction and increased disturbance attenuation capability). However, using PDLF may lead to a rapid increase in the computational cost as the size of control input $u(t)$ increases.

Remark 2.21 *Note that in Problem 1, a trade-off is struck between minimizing the attenuation level γ and enlarging the domain of attraction by maximizing λ^2. In order to attain a balance between them, maximizing λ^2 is identical to minimizing λ^{-2} and hence, the problem can be combined into minimizing the term $\lambda^{-2} + \gamma^2$. It is shown henceforth that Theorem 2.19 provides a larger λ value (maximal region of domain of attraction) and smaller γ value (better disturbance rejection capability) than constant Lyapunov function approach as discussed in the Remark above.*

Remark 2.22 *Theorems 2.18 and 2.19 provide new LMI criteria for SOF controller design for CT polytopic systems without and with actuator saturation, respectively. The proposed LMI conditions for $\mathcal{L}_2$ controller designs can be solved using LMI solvers, for e.g., using LMI control toolbox [158]. For solving these LMIs, one has to obtain the parameters α, ρ in Theorem 2.18, and $\bar{\alpha}$, $\bar{\rho}$ in Theorem 2.19. Obtaining these scalar parameters may require use of suitable algorithms for optimization of γ. For solving examples in this paper, the following approach is used : (i) First, the feasibility problem of the LMIs is solved to obtain a set of initial scalar parameters, (ii) Then, the numerical optimization algorithm 'fminsearch' [147] is used to obtain a locally convergent solution. Through gridding of the initial values of the scalar parameters, a large range of these parameters can be covered for obtaining a possible globally optimal solution. The same is used in design cases presented in the next section. Use of search functions for obtaining tuned LMI solutions is well known in LMI literature, for example, see [159], [160].*

Remark 2.23 *It is to be noted that the actuator saturates only occasionally. The control inputs remain within the limits when the states are well within the region of attraction. The polytopic representation of the saturation function used in this work is adopted from [18]. The polytope formation consists of two piecewise continuous curves— one, the linear region (defined by the nonsaturating straight line through the origin) and the other, the constant line having amplitude $\pm\bar{u}_j$, which is the saturating region. When the actuators are working properly, the control output lies in the linear region, which asserts that $\eta_i = 1$ for $i = 1$, without loss of generality and $\eta_i = 0$ for all other values of i, in* (1.35). *As* *we move into the nonlinear/saturating region, η_i becomes non-zero for finite values of i greater than one. Thus, the polytope takes care of the saturation function.*

Further, anti-windup compensation is often used [14, 161, 162] to deal with the saturation problem using the information of whether the actuator is saturated. This improves the transient response of the states while enlarging the region of attraction of the closed-loop system. This present work may be extended by incorporating anti-windup compensation.

2.3.4 Design cases

In this section, numerical examples are presented to demonstrate the efficacy of the proposed SOF design criteria. Comparative studies are carried out with the existing results. Additionally, a 2-DOF helicopter is considered to illustrate the efficacy of the proposed design method in Theorem 2.19 on an experimental setup. Note that for employing Theorem 2.18 and Theorem 2.19, decomposition of the auxiliary matrix variable G is made through Q as the null matrix of C and $R = C^T(CC^T)^{-1}$ (with $L_C = 0_{(n-p)\times p}$), see Remark 2.1 for more detail on this.

Example 2.24 *Consider a CT polytopic system without actuator saturation given in Dong et al [79]*

$$A_1 = \begin{bmatrix} -0.9896 & 17.41 & 96.15 \\ 0.2648 & -0.8512 & -11.39 \\ 0 & 0 & -30 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1.702 & 50.72 & 263.5 \\ 0.2201 & -1.418 & -31.99 \\ 0 & 0 & -30 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -85.09 \\ 0 \\ 30 \end{bmatrix}, B_1 = \begin{bmatrix} -97.78 \\ 0 \\ 30 \end{bmatrix}, \ B_{w_1} = B_{w_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ C_{z_1} = C_{z_2} = I,$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, D_{zw_1} = D_{zw_2} = D_{zu_1} = D_{zu_2} = D_{yw_1} = D_{yw_2} = 0.$$

For this system, the existing criteria in [43] (Lemma 2 (ii)), [67] (Theorem 4), [79] (Lemma 2) and [79] (Theorem 13) are considered for comparison with the proposed criteria in Theorem 2.18. These methods are referred here as M_1, M_2, M_3, M_4, M_5, respectively. The number of LMI variables (NLV) (resp. the number of optimization variables) in M_1, M_2, M_3, M_4 and M_5 are $0.5n^2 + p^2 + mp + 0.5n$ (resp. 0), $(0.5N + 2)n^2 + (N + 2)np + mp + (1 + 0.5N)p^2 + 0.5Nn + 0.5Np$ (resp. 4), $0.5Nn^2 + p^2 + mp + 0.5Nn$ (resp. 1),

$0.5Nn^2 + p^2 + mp + 0.5Nn$ *(resp.* 1*) and* $0.5Nn^2 + n^2 + mp + 0.5Nn$ *(resp.* 2*), respectively.* M_1 *does not require any scalar parameter for solving the respective LMI criteria, and hence are solvable in a single step. On the other hand,* M_2*,* M_3*,* M_4 *and* M_5 *involve obtaining scalar parameters for solving the LMIs. These indeed require obtaining locally optimized scalar values by solving the LMI problem several times. Hence, these methods (*M_2*,* M_3*,* M_4 *and* M_5*) result in better* γ *values, but with more computational burden.*

The dimensions associated with the system matrices are $n = 3$*,* $p = 2$ *and* $m = 1$*. The cardinality of the vertices of the polytope, i.e.,* $N = 2$*. The values of* Q *and* R *are chosen as* $Q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and the numerical results are listed in Table 2.2. The NLV for each criteria are also included along with the respective* γ *values. It can be seen that the present result using* M_5 *is less conservative than the others. It clearly illustrates that the proposed design technique provides a better design for the polytopic system without actuator saturation, although the NLVs are slightly more than* M_1*,* M_3 *and* M_4*, which results from the tradeoff between the computational complexity and performance obtained for the closed-loop system.*

Table 2.2: $\mathcal{L}_2$ based SOF Controller for Example 2.24

Methods	γ	NLV	No. of tuning parameters
M_1	9.7315	12	0
M_2	6.8028	66	4
M_3	7.0362 ($\beta = 0.11$)	18	1
M_4	2.3267 ($\tau = 0.05$)	18	1
M_5	2.0141 (CLF) ($\alpha = 410.19$, $\rho = 0.0008$)	17	2
	1.7435 (PDLF) ($\alpha = 58.1236$, $\rho = 0.0011$)	23	2

Next, a linear variation in $Q \in null(C)$ *is taken as* $Q = \delta[0\ 0\ 1]^T$ *to demonstrate the effect of the choice of* Q *on* γ*. Similarly, different choices of* R *are also made by choosing* L_C *in* $R = C^T(CC^T)^{-1} + [0\ 0\ 1]^T L_C$*. The different* γ *values obtained for different* δ *and* L_C *values for the CLF case are presented in Table 2.3. The variation in* Q*, though non-structural in nature, shows good variation in* $\mathcal{L}_2$ *performance as seen from Table 2.3.*

Table 2.3: Variation of design parameter for Example 2.24

$Q = \delta[0\ 0\ 1]^T$				$\bar{R}$	
δ	γ	δ	γ	L_C	γ
0.3	3.7059	1.1	2.1020	[0 1]	5.8847
0.4	2.9825	1.2	2.1498	[0.5 0]	6.0242
0.5	2.5697	1.3	2.1511	[0.5 0.5]	4.7569
0.6	2.3262	1.4	2.1534	[0 -0.5]	2.5092
0.7	2.1975	1.5	2.1592	[-0.1 0]	2.6859
0.8	2.1478	1.6	2.1652	[-0.1 -0.1]	2.6466
0.9	2.0552	1.7	2.2021	[0 -0.1]	2.1201
1.0	2.0141	1.8	2.2525	[0 0]	2.0141

However, a nominal choice ($\delta = 1$) as initially taken for this example also works well. Similarly, it can also be observed that R depends on the selection of L_C, which further affects γ. In this example, the choice $L_C = [0\ 0]$, yields the best result $\gamma = 2.0141$.

Example 2.25 *Next, consider a linear CT system with actuator saturation having the following matrices [93].*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1.8 \\ 0 & 240 & 1 & 10 \end{bmatrix}, B = \begin{bmatrix} 80 & 0 \\ 0 & 80 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; B_w = I_{4\times4}, \; C = \begin{bmatrix} 4 & -2 \\ -3 & 2 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}^T,$$

$$C_z = 10^{-2} \times \begin{bmatrix} 0.3 & 0.72 & 0 & 0 \\ 1.2 & 0 & 0 & 0 \\ 0.6 & 0.3 & 0 & 0 \\ 2.94 & 1.26 & 0 & 0 \end{bmatrix}, \; D_{zu} = \begin{bmatrix} 0.6204 & 0.7021 & -0.1352 & -0.3223 \\ 0 & -0.0808 & -0.9696 & 0.2309 \end{bmatrix}^T,$$

$$D_{zw} = D_{yw} = 0, \bar{u}_1 = \bar{u}_2 = 1.$$

We compare the respective domain of attractions (λ) and the $\mathcal{L}_2$ performances (γ) obtained in Theorem 2 of [93] and the Problem 1 in this work by using both constant Lyapunov function and PDLF. The results are given in Table 2.4. χ_Λ is considered as an ellipsoidal reference set with $\beta = 1$ and the weighting function is taken as $\Lambda = I_{4\times4}$. The values of

Q and R are taken as

$$Q = \begin{bmatrix} -0.6126 & -0.3925 \\ -0.0488 & -0.8230 \\ 0.7681 & -0.2996 \\ -0.1798 & 0.2807 \end{bmatrix}, \quad R = \begin{bmatrix} 0.1176 & 0 \\ -0.0327 & 0.1111 \\ 0.1438 & 0.1111 \\ 0.2222 & 0.4444 \end{bmatrix}.$$

The values of $(\bar{\alpha}, \bar{\rho})$ *are obtained as (28.3076, 0.0148) for the constant Lyapunov function (CLF) and (27.9452, 0.0144) for the PDLF, respectively. It can be observed from*

Table 2.4: $\mathcal{L}_2$ based SOF Controller for Example 2.25

Methods	γ_{min}	λ_{max}
Theorem 2 [93]	0.2949	1.2487
Problem 1 [Present] CLF	0.2042	2.1000
Problem 1 [Present] PDLF	0.1935	2.1286

Table 2.4 that the proposed methods yield smaller γ *with larger* λ *than previous work [93]. The corresponding controllers are :* $K_{CLF} = \begin{bmatrix} -0.0143 & 0.0273 \\ 0.0064 & -0.0348 \end{bmatrix}$, $K_{PDLF} = \begin{bmatrix} -0.0284 & 0.0592 \\ 0.0145 & -0.0902 \end{bmatrix}$. *Similarly, for the same design case, we have designed the SOF controller using Theorem 2.18 taking* $N = 1$ *without considering the actuator saturation and computed the* $\mathcal{L}_2$ *performance value as* $\gamma = 0.0522$ *for* $\alpha = 54.8755$ *and* $\rho = 0.2002$. *The corresponding controller gain is obatined as* $K = \begin{bmatrix} -0.0848 & 0.1162 \\ 0.0965 & -0.1968 \end{bmatrix}$.

2.4 Summary

In this chapter, LMI criteria for SOF design are derived for CT systems that can be used for both the centralized and the decentralized control design. The development involves suitable decomposition of Lyapunov matrices and deriving LMI criterion that ensures H_∞ performance. LMI criteria for pole-placement in LMI region are also derived. Next, the SOF design condition is extended to the PID controller design problem for the higher order MIMO systems using the transformation matrix. The design approach is more suitable for easy tuning of controller gains and implementation.

The SOF design problem is extended to the class of uncertain polytopic systems. Sufficient conditions are derived for designing SOF controller for systems with matched output matrix containing polytopic uncertainties both with and without actuator saturation by introducing an auxiliary matrix variable and decomposing this variable instead of the Lyapunov matrix. The derived conditions employ a PDLF approach, which gives less conservative results than the constant Lyapunov function one. The efficacy of the proposed design criteria are illustrated through numerical examples.

In the next chapter, we explore the SOF design problems for DT systems.