Chapter 1

Introduction

Feedback control is mainly used to ensure robustness including stabilizing unstable systems whilst improving dynamic performance as well. However, if a feedback strategy is not designed appropriately, it can degrade the system performance and even may lead to instability. In particular, it is well known that under certain conditions it is possible to assign the system eigenvalues to arbitrary values by feedback, allowing us to design the dynamics of a system.

In general, two types of feedback control strategies have evolved in literature, namely, the state and output feedback. While the former one is simpler to design but it requires the measurement of full state vector. Hence, it is difficult and costly to implement. This is because direct measurement of all system state variables may either be impossible or impractical to measure in real-time. To address this shortcoming, observer design has been looked into, which uses the measurement of the inputs and outputs of the system, along with knowledge of the system's dynamic model, to estimate the unavailable (or all) states. However, this design method becomes tedious and complex since it involves the dynamics of both the system as well as the observer itself. To alleviate to this problem, output feedback is generally used for practical applications. With output feedback, only the information of measurable output is required. Output feedback can be of two types, static output feedback (SOF) and dynamic output feedback (DOF). SOF is easy to implement and also reduces the real-time implementation cost, whereas, in DOF controller, the dynamics of the controller is also involved. However, it is a tedious and challenging task to design the SOF controller. The DOF controller design problem can be cast as the SOF design problem. In this work, the issues and difficulties arising while designing the SOF controller for both the continuous-time as well as discrete-time LTI systems are discussed.

1.1 Static output feedback control design problem

Discussions on the challenges and issues associated with the SOF design problem is presented in this section. Performance criteria such as H_2, H_∞ and pole-placement are presented. Existing methods available in the literature for solving the SOF problem are discussed.

1.1.1 Continuous-time system

Consider the continuous-time LTI systems described as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ represent the input and output vector, respectively. A, B and C are the system matrices of appropriate dimension. For simplicity, the stabilization problem of the system (1.1) (also referred to as the nominal system because the dynamics are free of any disturbances) is considered first. It is assumed throughout this thesis work that the system (1.1) is reachable and observable. For system (1.1), consider the control law as :

$$u(t) = Ky(t) \tag{1.2}$$

where K is the SOF controller gain. Then the closed-loop system for (1.1) is given as:

$$\dot{x}(t) = (A + BKC)x(t). \tag{1.3}$$

The control objective for stabilization problem is to design the SOF gain K such that (1.3) is asymptotically stable. Now, for stability analysis of the system (1.3), we use the Lyapunov stability theory, for which we consider the quadratic Lyapunov function $V = x^T(t)Px(t)$ with $P = P^T > 0$. This amounts to finding the matrix such that

$$(A + BKC)^T P + P(A + BKC) < 0 \tag{1.4}$$

Pre- and post-multiplying (1.4) by P^{-1} and its transpose, respectively, and substituting $X = P^{-1}$, one can obtain

$$XA^T + AX + XC^T K^T B^T + BKCX < 0. \tag{1.5}$$

We can now see that (1.4) and (1.5) are not linear, which poses a Bilinear Matrix Inequality (BMI) problem due to the term $BKCX$ and its transpose. Such a problem is proved to be non-convex and NP-hard [1] in nature. Thus, developing general SOF controller design problem is challenging in practice.

A lot of research has been carried out in the past decades to solve the above mentioned SOF problem. These have been discussed in the literature review in more details in the upcoming section.

Next, consider the generalized plant as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_w w(t), \\ z(t) &= C_z x(t) + D_{zu} u(t) + D_{zw} w(t), \\ y(t) &= Cx(t) + D_{yu} u(t) + D_{yw} w(t), \end{aligned} \tag{1.6}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input and $z(t) \in \mathbb{R}^{p_1}$ is the controlled output of the system. $w(t) \in \mathbb{R}^{m_1}$ and $y(t) \in \mathbb{R}^p$ are disturbance input and measured output, respectively. $A, B, B_w, C_z, D_{zu}, D_{zw}, C, D_{yw}$ are matrices of appropriate dimensions. Such a model is important since it captures how modeling uncertainties, noise and disturbances acting on the system are modeled into the dynamics [2]. It should be kept in mind that control engineers role is not merely one of designing the controller for systems with nominal model. It also involves assisting in the choice and configuration of hardware by taking a system-wide view of performance. That is why it is necessary to develop the theory that not only lead to good designs when these are possible, but also indicate directly and unambiguously when the performance objectives cannot be met. It is also important to mention that most of the practical problems have uncertain, nonminimum–phase plants (non-minimum-phase means the existence of right half-plane zeros, so the inverse is unstable); that there are inevitably unmodeled dynamics that produces substantial uncertainty, usually at high frequency; and that sensor noise and input signal level constraints limit the achievable benefits of feedback [2]. Usually, these practical issues are ignored while developing the controller design methods. Nevertheless, any general theory should be able to treat all these issues explicitly and give quantitative and qualitative results about their impact on the system performance. Taking this point into consideration, we used the generalized model of the control system considering the exogenous input, $w(t)$ (disturbances, sensor noises, and so on) and controlled output variable $z(t)$ to compute the performance of the system [2]. A block diagram of the

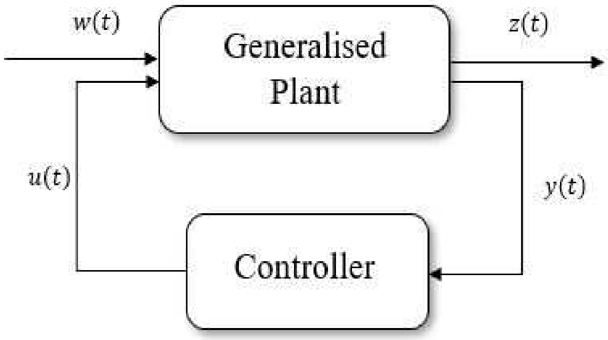


Figure 1.1: Generalized Control System

generalized control system model is shown in Fig. 1.1. For system (1.6), consider the same SOF controller [1] as in (1.2). The closed-loop system is then given by

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \tag{1.7}$$

where, $A_{cl} = A+BKC, B_{cl} = B_w+BKD_{yw}, C_{cl} = C_z+D_{zu}KC$ and $D_{cl} = D_{zw}+D_{zu}KD_{yw}$ are the closed-loop system matrices. The closed-loop transfer function matrix from $w(t)$ to $z(t)$ is

$$T_{zw}(s) = C_{cl}(sI - A_{cl})^{-1}B_{cl} + D_{cl}. \tag{1.8}$$

For more detail on uncertainty modeling on system representations and transformations, one may refer to the well-versed books on robust control, e.g. [2–4]. However, these aspects will not be discussed in this thesis, rather we continue with the generalized plant representations as above not bothering about how such a system model can be obtained.

[1] The generic case of centralized control is considered for the developments. The decentralized and other restricted feedback cases can be incorporated by imposing restrictions in the SOF gain matrix and corresponding matrix variables in the design that are discussed later.

1.1.1.1 Performance Measure

The robust performance of the control system is defined in terms of norm of the signals. In general, the input/output behavior from $w(t)$ to $z(t)$ gives the measure of performance of the system. In this thesis work, we consider the H_2 and H_∞ norm criteria, which are defined below.

Definition 1.1 ([5]) *Given a stable and strictly proper LTI transfer function $T_{wz}(s)$, the H_2 norm is defined as :*

$$\|T_{wz}\|_2 := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} Trace(T_{wz}(j\omega)T_{wz}^*(j\omega))d\omega \right)^{1/2} \tag{1.9}$$

H_2 norm is also defined as root mean-square (RMS) of the impulse response of the system, i.e.,

$$\|T_{wz}\|_2 := \|z(t)\|_2$$

In case of white noise input $w(t)$ with unit variance, the H_2 norm represents the expected (E) RMS-value of the output of system as:

$$\|T_{wz}\|_2 := \left\{ \mathrm{E}\left(\frac{1}{T} \int_0^{\infty} z^T(t)z(t)dt \right) \right\}^{1/2} \tag{1.10}$$

Definition 1.2 ([5]) *Given a stable proper LTI transfer function $T_{wz}(s)$, its H_∞ norm is defined as :*

$$\begin{aligned} \|T_{wz}\|_\infty &:= \sup_{\omega \in \mathbb{R}} \sigma_{max}(T_{wz}(j\omega)) \\ &:= \sup_{\|w(t)\|_2 \neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} \end{aligned} \tag{1.11}$$

where $\|z(t)\|_2^2 = \int_0^\infty z(t)^T z(t)dt, \;\; \|w(t)\|_2^2 = \int_0^\infty w(t)^T w(t)dt.$

The H_∞ norm represents the largest possible frequency gain, which corresponds to the maximum of the largest singular value of $T_{wz}(j\omega)$ for the MIMO system. In the case of a SISO system, $\|T_{wz}\|_\infty$ is the maximum of $|T_{wz}(j\omega)|$, i.e., $\|T_{wz}\|_\infty := \max_w |T_{wz}(j\omega)|$.

Following the above, the system performance for the H_∞ control is given as:

$$\|z(t)\|_2 \leq \gamma \|w(t)\|_2.$$

where γ refers to the performance level of the closed-loop system under the effect of exogenous signals. Other way [2], it can also be defined as the following.

[2] If the system is not LTI then we use $\mathcal{L}_2$ norm in terms of the input and output signals of the system as a measure of performance.

Definition 1.3 *[3] Let the closed-loop system* (1.29) *be asymptotically stable and satisfies*

$$\int_0^\infty z^T(t)z(t)dt < \gamma^2 \int_0^\infty w^T(t)w(t)dt, \tag{1.12}$$

for $x(0) = 0$*. Then the system* (1.29) *has a* $\mathcal{L}_2$ *performance of* γ*.*

Note that throughout this work, It is assumed zero initial conditions. However, in the case where $x(0) \neq 0$, the calculated H_∞ performance estimates (i.e., γ) assuming zero initial conditions will serve as more conservative estimates (or overestimates). This entails that the robust performance region calculated (using $x(0) = 0$) is a superset of the new actual region (which should be calculated by taking initial conditions into account). This is because for $x(0) \neq 0$, we may need to introduce a new performance measure that is essentially the worst-case norm of the regulated outputs over all exogenous signals and initial conditions. Precisely, this takes the form $\sup\limits_{x(0)\in\mathbb{R}^n, w\in\mathcal{L}_2[0,\infty), \|w(t)\|^2+x(0)^TRx(0)\neq 0} \frac{\|z(t)\|^2}{\|w(t)\|^2+x(0)^TRx(0)}$. Note here the weighting matrix R can be thought of as a measure of relative in initial conditions vis-a-vis the uncertainty in $w(t)$. A more detailed analysis on this can be found in [6–8].

1.1.1.2 Performance Criteria

For H_2 and H_∞ performance of system (1.8), the below lemmas are well known.

Lemma 1.4 ([3, 4]) *System* (1.8) *satisfies the following performance criteria.*

1. *Assume* $D_{cl} = 0$ $(D_{zw} = 0, D_{yw} = 0)$*. The* H_2 *performance defined as* $\|T_{wz}(s)\|_2^2 < \mu$ *is guaranteed if there exist matrices* $P = P^T > 0$, $\mathcal{W} = \mathcal{W}^T > 0$ *satisfying*

$$trace(\mathcal{W}) < \mu, \quad \begin{bmatrix} P & * \\ C_{cl} & \mathcal{W} \end{bmatrix} > 0, \tag{1.13}$$

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} & * \\ B_{cl}^T P & -I \end{bmatrix} < 0. \tag{1.14}$$

Lemma 1.5 (Bounded Real Lemma [3, 9]) *The following statements are equivalent for* $\gamma > 0$*.*

1. $\|T_{zw}(s)\| < \gamma$ *and* A_{cl} *is Hurwitz.*

2. There exists $X = X^T > 0$ satisfying

$$\begin{bmatrix} Sym\{A_{cl}X\} & * & * \\ B_{cl}^T & -\gamma^2 I & * \\ C_{cl}X & D_{cl} & -I \end{bmatrix} < 0. \tag{1.15}$$

3. There exists $P = P^T > 0$ satisfying

$$\begin{bmatrix} Sym\{PA_{cl}\} & * & * \\ B_{cl}^T P & -\gamma^2 I & * \\ C_{cl} & D_{cl} & -I \end{bmatrix} < 0. \tag{1.16}$$

Conditions (1.13), (1.14), (1.15) and (1.16) are BMIs due to the involvement of the terms $BKCX$ or $PBKC$. Since these are NP-hard in nature, these are difficult to solve.

1.1.2 Continuous-time pole-placement

We now consider the technique of pole-placement in damping regions [10] as our next performance criteria.

It is well known that, though the H_∞ control in the previous section yields good robust performance, it lacks ensuring transient performance. The transient performance of the closed-loop system can be improved through locating the closed-loop poles in specified regions for which transient performance guaranteeing minimum performance criteria, such as damping ratio, decay rate, are known. To proceed further, consider the closed-loop system of (1.7) as:

$$\dot{x}(t) = A_{cl}x(t) \tag{1.17}$$

with $A_{cl} = A + BKC$ as in (1.6). The objective is to design the SOF gain K so that the closed-loop poles corresponding to (1.17) are placed in specified damping region.

1.1.2.1 LMI region

The well-known damping region criterion can be expressed as LMI [3] and thereby as LMI region as follows.

Let $\mathcal{V}$ be a sub-region of the left-half of the complex s-plane. The system (1.17) is said to be $\mathcal{V}$-stable if all the eigenvalues of the matrix A_{cl} lie in a specified $\mathcal{V}$-region. Gutman

[3] For a brief introduction of LMIs, please see Appendix A

in [11] extended the Lyapunov stability condition to a variety of regions in polynomial form that are further extended to LMI regions in [12]. The following definition is recalled from [12] for LMI regions.

Definition 1.6 *Any subset $\mathcal{V}$ of the complex plane is called an LMI region if and only if there exists a symmetric matrix $N \in \mathbb{R}^{n\times n}$ and a matrix $M \in \mathbb{R}^{n\times n}$ satisfying*

$$\mathcal{V} = \{s \in \mathbb{C}\colon f_{\mathcal{D}}(s) < 0\}, \tag{1.18}$$

with

$$f_{\mathcal{V}}(s) = N + sM + \bar{s}M^T, \tag{1.19}$$

where $\bar{s}$ is the complex conjugate of any point s in $\mathcal{V}$. The function $f_{\mathcal{V}}(\cdot)$ is known as the characteristics function of $\mathcal{V}$ and takes value in the space of $n \times n$ Hermitian matrices. It is to be further noted that LMI regions are convex and symmetric w.r.t. the real axis since $f_{\mathcal{V}}(\bar{s})=\bar{f}_{\mathcal{V}}(s) < 0$ for any $s \in \mathcal{V}$. Some of the commonly used LMI regions [12] with their characteristic functions are given below.

- Left-half plane $Re(s) < -\alpha$: $f_{\mathcal{V}}(s) = s + \bar{s} + 2\alpha$

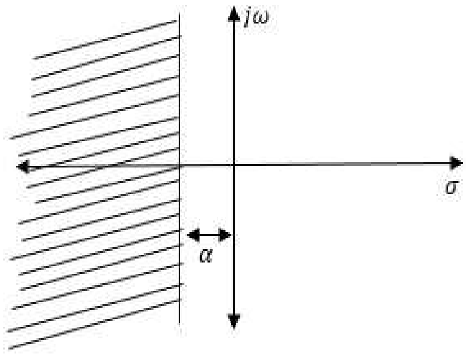


Figure 1.2: Half-plane region

- Conic sector with apex at the origin and inner angle 2θ:

$$f_{\mathcal{V}}(s) = \begin{bmatrix} \sin\theta(s+\bar{s}) & \cos\theta(s-\bar{s}) \\ \cos\theta(\bar{s}-s) & \sin\theta(s+\bar{s}) \end{bmatrix} < 0.$$

- Circular disk centered at $(-q, 0)$ and radius r:

$$f_{\mathcal{V}}(s) = \begin{bmatrix} -r & q+s \\ q+\bar{s} & -r \end{bmatrix} < 0.$$

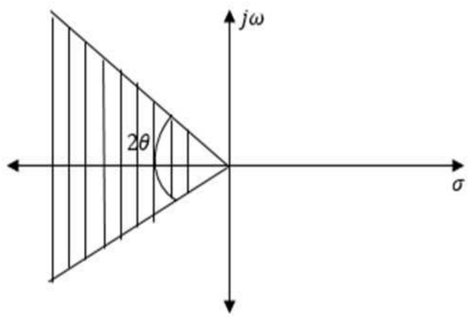


Figure 1.3: Conic sector region

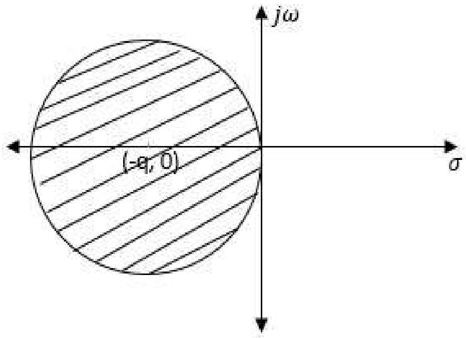


Figure 1.4: Circular disc region

For system (1.17), LMI conditions can be drawn corresponding to the above characteristic functions as given in the below result from [12].

Theorem 1.7 *The system (1.17) is $\mathcal{V}$-stable if and only if there exists a symmetric matrix $X > 0$ such that*

$$\mathcal{M}_{\mathcal{V}}(A, X) < 0, \tag{1.20}$$

with

$$\mathcal{M}_{\mathcal{V}}(A_{cl}, X) := N \otimes X + M \otimes (A_{cl}X) + M^T \otimes (A_{cl}X)^T \tag{1.21}$$

where $\otimes$ denotes the Kronecker product.

Comparing (1.19) and (1.21), an important mapping can be noted for deriving LMI condition corresponding to a given characteristic function (1.19) [12] as:

$$(1, s, \bar{s}) \rightarrow (X, A_{cl}X, XA_{cl}^T) \tag{1.22}$$

LMI conditions corresponding to the left-half plane and the conic sector (a region which is exploited for pole placement in this thesis work) is presented in the following. Similar methodology can be followed for representing other regions, such as circular disk etc.

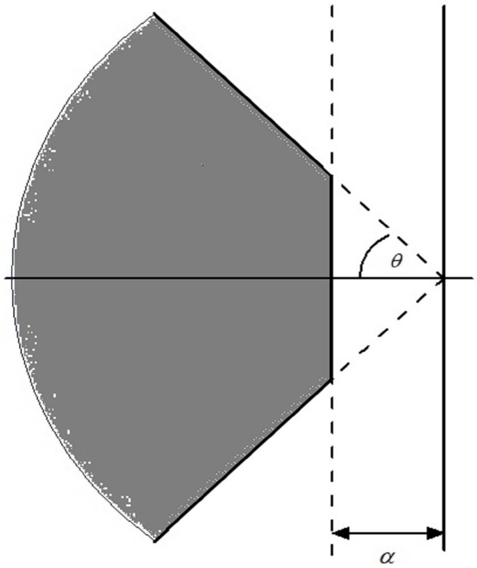


Figure 1.5: The pole location region $\mathcal{D}_C(\alpha, \theta)$

Consider an LMI region $\mathcal{D}_C(\alpha, \theta)$, which consists of a set of complex numbers $p + jq$ such that $p < -\alpha < 0$ and $p \tan\theta < -|q|$ as shown in Figure.1.5. The corresponding characteristic equation [10] is

$$f_{\mathcal{V}}(s) = \begin{bmatrix} s_\theta(s + \bar{s} + 2\alpha) & c_\theta(s - \bar{s}) \\ c_\theta(\bar{s} - s) & s_\theta(s + \bar{s} + 2\alpha) \end{bmatrix} < 0, \tag{1.23}$$

where $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. Then, the matrix inequality criterion for placing the closed-loop poles of (1.17) in $\mathcal{D}_C(\alpha, \theta)$ can be derived using the mapping in (1.22) as:

$$\begin{bmatrix} s_\theta(Sym\{A_{cl}X\} + 2\alpha X) & c_\theta(A_{cl}X - XA_{cl}^T) \\ c_\theta(XA_{cl}^T - A_{cl}X) & s_\theta(Sym\{A_{cl}X\} + 2\alpha X) \end{bmatrix} < 0. \tag{1.24}$$

Now that we have discussed the LMI regions, it is easy to see that in contrast to the pointwise pole-placement, regional pole placement technique assigns the closed-loop eigenvalues to a region of the complex plane. This can be even useful when the underlying system is uncertain. Taking this into consideration, in the upcoming section, discussion is made on the system uncertainties in brief and the necessary preliminaries used for handling those uncertainties while designing the SOF controller for the continuous-time systems.

1.1.3 Continuous-time system with parametric uncertainties

An important use of feedback control is to reduce the effects of plant uncertainties, noise, and disturbances while improving command tracking. Such a problem constitutes robust controller design. Mathematical models describing the system dynamics suffer from inaccuracies that result from the measurement error, modeling error, parameter variation or inability to capture exogenous phenomena. Therefore, there is always an mismatch between the mathematical model considered and the actual system. Hence, the main objective of robust control design is to take these uncertainties into account in a systematic fashion for controller design. The system uncertainty is categorized under two subheadings, namely, unstructured and structured uncertainty. While the former arises due to unmodeled dynamics in the system, the latter generally occurs due to parameter variations in the system. Unstructured uncertainty can be of feed-forward forms (additive, multiplicative input and multiplicative output) and feedback or inverse forms (inverse additive, inverse multiplicative input and inverse multiplicative output) [2]. For systems with structured uncertainties, we consider norm-bounded and polytopic representations. These have been discussed as follows.

1.1.3.1 Systems with norm-bounded uncertainty

Consider an uncertain system with norm-bounded uncertainties in the input matrix, represented as:

$$\dot{x}(t) = Ax(t) + B_w w(t) + (B + \Delta B(t))u(t) \tag{1.25}$$

with $x(t)$, $u(t)$, $w(t)$, A, B, B_w as defined in (1.6). The controlled output $z(t)$ and the measured output $y(t)$ equations are the same as in (1.6). $\Delta B(t)$ captures possibly time-varying parameter uncertainties in B and it can be decomposed under the norm-bounded assumption as follows:

$$\Delta B(t) = DF(t)E, \ \|F(t)\| \leq 1 \tag{1.26}$$

where D and E are constant matrices of appropriate dimensions and $F(t)$ takes care of the time variations in $\Delta B(t)$.

The following Lemma is borrowed from [3, page no. 110] that ensures H_∞ performance of (1.25) under uncertainty (1.26).

Lemma 1.8 ([3]) *The following statements are equivalent for* $\gamma > 0$.

1. $\|T_{zw}(s)\| < \gamma$ *and* A_{cl} *is Hurwitz.*

2. *For known real matrices* E *and* D *with appropriate dimensions and scalar* $\epsilon > 0$, *there exists* $X = X^T > 0$ *satisfying*

$$\begin{bmatrix} Sym\{A_{cl}X\} + \epsilon DD^T & * & * & * \\ B_{cl}^T & -\gamma^2 I & * & * \\ C_{cl}X & D_{cl} & -I & * \\ EKCX & EKD_{yw} & 0 & -\epsilon I \end{bmatrix} < 0. \tag{1.27}$$

The condition (1.27) is a BMI problem due to the involvement of the term $EKCX$ and its transpose . This problem is again considered in chapter 2 of this thesis.

1.1.3.2 Systems with polytopic uncertainty

Consider a linear CT system with polytopic uncertainties described as:

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B(\theta)u(t) + B_w(\theta)w(t) \\ z(t) &= C_z(\theta)x(t) + D_{zu}(\theta)u(t) + D_{zw}(\theta)w(t) \\ y(t) &= C(\theta)x(t) + D_{yw}(\theta)w(t) \end{aligned} \tag{1.28}$$

where $x(t)$, $u(t)$, $w(t)$, $z(t)$ and $y(t)$ as defined in (1.6). The parameter $\theta{=}[\theta_1, \theta_2, \cdots, \theta_r]^T \in \mathbb{R}^r$ is uncertain. The parameter dependent matrices $A(\theta)$, $B(\theta)$, $B_w(\theta)$, $C_z(\theta)$, $D_{zu}(\theta)$, $D_{zw}(\theta)$, $D_{yw}(\theta)$ and $C(\theta)$ are assumed to be continuous in their arguments and bounded, having an affine dependence on θ. Further, the parameter θ ranges in a polytope Λ as:

$$\theta \in \Lambda := Co\{\Omega_1, \Omega_2, \cdots, \Omega_N\} = \left\{ \sum_{i=1}^{N=2^r} \zeta_i \Omega_i : \zeta_i \geq 0, \ \sum_{i=1}^{N} \zeta_i = 1 \right\}$$

where N represents the cardinality of the vertices of the polytope. Again, with the controller as in (1.2), the closed-loop dynamics for (1.28) are given as:

$$\begin{aligned} \dot{x}(t) &= A_o(\theta)x(t) + B_o(\theta)w(t) \\ z(t) &= C_o(\theta)x(t) + D_o(\theta)w(t), \end{aligned} \tag{1.29}$$

with $A_o(\theta) = A(\theta){+}B(\theta)KC$, $B_o(\theta) = B_w(\theta){+}B(\theta)KD_{yw}(\theta)$, $C_o(\theta) = C_z(\theta){+}D_{zu}(\theta)KC$, $D_o(\theta) = D_{zw}(\theta) + D_{zu}(\theta)KD_{yw}(\theta)$.

The objective is to design the SOF controller (1.2) such that the system (1.29) attains $\mathcal{L}_2$ performance as given in Definition 1.3. Note that, since the system (1.28) is not an

LTI but a polytopic one, we use time domain-based $\mathcal{L}_2$ gain criteria rather than the H_∞ performance measure.

Now, we recall the following Lemma for evaluating the $\mathcal{L}_2$ performance of (1.29).

Lemma 1.9 *[3, 13] The closed-loop system* (1.29) *satisfies the $\mathcal{L}_2$ performance* (1.12) *if there exists $X(\theta) = X^T(\theta) > 0$ and K satisfying*

$$\begin{bmatrix} Sym\{A_o(\theta)X(\theta)\} & * & * \\ B_o(\theta)^T & -\gamma^2 I & * \\ C_o(\theta)X(\theta) & D_o(\theta) & -I \end{bmatrix} < 0. \quad (1.30)$$

The condition (1.30) is also an BMI due to the involvement of the terms $B(\theta)KCX(\theta)$). This problem is considered in chapter 2 of this thesis.

Next, the problem of designing controller for system with control input saturation is discussed.

1.1.4 Continuous-time system with actuator saturation

This section highlights the effects of the actuator saturation on the system and motivation behind considering this as an addition constraint while designing the SOF controller.

Designing SOF controller under actuator saturation is a challenging task for control engineers since actuator saturation may lead to degradation of system performance, occurrence of limit cycles, multiple equilibria and even causes unstable closed-loop system operation [14, 15]. Actuator saturation in the form of control limitation is present in almost all control systems. It limits the magnitude of the control signal due to the physical limitation of actuators. Such limitation can be found out in every devices used in process control industries, such as heating actuators, proportional valves, electromechanical actuators, power amplifiers etc, even in limit circuit such as voltage limits in electrical actuators, the limits on flow volume or rate in hydraulic actuators, deflection limits in aircraft actuators and so many. It restricts the achievable performance of the closed-loop system. Such a constraint can be modeled as a saturation non-linearity in the control loop, which introduces a nonlinear behavior in the closed-loop system, even if the open-loop system is linear. Therefore, it is necessary for the designers to take into account the effects of actuator saturation non-linearities.

Although the saturation function looks like a simple nonlinearity, but its mathematical treatment to obtain stability and stabilization conditions is rather complicated. Therefore, it is considered as hard nonlinearity [4] as is the case of the dead–zone, the hysteresis, the backlash, etc [14]. Hence, to derive tractable conditions and provide efficient solutions to this problem, appropriate overboundings and representations of the saturation function is required. For these purposes, four useful representations are available in literature for the closed-loop system in the presence of saturation. They are listed below.

- The first representation is based on the use of polytopic differential inclusions. It involves a local description of the saturated closed-loop system through a polytopic model. This allows stability and stabilization problems to be treated using robust control approaches. Three types of polytopic modeling are given. a) Polytopic model I [16], which is based on the use of differential inclusions [17], b) Polytopic Model II [18], which is based on using an auxiliary vector variable h, and to compose the output of the saturation function as a convex combination of the actual control signals u and h, c) Polytopic Model III [19], which is a generalization of the polytopic approach II.

- The second one involves re-writing the closed-loop system and replacing the saturation term by a dead-zone nonlinearity. Hence, sector conditions, locally or globally valid, can be used to relax stability and stabilization conditions. In particular, two types of sector conditions are presented. They are a) Classical Sector Condition, b) Generalized Sector Condition [20, 21]. Generalized sector condition yields less conservative results than the use of classical sector conditions.

- The third representation [22, 23] involves dividing the state space into regions of saturation. Inside each one of these regions, the system dynamics is described by an affine linear system. This allows the problem to be treated using tools from hybrid systems.

- The fourth one [24, 25] consists of modeling the saturation as norm-bounded uncertainty. In this case, the closed-loop system with saturation can be written in an linear fractional transformation (LFT) form.

[4] Nonlinear faults in control valves, such as stiction, deadband, backlash and hysteresis, are called as hard nonlinearities because of their dominant nonlinear characteristics.

Throughout this thesis work, we have considered the second representation, i.e, polytopic model II to approximate the nonlinearity caused due to saturation function. The saturation function is approximated by the convex combination of the actual control signals u and h. The representation is shown below.

Consider the CT polytopic system (1.28) subject to actuator saturation as:

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B(\theta)\text{sat}\{u(t)\} + B_w(\theta)w(t) \\ z(t) &= C_z(\theta)x(t) + D_{zu}(\theta)\text{sat}\{u(t)\} + D_{zw}(\theta)w(t) \\ y(t) &= Cx(t) + D_{yw}(\theta)w(t) \end{aligned} \tag{1.31}$$

where the vectors and matrices are as defined above and the saturation function $sat\{\cdot\} : \mathbb{R}^m \to \mathbb{R}^m$ is defined as $sat\{u(t)\} := [sat\{u_1(t)\}, \cdots, sat\{u_m(t)\}]^T$, where $sat\{u_i(t)\} = sign(u_i(t)) \min\{\bar{u}_i, |u_i(t)|\}$, $\bar{u}_i > 0$ is the saturation amplitude of the i^{th} input. is defined as

$$sat\{u_j(t)\} = \begin{cases} \bar{u}_j & u_j(t) \geq \bar{u}_j \\ u_j(t), & u_j(t) \in [-\bar{u}_j, \bar{u}_j], \; j = 1, \cdots, m \\ -\bar{u}_j, & u_j(t) \leq -\bar{u}_j \end{cases} \tag{1.32}$$

where $\underline{u}_j$ and $\bar{u}_j$ are the saturation amplitudes of the j^{th} input. Note that the notation $sat\{\cdot\}$ is used for both the scalar and vector valued functions.

The following Definition and Lemma are useful for taking care of the saturated control input that introduces nonlinearity in the system dynamics and requires convex approximation for quadratic analysis.

Definition 1.10 *[14] For a matrix $H \in \mathbb{R}^{m\times p}$, $Hy(t)$ is said to be unsaturated if the following condition is satisfied*

$$|h_j y(t)| \leq \bar{u}_j, \; \forall j = 1, \cdots, m, \tag{1.33}$$

where h_j represents the j^{th} row vector of H.

A set defined as $\sigma(H, u_j) = \{y(t) \in \mathbb{R}^p : |h_j y(t)| \leq \bar{u}_j, \forall j = 1, \cdots, m\}$ represents the region in which the control input is unsaturated.

Define $\mathcal{D}_m$ as the set of $\mathbb{R}^{m\times m}$ diagonal matrices whose diagonal entries are allowed to take binary values, i.e., either 1 or 0. It is thus easy to see that the cardinality of $\mathcal{D}_m$ is 2^m. Label each element of $\mathcal{D}_m$ as $\prod_i$, $i = 1, \cdots, 2^m$ and let $\bar{\prod}_i = I - \prod_i$. We can see that $\bar{\prod}_i$ is also an element of $\mathcal{D}_m$. It can be shown following [18] that the saturated control input can be written as a convex function as in the Lemma below.

Lemma 1.11 *[18] For the feedback gain matrix K and auxiliary variable matrix H, if $y(t) \in \sigma(H, u_j)$, then*

$$sat\{Ky(t)\} \in Co\left\{\left(\prod_i Ky(t) + \bar{\prod}_i Hy(t)\right) : i = 1, \cdots, 2^m\right\}. \tag{1.34}$$

Using Lemma 1.11, sat$\{u(t)\}$ can be rewritten as

$$\text{sat}\{Ky(t)\} = \sum_{i=1}^{2^m} \eta_i \left(\prod_i K + \bar{\prod}_i H\right) y(t),\ 0 \leq \eta_i \leq 1,\ \sum_{i=1}^{2^m} \eta_i = 1. \tag{1.35}$$

The closed-loop system for (1.31) with SOF controller (1.2) is given as:

$$\begin{aligned} \dot{x}(t) &= A_o(\theta)x(t) + B_o(\theta)w(t) \\ z(t) &= C_o(\theta)x(t) + D_o(\theta)w(t), \end{aligned} \tag{1.36}$$

with

$$\begin{aligned} A_o(\theta) &= A(\theta) + B(\theta)\left\{\sum_{i=1}^{2^m} \eta_i \left(\prod_i K + \bar{\prod}_i H\right)\right\} C, \\ B_o(\theta) &= B_w(\theta) + B(\theta)\left\{\sum_{i=1}^{2^m} \eta_i \left(\prod_i K + \bar{\prod}_i H\right)\right\} D_{yw}(\theta), \\ C_o(\theta) &= C_z(\theta) + D_{zu}(\theta)\left\{\sum_{i=1}^{2^m} \eta_i \left(\prod_i K + \bar{\prod}_i H\right)\right\} C, \\ D_o(\theta) &= D_{zw}(\theta) + D_{zu}(\theta)\left\{\sum_{i=1}^{2^m} \eta_i \left(\prod_i K + \bar{\prod}_i H\right)\right\} D_{yw}(\theta). \end{aligned}$$

For designing the SOF controller with $\mathcal{L}_2$ performance for the system (1.31), Lemma 1.9 can be used. While evaluating the $\mathcal{L}_2$ performance of the system (1.31), it can be seen that the developed conditions are BMIs due to the nonlinear term arising through the multiplication of Lyapunov matrix and the controller gain matrix. Limited results are available in the literature to overcome this issue. These has been discussed in the literature review in brief. The same problem has also been addressed in this work later in chapter 2.

Next, we will consider the linear discrete-time (DT) systems, and discuss the SOF problem along with the issues arising while designing SOF controller for the same.

1.1.5 Discrete-time system

This section presents the SOF design problem for DT systems. Performance criteria, such as, H_2, H_∞ and pole-placement, are presented. Existing methods available in the literature for solving SOF problem are discussed.

1.1.5.1 Challenges and Issues

Consider the LTI systems described as:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1.37}$$

where k denotes the sampling instant, $x(k) \in \mathbb{R}^n$ is the state vector. $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ represent the input and output vector, respectively. A, B and C are the system matrices of appropriate dimensions. For the sake of simplicity, the stabilization problem of the nominal system is considered first. It is assumed that the system (1.37) is reachable and observable. Our control objective is to design the SOF control law given as:

$$u(k) = Kx(k) \tag{1.38}$$

where K is the SOF controller gain so that the closed-loop system

$$x(k+1) = (A + BKC)xk \tag{1.39}$$

is asymptotically stable. K is the SOF controller gain to be designed. Now, for stability analysis of the system (1.37), we use the theory of Lyapunov stability by considering the quadratic Lyapunov function, i.e., $V(k) = x^T(k)Px(k)$. This leads to finding $P = P^T > 0$ such that

$$(A + BKC)^T P(A + BKC) - P < 0, \tag{1.40}$$

Applying Schur complement [3], one can rewrite (1.40) as :

$$\begin{bmatrix} -P & * \\ A + BKC & -P^{-1} \end{bmatrix} < 0 \tag{1.41}$$

Now, pre- and post multiplying (1.41) by diag$\{I, G\}$ and its transpose on both the sides and using bounding condition, $-GP^{-1}G^T \leq -G - G^T + P$, then one can obtain

$$\begin{bmatrix} -P & * \\ GA + GBKC & -G - G^T + P \end{bmatrix} < 0 \tag{1.42}$$

The inequality (1.42) is not linear and is also a BMI problem due to the term $GBKC$.

Next, we will consider the DT generalized control systems with disturbances acting on the system. Consider a DT LTI system represented as:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_w w(k), \\ z(k) &= C_z x(k) + D_{zu} u(k) + D_{zw} w(k), \\ y(k) &= Cx(k) + D_{yw} w(k) \end{aligned} \tag{1.43}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the input and $z(k) \in \mathbb{R}^q$ is the controlled output of the system. $w(k) \in \mathbb{R}^r$ and $y(k) \in \mathbb{R}^p$ are the disturbance input and the measured output, respectively. $A, B, B_w, C_z, D_{zu}, D_{zw}, C, D_{yw}$ are matrices of appropriate dimensions. For system (1.43), consider a SOF controller as (1.38). The closed-loop system is given by

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \left(\bar{A} + \bar{B}K\bar{C}\right) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{1.44}$$

where

$$\bar{A} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ D_{zu} \end{bmatrix}, \ \bar{C} = \begin{bmatrix} C & D_{yw} \end{bmatrix}$$

The closed-loop pulse transfer function matrix from $w(k)$ to $z(k)$ is

$$T_{wz}(\zeta) = C_{cl}(\zeta I - A_{cl})^{-1} B_{cl} + D_{cl} \tag{1.45}$$

where ζ^{-1} is the DT unit delay operator and $A_{cl} = A + BKC$, $B_{cl} = B_w + BKD_{yw}$, $C_{cl} = C_z + D_{zu}KC$ and $D_{cl} = D_{zw} + D_{zu}KD_{yw}$. Given the above system description, the objective of this work is to design the SOF controller K while the design performances are presented in the next section.

1.1.5.2 Performance criteria

The below lemma can be constructed from the well known results on H_2 and H_∞ performance of (1.44). The results are in the similar line as obtained in [26] with appropriate considerations to suit the present developments.

Lemma 1.12 *System* (1.44) *satisfies the following performance criteria.*

1. *Assume* $D_{cl} = 0$ $(D_{zw} = 0, D_{yw} = 0)$. *The* H_2 *performance defined as* $\|T_{wz}(\zeta)\|_2^2 < \mu$ *is guaranteed if there exist matrices* $P = P^T > 0$, $\mathcal{W} = \mathcal{W}^T > 0$ *and* G *satisfying*

$$trace(\mathcal{W}) < \mu, \quad \begin{bmatrix} -\mathcal{W} & * \\ GB_{cl} & -Sym\{G\} + P \end{bmatrix} < 0, \tag{1.46}$$

$$\begin{bmatrix} -P & * & * \\ GA_{cl} & -Sym\{G\} + P & * \\ C_{cl} & 0 & -I \end{bmatrix} < 0. \tag{1.47}$$

2. *The* H_∞ *performance defined as* $\|T_{wz}(\zeta)\|_\infty < \gamma$ *is guaranteed and* A_{cl} *is Hurwitz, if there exist matrices* $\bar{P} = \bar{P}^T > 0$ *and* $\bar{G}$ *satisfying*

$$\begin{bmatrix} -\bar{P} & * & * & * \\ 0 & -\gamma I & * & * \\ \bar{G}A_{cl} & \bar{G}B_{cl} & -Sym\{\bar{G}\} + \bar{P} & * \\ C_{cl} & D_{cl} & 0 & -\gamma I \end{bmatrix} < 0. \tag{1.48}$$

Note that this result is not directly available in literature, particularly involving the auxiliary matrix variable G and $\bar{G}$ although its equivalent variations are available. Hence, a proof is presented below.

Proof : (*1*) The H_2 performance can be guaranteed [4] if there exist P and $\mathcal{W}$ satisfying the following inequalities:

$$trace(\mathcal{W}) < \mu, \; B_{cl}^T P B_{cl} - \mathcal{W} < 0, \tag{1.49}$$

$$A_{cl}^T P A_{cl} - P + C_{cl}^T C_{cl} < 0 \tag{1.50}$$

Taking Schur complement on the second inequality of (1.49), one obtains

$$\begin{bmatrix} -\mathcal{W} & * \\ B_{cl} & -P^{-1} \end{bmatrix} < 0 \tag{1.51}$$

Then, pre- and post-multiplying by $diag\{I, G\}$ and $diag\{I, G^T\}$, respectively, and thereby replacing $-GP^{-1}G^T$ with the bounding inequality, $-GP^{-1}G^T \leq -\text{Sym}\{G\} + P$ [26], one obtains (1.46). Using similar steps, it can be shown that (1.47) is sufficient for (1.50).

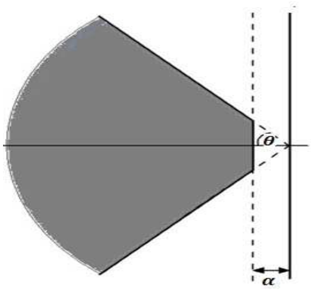
(*2*) The H_∞ performance is guaranteed [27] if there exists $\bar{P}$ satisfying

$$\begin{bmatrix} -\bar{P} & * & * & * \\ 0 & -\gamma I & * & * \\ A_{cl} & B_{cl} & -\bar{P}^{-1} & * \\ C_{cl} & D_{cl} & 0 & -\gamma I \end{bmatrix} < 0 \tag{1.52}$$

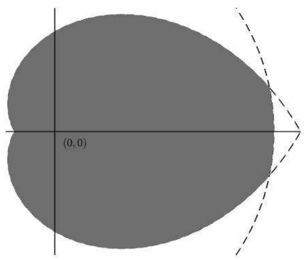
Pre- and post-multiplying the above inequality by $diag\{I, I, \bar{G}, I\}$ and $diag\{I, I, \bar{G}^T, I\}$, respectively, and thereby replacing $-\bar{G}\bar{P}^{-1}\bar{G}^T$ with the bounding inequality, $-\bar{G}\bar{P}^{-1}\bar{G}^T \leq -\text{Sym}\{\bar{G}\} + \bar{P}$, one obtains (1.48). In the above, μ and γ represent the H_2 and H_∞ performances, respectively. Conditions (1.46), (1.47) and (1.48) are BMIs and therefore NP-hard [1] due to the involvement of the terms $GBKC$ and $\bar{G}BKC$. This problem is considered in Chapter 3 of this thesis work.

1.1.6 Discrete-time pole-placement

This section presents the problem of pole-placement for DT systems. Before we move to DT doamin, consider its CT counterpart $\mathcal{D}_C(\alpha, \theta)$ that represents the set of complex numbers $s = \sigma + j\omega$ such that $\sigma \leq -\alpha < 0$ and $\sigma \tan\theta \leq -|\omega|$ as shown in Fig. 1.6 (a). Such a region ensures minimum decay rate of α and a minimum damping of $\zeta = \cos\theta$ for CT systems. It is an LMI region for CT systems and the result of [28] have been extensively used in different applications [29, 30]. The constant damping ratio line is



(a) $\mathcal{D}_C(\alpha, \theta)$ in CT



(b) $\mathcal{D}_D(\alpha, \theta)$ in DT

Figure 1.6: The constant damping regions in continuous and discrete-domain

represented by $\sigma = -cotan(cos^{-1}(\zeta))|\omega|$ with $-cotan(cos^{-1}(\zeta))$ defining the slope of the line. For DT systems, the constant damping ratio loci in the z-plane can be written as :

$$z = e^{-cotan(cos^{-1}(\zeta))|\omega|T} e^{j\omega T}, \tag{1.53}$$

where T is the sampling time. The DT region corresponding to $\mathcal{D}_C(\alpha, \theta)$ is denoted as $\mathcal{D}_D(\alpha, \theta)$ a representation of which is shown in Fig. 1.6 (b). The constant ζ locus is a logarithmic spiral, and, hence, a non-convex region in the z-domain, which cannot be explicitly represented as an LMI region. In order to take care of such non-convexity, construction of approximate LMI regions are discussed in chapter 3 of this thesis work.

Different sub-regions are constructed in literature for approximating the $\mathcal{D}_D(\alpha, \theta)$-region. In [31], circular regions are adopted. An inner-ellipse approximation [32] and a conic sector approximation [33, 34] have also been attempted. Note that, all these works used contracted area of the actual $\mathcal{D}_D$-region. This, combined with fact that LMI synthesis mostly deal with sufficient criteria, available designs are much restrictive. Alternate approach would be to consider non-contracting regions. This forms the conceiving idea behind the pole-placement work presented in Chapter 3 of this thesis.

1.2 Literature review on static output feedback (SOF) controller design

1.2.1 Continuous-time system

Designing SOF controller still remains one of the challenging problems in the area of control engineering. It has uses not only in finding centralized SOF controllers, but also for several other design problems, such as, design of PI/PID controllers [35], restricted feedback controllers [36]. A study on the same has been carried out in the work by [1] demonstrating its non-convexity and NP-hardness. Several methods have been developed in literature to derive computationally effective SOF design criteria (see survey papers by [37–39] and references therein). Initial results are based on Riccati equation [40], convex approaches using optimization techniques (min-max algorithm) [41] and iterative linear matrix inequalities ([42]). Also, computationally efficient but sufficient criteria based on LMIs have been reported in [43]. Apart from the iterative approaches, sufficient conditions for designing SOF controllers in terms of LMIs have been developed in [44, 45]. Although only sufficient, it has advantages of being convex and, thereby, easily tractable by standard optimization techniques. In contrast to the iterative algorithms or Bilinear Matrix Inequality based algorithms [46], such convex approaches are either single-step or involves choosing few scalar parameters to obtain the LMIs [47].

Indeed, there exist special classes of systems for which the LMI criteria for designing SOF is guaranteed to exist for the stabilization case. One class of such systems is that of square (equal number of inputs and outputs) and minimum phase systems considered in [39], where the necessary and sufficient conditions for existence of output feedback controller is derived in terms of solvability of a set of matrix inequalities. Another class is that of non–zero high frequency gain systems [48], where under the assumptions of minimum phase and non-singular high-frequency gain for the nominal plant, stabilizing SOF control gain design is achievable. Although the SOF solution does exist for these special classes of systems, however, in general, necessary and sufficient condition for SOF design is not computationally tractable. This work considers such general class of systems for SOF design.

Restricted feedback control (perfect decentralized and overlapping ones), both the state and the output feedback, can be formulated only as SOF control with restricted feedback gain matrix. The class of SOF for which the controller gain structure is predefined is known as restricted SOF (RSOF). RSOF controller design using convex programming has been reported in [49–51]. A survey of RSOF designs is presented in [38]. Traditional controller synthesis methods, based on solving AREs does not work when additional constraints are imposed on the structure of the controller matrix making them NP hard problem [1]. RSOF design using a gradient based iterative method and a penalty function that uses gain matrix has been studied in [52]. LMI based conditions with rank constraints has been proposed in [53] to design RSOF using an alternating projection method. Design of optimal state feedback gain for a pre-specified structure known as sparsity constrained controller is synthesized using augmented Lagrangian method in [54]. In [55], a rank constrained optimization method is exploited to solve the sparse output feedback problem. Stabilizing RSOF controller with BMI constraints has been solved using iterative generalized bender decomposition and gradient projection method in [56].

In recent years, there are important development in solving RSOF control problem using non-iterative LMI algorithms [36, 47]. In [57] a comparatively simpler expression for controller gain matrix is derived imposing structural constraint on the LMI variables and similar approach has been applied to building structure and wind turbine control problems [58, 59].

Challenges in designing SOF controllers are consistent from mere stabilization to guaranteed performance based designs. Many robust control problems, e.g., disturbance rejection, robust stabilization, can be formulated as H_∞ control problem [4, 60]. H_∞ performance is well known for formulating robust control specifications [61]. Though there exist celebrated LMI-based results for H_∞ control using dynamic controller [62–64], only a handful of results available for the SOF design [36, 65]. Sufficient conditions for SOF controller design have also been developed by enforcing constraints on the Lyapunov matrix [43] or assigning special structure to Lyapunov matrix [66]. A parameter-dependent stabilization method has been used in the work by [67] for designing SOF based H_∞ controllers. Another sufficient condition for SOF design have been proposed in [68] by using parameter-independent slack variables and exploring null space of the output matrix. Robust H_∞ SOF controllers for continuous and discrete-time system have been designed in [69, 70] in terms of LMI but using two iterative stages. Simultaneous H_∞ control and pole placement design results have been obtained in [64, 71].

Industrial processes commonly use PID controllers due to the inherent ease in implementation and tuning. Potential areas of application of such controllers include chemical process industries, food processing industries, aerospace industry, robotic industry and numerous other engineering domains. Due to their popularity, a host of tuning methods have been proposed in literature, such as [72], [73], which considered modeling and autotuning techniques to determine the gains of the PID controllers associated with the proportional, integral and derivative inputs. Classical design methods for tuning the P, I and D gains, e.g. root locus, bode plots, Ziegler-Nichols method, do not account for appropriate consideration of uncertain models. Though there exist a spectrum of design methods that give different types of performances [72], unfortunately, a single tuning method usually does not satisfy a variety of practical issues such as load disturbances, sensitivity of the system to measurement noise and model uncertainties. Also, with the growing need for improved process control, we need to design controllers in a robust way to extract satisfactory performance even in uncertain environments. A good robust controller should ensure stability of the overall closed-loop system and performance over the entire uncertainty domain [72, 74].

Therefore, to overcome the above-mentioned shortcomings, LMI based controller design [3] is one of the widely used approach for designing robust controllers since it provides solution to a large set of convex problems effectively. Additionally, it provides simplicity and flexibility in tuning the controller gain parameters rather than other techniques such as Ziegler-Nichols, Model Reference Adaptive Control (MRAC), Particle Swarm Optimization (PSO), Adaptive Particle Swarm Optimization (APSO) and the like. Motivated by the above problems and scope of solution in the design of controllers in the framework of LMIs, the problem of PID controller design is transformed into design of robust SOF controller using the transformations given in [35]. Various iterative algorithms have been developed by [9, 65, 75, 76] in the LMI framework, which are widely used in the design of PID controllers. But such iterative algorithms are complex and require large computation time.

The SOF problem has been extended to robust control design incorporating uncertainties by [68, 77, 78]. Systems with parametric uncertainties pertaining to a convex polyhedron [79] are represented as polytopic systems. Such systems form a widely studied class of systems incorporating uncertainties and subsequently allowing robust control design. Robust SOF control design problem for such systems, being non-convex problem, requires numerical algorithms (possibly LMI based) for obtaining a solution. Several design methods have been proposed in literature for SOF design for polytopic systems. One such method involves a two-step algorithm with the state feedback gain as the initial value [80, 81]. Another class of solutions includes equality based constraints along with non-strict LMIs, often requiring one vertex of the output matrices to be full row rank, which may not always be the case for uncertain systems [43]. Another set of methods involves introducing a slack variable with restricted structure, such as diagonal or subtriangular, for SOF controller design [26, 82]. A combination of the above techniques have been employed to compute the SOF controller for matched output matrix condition, where the output matrix C is assumed to be constant [26, 83] with its prospect of numerous applications in autonomous, airborne and underwater vehicles and power electronics (see [84–88]). Next, under the category of SOF design solutions that allow the output matrix to be non-row full rank, certain constraints need to be imposed on the system matrices [89], which make them inapplicable to a general class of uncertain systems. Multi-objective output feedback controller design via linearizing change of variables has been studied in the works by [64,90], but it does not lead to a solution of output controller for polytopic uncertain systems. This is because to linearize the matrix inequality, the introduced new variables will have to be vertex-dependent and involve the controller parameters to be sought, which implies that the controller parameters cannot be computed from the introduced variables [91]. In addition, while in literature the term H_∞ has been considered for specified performance for polytopic systems, we here use the time-domain $\mathcal{L}_2$ performance definition for controller design.

Apart from the above, Dong et al [79] proposed new sufficient LMI conditions for SOF H_∞ controller design, where a line search was employed for tuning a scalar variable and the uncertain output matrix of the underlying system was permitted to be rank deficient. In contrast to previous methods, sufficient conditions were proposed in [13] for designing H_∞ SOF controllers for systems with polytopic uncertainties without imposing any restrictions on the system matrices using parameter-dependent Lyapunov functions (PDLF).

To this end, designing controllers with actuator saturation is an attractive problem due to its practical importance. Unaccounted actuator saturation in a closed-loop system may lead to degradation of system performance, occurrence of limit cycles, multiple equilibria and even instability [14, 15]. Results for SOF synthesis with actuator saturation have been obtained in [18, 92]. To deal with this problem, low control gains are commonly designed that remain within the limits of saturation and allow the system to remain in a subspace known as Region of Attraction (ROA). Itagaki [92] used a hyperbolic tangential function to model the saturation behaviour. In [18], the saturation behaviour was expressed as a convex function. In [93], H_∞ SOF controller design for systems with actuator saturation is considered but not for polytopic systems. However, there are limited results considering SOF controller design for polytopic systems with actuator saturation [94]. In [94], only stabilization of multiple models under actuator saturation analysis via a SOF control law has been considered without any performance criteria.

Pole-placement in sub-regions of the complex plane has been conventionally used as one of the primary design criteria to improve the transient behavior of the closed-loop system. Necessary and sufficient algebraic conditions for regional pole locations in sub-regions of complex plane were derived in [11]. Some of these criteria extended to state-feedback controller design for CT uncertain systems using Riccati equations [95–98]. Later on, LMI criteria have been developed due to its superiority in genericness and facilitating multi-objective based design [28] leading to defining LMI regions, for example, conic sector, circular disk, vertical-horizontal strip and a half-plane. The celebrated result of [10,28,99–101] on output feedback controller synthesis for pole placement in LMI region is still used extensively in different applications for CT systems [29, 30, 102–104].

1.2.2 Discrete-time system

Similar to CT systems, SOF controller design for DT systems is also one of the challenging problems in control theory and practice. The SOF control design problem, i.e., designing K is well known as an NP-hard problem [1] that has attracted considerable attention in the past decades and has been studied exhaustively in literature [37, 38, 105–107]. Many earlier attempts that have been made to solve the SOF design problem can be grouped into two categories. The first includes methods invoking the Lyapunov theory such as those based on Riccati equations [40, 108], convex approaches [41] using optimization technique (min/max algorithm), BMI using iterative LMI [9, 109, 110] and so on. The second category includes non-Lyapunov-theory based methods (see [111–113]) using optimization tools such as hinfstruct [114], HIFOO [115, 116] and HIFOO-D [111]. Although the latter approaches are more efficient than the former ones, the drawbacks of non-Lyapunov based methods are that they cannot handle robustness issues in a guaranteed manner as the Lyapunov theory based approaches [38].

Extensive literature (see for example, [45, 117–123]) exist where sufficient LMI conditions have been derived to obtain efficient solutions for the SOF design. Further, such methods have been extended to solve SOF control problems with uncertainties [82]. Various robust SOF design problems with uncertainties have been studied in [124], [125], [126].

In [43], LMI conditions have been derived for designing H_∞ and H_2 output feedback controllers by restricting the Lyapunov matrix within certain structural limits. Later, an auxiliary variable has been introduced to free the Lyapunov matrix variable in [26]. In [122], sufficient LMI condition has been obtained by introducing auxiliary variable to allow decoupling between the Lyapunov matrix variable and the controller gain matrix. It includes two step design procedure – first step is devoted to state feedback controller design and the second one is solving an LMI problem. Further, a matrix transformation has been introduced in [68] with a variable that results in less conservative sufficient condition.

The SOF design methods have also been extended to SOF based robust control problems, e.g. H_∞ [61] and H_2 [127] control. Note that, unlike the stabilization problem, it is easier to directly compare the effectiveness of the developed design methods for such performance based designs.

Earlier results on sufficient LMI conditions for H_∞ SOF controller design have been derived in [62], [43], [128] and [82]. In [26], an LMI based design criterion has been proposed, where the variables associated with the controller parameters were made independent of the Lyapunov matrix. By introducing auxiliary matrix variable with structural restriction, another LMI based design criterion has been proposed in [27]. However, it requires the auxiliary variable to be diagonal in structure. Such a restriction on the auxiliary variable has been avoided in the design criteria presented in [89]. In [89], conditions for designing of SOF, dynamic output feedback (DOF) and observer-based output feedback control, have been presented in terms of an unified LMI representations that seems to not involve any restriction on the variables. However, nonlinear coupling terms evolved due to the multiplication of system matrix and auxiliary matrix variable are approximated by bounding inequalities that lead to conservativeness in the development. A similar approach has been adopted in [129] for designing H_∞ SOF controller where sufficient LMI conditions with two scalar parameters have been presented using a matrix decoupling technique. Similarly, some of the above methods have also been extended to H_2 SOF controller design problem [26, 68, 130, 131].

It is well known that the pole-placement is another primary criteria to improve the performance of the system. Compared to CT systems, similar developments on pole placement inside LMI regions have been carried out for DT systems. A state feedback controller has been designed for DT systems to place the closed-loop poles inside a circular disk region in [132]. There are other works which considered LMI based design, such as [133, 134]. Although extensive literature exists on the theme of regional pole placement in damping regions for CT systems, few results have been developed to handle the non-convexity posed by the logarithmic spiral of the constant damping ratio loci for DT systems. In [31], the non-convex region has been approximated with a circular sub-regions by fitting a circle of maximal radius within the spiral. LMI constraints have been developed in [33] and [34] with inner conic sector approximation using non-iterative and iterative algorithms. Another set of LMI criteria were derived in [32] with an inner full ellipse sub-regions, which are shown to better approximate the non-convexity due to logarithmic spiral near the region closer to $(1,0)$ point in the z-plane when compared to circular sub-regions. A brief survey of the contracted inner convex approximations of the non-convex region has been presented in [135]. All these results are quite conservative mainly due to inappropriate convex approximation of the damping region. Moreover, due to this gap, although DT control design often carried out for digital implementation of the controller [136–139], the damping region performance based design is mostly not adopted in applications. Only few literature recently attempted to address this gap, e.g. [33, 140, 141].

1.3 Projection Lemma and some equivalences of matrix inequalities

In this section, we state the following Lemmas that will be useful in deriving the main results in this thesis.

Lemma 1.13 (Projection Lemma [142]) *Given matrices* Z, U *and* V *with appropriate dimensions, there exist a matrix* W *such that the inequality* $Z+U^TWV+V^TW^TU<0$ *holds if and only if the inequalities* $N_U^TZN_U<0$ *and* $N_V^TZN_V<0$ *are satisfied, where* N_U *and* N_V *represent the right orthogonal complement of* U *and* V*, respectively, i.e.,* $UN_U=0$ *and* $VN_V=0$.

Projection Lemma is useful for obtaining an LMI approximation of BMIs. By appropriate choices of N_U and N_V equivalencies among different inequalities can be drawn. Equivalency of several such inequalities has been shown through using projection lemma in [142].

Lemma 1.14 ([4]) *For arbitrary matrices* U, V *and* $W=W^T>0$*, the following holds:*

$$UV+V^TU^T\leq UWU^T+V^TW^{-1}V. \tag{1.54}$$

The following Lemma is taken from [89]. An alternate proof (to that of [89]) of the Lemma using projection Lemma [142] is also given below.

Lemma 1.15 *For the matrices* $\mathcal{T}$, R, L *and* M*, the condition*

$$\begin{bmatrix} \mathcal{T} & M^TL^T \\ LM & -L-L^T+R \end{bmatrix}<0\Leftrightarrow\mathcal{T}<0,\ \mathcal{T}+M^TRM<0. \tag{1.55}$$

Proof: The proof directly follows from projection Lemma. By substituting $Z = \begin{bmatrix} \mathcal{T} & 0 \\ 0 & R \end{bmatrix}$, $U = [M \quad -I]$, $V = [0 \quad I]$, $N_U = [I \quad M^T]^T$, $N_V = [I \quad 0]^T$ and $W = L^T$ in Lemma 1.13, (1.55) can be obtained. □

The Lemma presented below is taken from [143, 144]. An alternate proof using the projection lemma is also given.

Lemma 1.16 *For the matrices $\mathcal{T}$, S, L and M of appropriate dimensions and a scalar β, the condition*

$$\begin{bmatrix} \mathcal{T} & * \\ \beta S^T + LM & -Sym\{\beta L\} \end{bmatrix} < 0 \Leftrightarrow \mathcal{T} + Sym\{SN\} < 0, \quad U < 0 \tag{1.56}$$

.

Proof: The proof of Lemma 1.16 can be directly followed from Lemma 6 in [144]. However, it can be derived from Lemma 1.13. By substituting $Z = \begin{bmatrix} \mathbb{T} & \mathbb{S} \\ \mathbb{S}^T & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & I \end{bmatrix}$, $U = \begin{bmatrix} M & -I \end{bmatrix}$, $W = L^T$, $N_U = \begin{bmatrix} I \\ M \end{bmatrix}$ and $N_V = \begin{bmatrix} I \\ 0 \end{bmatrix}$, such that $UN_U = 0$ and $VN_V = 0$, one obtains (1.56). □

1.4 Motivation

From the review, it is clear that designing of SOF controller for both the CT and DT systems is still open problem in control theory due to the absence of computable necessary and sufficient conditions. Numerous design methods are available in literature for solving the SOF problem yet there are scopes for improvement. The work carried out in [57], where the sufficient LMI criteria has been presented for designing the SOF controller for the CT systems by decomposition of the Lyapunov matrix variable is one of the motivation of this work. In [57], a diagonal decomposition of the Lyapunov matrix is considered. However, the off-diagonal terms of the Lyapunov matrix have been neglected therein. This resulted in restrictions in the developed criteria. Appropriate off-diagonal terms in the Lyapunov matrix are considered in our earlier work [145] that yields improvement over the work of [57]. In addition, in order to improve the transient behavior of the considered CT systems, it is desired to place the closed-loop system in some LMI region.

So, the regional pole-placement criteria in a LMI region of conic sector is added in the design. Robustness as well as good transient feature, both are ensured. However, the decomposition and thereby obtaining LMI is not applied to other classes of SOF problem.

It is desired to provide the better tuning methods so that the controller gains of the PID controller can be obtained satisfying robust performance criterion. Hence, to obtain these, PID control problem is first framed into the SOF design problem using the transformation given in [35] and then same SOF design criteria, the required controller gains of the PID controller are obtained. Although such an approach has been already discussed in the literature but mostly involving the iterative methods, which require more computation time.

Next, the SOF design for a class of CT polytopic systems (1.31) ensuring $\mathcal{L}_2$ performance subject to actuator saturation has been less investigated in literature. Limited results are available in the literature as already discussed in the previous section. In [94], only stabilization of multiple models under actuator saturation analysis via a SOF control law has been considered without any performance criteria. To this end, designing controllers with actuator saturation is an attractive problem due to its practical importance. Unaccounted actuator saturation in a closed-loop system may lead to degradation of system performance, occurrence of limit cycles, multiple equilibria and even instability [14, 15]. Motivated by these, the problem of SOF design for CT polytopic systems with actuator saturation is investigated. Although, the idea of convex handling of the constraints is borrowed from [18], the proposed conditions are different since the actuator saturation problem solved by these authors cater to the state feedback case, which is in contrast to the output feedback/reduced order controller cases that we consider here.

It is discussed that the SOF design problem for the DT system is also a BMI problem and is non-convex. In [89], conditions for designing of SOF, dynamic output feedback (DOF) and observer-based output feedback control, have been presented in terms of an unified LMI representations that seems to not involve any restriction on the variables. However, nonlinear coupling terms evolved due to the multiplication of system matrix and auxiliary matrix variable are approximated by bounding inequalities that lead to conservativeness in the development. Similar attempts has been made in [129] for SOF design considering new bounding condition for handling nonlinear terms with introduction of extra scalar variables in the design criteria. Motivated by the work presented in [89,129], new sufficient criteria for designing H_2 and H_∞ based SOF controller for linear DT systems with and without decomposition of matrix variable is investigated in a similar fashion to what improved the results for CT systems.

H_2 and H_∞ performance criteria yields a good robust performance, desired transient performance is, however, lacking. It is desired to place the closed-loop poles in some LMI region, which is term as pole-placement technique. The location of closed-loop poles ensure minimum decay rate (α) and damping ratio (ζ). Therefore, pole-placement criterion is introduced in the design to improve the transient behavior of the closed-loop system. While the pole-placement for CT systems has been studied in the past and its solutions exist due to convexity but pole-placement,it is an open problem due to non convexity of the logarithmic spirals of the constant damping locus for DT systems. Different subregions are constructed in literature for approximating the non convexity. In [31], circular regions are adopted. An inner-ellipse approximation [32] and a conic sector approximation [33, 34] have also been attempted. Note that, all these works used contracted area of the actual non convex region. This, combined with fact that LMI synthesis mostly deal with sufficient criteria, available designs are much restrictive. Alternate approach would be to consider non-contracting regions. This forms the conceiving idea behind this work.

Often the theoretical works developed are validated through the numerical examples to show its effectiveness, it is always interesting to verify the developed results on the real-time models. This inspires to carry out demonstration of the developed controller design methods through the implementation on the hardware prototypes. For this, two real-time applications designed by Quanser have been considered in this work, i.e., a coupled tank system (second order system) and 2-DOF helicopter (fourth order system, also called as Twin Rotor MIMO System (TRMS)). Note that systems are nonlinear in nature. Hence, the systems are required to be linearized. In conventional linearization approach, the system model is linearized about an operating point, neglecting the higher-order dynamics. This could result in drastic changes in the transient behavior of the system for which the controller may fail to ensure robustness and a good tracking response. Also, the existing linear controllers such as LQR, PID, etc have limitations such as selection of weighting matrices Q and R in the LQR method is a tedious task, difficulty for gain tuning of PID controller, deterioration of transient response due to linearization and failure to ensure robustness in the presence of uncertainty in the equilibrium point. To overcome these shortcomings, robust PI controller is designed with pole-placement to improve transient behavior ensuring performance criteria by considering system nonlinearities as an uncertain parameter, thereby representing the nonlinear model in the form of polytopic one.

Sometimes the uncertain parameters considered in the design are time-varying and are the function of state variables of the system dynamics. In cases, the variation in the uncertain parameter is dependent on the system states and available for the real-time measurements. Thus, it is necessary for the practical systems to involve this time-varying parameter in the system dynamics and controller gain is made dependent on this varying uncertain parameter so that it can capture the complete change occurring in the system and adjust according to it. This motivates to carry out a work in the direction of designing quasi-LPV based PI controller design for the TRMS.

1.5 Thesis objectives

The main objectives of the thesis are described as follows:

- To investigate the SOF problem for CT systems and design the control law, that can be used for both the centralized and the decentralized control design and yields less conservative result.

- To improve the transient behavior of the CT systems by incorporating the pole-placement criterion in the design.

- To develop SOF design conditions to compute the PID controller gains for higher order MIMO systems using appropriate matrix transformation conditions.

- To look into the problem of SOF design for the class of uncertain polytopic systems with matched output matrices containing polytopic uncertainties both with and without actuator saturation using parameter dependent Lyapunov function.

- To address the SOF problem for DT systems, similar to CT systems and design the control law ensuring H_2 and H_∞ performances using appropriate use of bounding inequalities and matrix decomposition method, which yield less conservative results.

- To develop new transformation framework for designing dynamic output feedback controller using the SOF design method.

- To investigate the problem of DT pole-placement in constant damping loci and develop new approximation for non-convexity caused by the constant damping ratio loci.

- To derive LMI criteria for state, output feedback (static as well as dynamic) and PI controllers to locate the closed-loop poles of DT uncertain polytopic systems in constant damping loci.

- To apply the proposed theories on prototypes in real-time for experimental validation.

- To design robust quasi–LPV PI controller for tracking control of a TRMS by incorporating actuator saturation criterion at the input.

1.6 Organization of the Thesis

The thesis consists of five chapters. The present chapter, as seen, provides a brief review on SOF controller design for both CT and DT systems, It provides the motivation behind this work, detailed literature survey to show the research gap and the main contributions.

Chapter 2 investigates the problem of designing the SOF controller for CT systems with regional pole-placement in a desired LMI region of conic sector ensuring H_∞ performance and improved transient behavior. This work has been published in [145] and earlier reported in [146]. Here, the same result is included for completeness of the presentation Then, the PID problem is framed into the SOF framework to obtain the controller gains of the PID controller for the MIMO system. The SOF design work is extended to general class of CT systems with polytopic uncertainties subject to actuator saturation ensuring $\mathcal{L}_2$ performance. The effectiveness of the proposed design methods is illustrated through the numerical examples.

In Chapter 3, new sufficient criteria for H_2 and H_∞ based SOF controller design for the DT LTI systems is presented. Two design criteria are proposed. One, the bounding condition without decomposition of any matrix variable and the second one is based on the decomposition of an auxiliary matrix variable. Through the numerical examples, it is shown that the design criteria are less conservative than the existing ones. A transformation framework is also proposed for designing DOF controller using the SOF design method. A new convex approximation of the constant damping ratio loci by an elliptical segment is proposed to handle the non-convexity of the constant damping loci arising in the DT domain for pole-placement. LMI criteria for designing controllers (state, output feedback of static and dynamic types, PI) to place the closed loop poles of DT uncertain system into a new convex LMI region. The efficacy of the proposed criteria is shown through numerical examples.

Chapter 4 focuses on application of the theories on prototypes in real-time. A robust PI controller is designed through polytopic modeling of coupled tank system and implemented on it. Experimental results are provided to validate the results and compared with the existing controllers. The SOF criteria designed for polytopic systems in the presence of actuator saturation in chapter 2 is demonstrated through the implementation on the real–time 2–DOF helicopter model. Also, a quasi-LPV polytopic modeling of 2-DOF helicopter is carried out by including the dynamics of the uncertain system into the model through real-time measurement and then PI controller is designed and implemented on the same helicopter setup to obtain better tracking response and improved transient behavior. Pole-placement criteria developed in chapter 3 is now implemented on the boost converter to improve its transient behavior subject to variation in input voltage and load resistance. Simulation results are provided to validate the results.

Finally, Chapter 5 states the overall conclusions of the works along with future perspectives led by the present work.