CHAPTER 4

TEMPERATURE-RATE DEPENDENT TWO-TEMPERATURE THEORY OF THERMOELASTICITY

4.1 Theory of Temperature-rate Dependent Two- Temperature Thermoelasticity for Anisotropic Medium[1]

4.1.1 Introduction

This chapter is concerned with the study of the temperature-rate dependent two-temperature (TRDTT) theory of thermoelasticity. It starts with the present section that aims to formulate the temperature-rate dependent two-temperature (TRDTT) theory of thermoelasticity. The two-temperature thermoelasticity theory and the temperature-rate dependent thermoelasticity theory are two well-established theories, which are developed from the generalized thermodynamic principles independently. Although the constitutive equations for TRDTT theory have been introduced, the formulation for the theory from the thermodynamic principles is not yet derived. Therefore, it is worth deriving the theory from the generalized laws of thermodynamics and derive all the governing equations and constitutive relations for the theory.

The concept of the two-temperature theory has been introduced by Gurtin and

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Williams (1966). They have suggested that the entropy inequality needs a modification based on the distinction between heat flux inside the body and the external heat supply. Due to these two different mechanisms, the entropy flow can be separated in the second law of thermodynamics. By the same factor of proportionality, two different temperatures can be assumed. Further, by following this modified entropy inequality, Chen et al. (1969) have derived the governing equations of the two-temperature theory of thermoelasticity from the fundamental laws of thermodynamics. The two-temperature generalization of LS theory has been introduced by Youssef (2006b) by including one thermal relaxation parameter. Youssef (2006b) has derived the constitutive relations from the energy equation and the first law of thermodynamics by keeping the two-temperature relation suggested by Chen et al. (1969) unchanged. The effect of this generalization has been analyzed by many authors. Youssef and Al-Lehaibi (2007) have examined the two-temperature LS theory (TTLS) for one dimensional thermoelastic problem. Kumar and Mukhopadhyay (2010b) have shown the effect of relaxation time on the plane wave propagation under this theory. Theoretical development of two-temperature thermoelasticity and some interesting investigations in this respect has been already discussed in chapter 1 of the present thesis. This theory has drawn serious attention during the last two decades and finds relevance in several areas. Chakrabarti (1973) has examined the harmonic thermoelastic plane waves and Rayleigh waves in the context of the two-temperature theory. Puri and Jordan (2006) have extended the work of Chakrabarti (1973) and studied the plane harmonic wave for media described for two-temperature under LS theory. They have determined critical values of physical parameters and reported that the effect of the two-temperature parameters is very much prominent at higher frequencies. Quintanilla (2004b) has discussed the structural stability, convergence, and spatial behaviour of thermoelastic problems under the two-temperature theory.

While formulating the two-temperature generalized theory, Youssef (2006b) has stated the governing equations for two-temperature GL (TTGL) theory that includes the classical two-temperature relation derived by Chen et al. (1969). Later on, Magana and Quintanilla (2009) discussed the uniqueness and growth of the solutions for two-temperature versions of LS and GL theories. Several authors, including Kumar et al. (2016), Kumar et al. (2017), Quintanilla and Jordan (2009), Kumari and Singh (2019), have reported some investigations and analyses on TTGL (or TRDTT) theory. However, the development of the governing equations for TRDTT theory from the fundamental laws of irreversible thermodynamics is not available in the literature.

Therefore, this subchapter is motivated to formulate a temperature-rate dependent two-temperature thermoelasticity theory from the fundamental laws of thermodynamics. All the basic governing equations and constitutive relations for homogeneous and anisotropic media are derived. A new and more general two-temperature relation is derived that involves the temperature-rate terms of conductive and thermodynamic temperatures. Further, it is observed that this relation is different from the two-temperature relation reported in the literature. The present relation involves the temperature-rate terms of conductive and thermodynamic temperatures and is reduced to the two-temperature relation given by Youssef (2006b) in a special case. Further, a uniqueness theorem for a mixed initial and boundary value problem is established for the anisotropic medium based on the present formulation. A one dimensional half space problem is investigated to examine the effect of modified temperature-rate dependent two-temperature thermoelasticity (MTRDTT) theory. This problem is studied under four different theories: Biot theory, GL theory, Two-temperature GL theory (Youssef (2006b)) and the present two-temperature theory (MTRDTT) in a unified way to compare the results of the present theory with the corresponding results of other existing theories. Distinct features of the present modification are highlighted with numerical results of the problem.

4.1.2 Formulation of the Theory

In this section, a generalized temperature-rate dependent two-temperature thermoelasticity theory with two relaxation parameters is derived by using the generalized laws of thermodynamics (see refs. Chen and Gurtin (1968); Chen et al. (1969); Youssef (2006b)). We consider a finite thermoelastic body with volume V bounded by a closed surface A. The standard equation of motion can be stated as

$$\sigma_{ij,j} + \rho b_i = \rho \ddot{u}_i \tag{4.1.1}$$

In view of the first law of thermodynamics and the principle of virtual work done, for any thermoelastic body, a relation in the integral form is obtained as

$$\int_V \sigma_{ij}\dot{e}_{ij} dV - \int_A q_i n_i dA = \int_V \rho(\dot{U} - Q) dV \tag{4.1.2}$$

where e_{ij} is the strain tensor; U is internal energy per unit mass; q_i is the component of heat flux vector; Q is the external heat source per unit mass per unit volume; n_i is the unit normal to the surface A.

The Eq. (4.1.2) holds for any arbitrary volume V and therefore we deduce the relation

$$\sigma_{ij}\dot{e}_{ij} - q_{i,i} = \rho(\dot{U} - Q) \tag{4.1.3}$$

Following the second law of thermodynamics, the modified form of Clausius inequality is taken as (see ref. Gurtin and Willams (1966))

$$\frac{d}{dt}\int_V \rho S dV \geq \int_V \rho \frac{Q}{\theta} dV - \int_A \frac{q_i n_i}{\phi} dA \tag{4.1.4}$$

In this subchapter, θ and ϕ will be used to denote the thermodynamic temperature and conductive temperature, respectively.

In the present work, we aim to construct a theory (two-temperature theory) involving non-simple material for which the thermodynamic temperature and conducive temperatures do not coincide.

Green and Laws (1972) have suggested that at equilibrium, it is acceptable that the specific entropy supply, $\frac{Q}{\theta}$ is associated with a specific external volume supply of heat, Q. On the other hand, for the non-equilibrium situations, the entropy supply would not be same. It is more reasonable to assume that the specific entropy supply will be $\frac{Q}{T}$, where T is a function which requires a constitutive equation with the restriction $T > 0$ and on equilibrium $T = \theta$, i.e. T can be assumed as a function of θ, $\dot{\theta}$, $\theta_{,i}$. Here, we recall the fact that in time-independent cases, the difference of two temperatures will be proportional to the external heat supply and in the absence of the external heat source, the two temperatures would be same, while in the time dependent situations two temperatures will always be different (see refs. Chen and Gurtin (1968); Chen et al. (1969); Youssef (2006b)). As in the present case, the situation is time-dependent and the external heat source (Q) is nonzero, the two temperatures are not the same. Hence, in view of these facts the surface relevant entropy change can be assumed in terms of a new function ($\Phi > 0$) as $\frac{q_i n_i}{\Phi}$, where this function Φ will be the functions of ϕ, $\dot{\phi}$, $\phi_{,i}$ and we refer Φ as generalized conductive temperature. Therefore, we postulate the generalized form of Eq. (4.1.4) as

$$\frac{d}{dt}\int_V \rho S dV \geq \int_V \rho \frac{Q}{T} dV - \int_A \frac{q_i n_i}{\Phi} dA \tag{4.1.5}$$

Now, from the above equation, we have

$$\rho(T\dot{S} - R) + \frac{T}{\Phi} q_{i,i} - \frac{T}{\Phi^2}\Phi_{,i} \geq 0 \tag{4.1.6}$$

4.1.3 Thermoelastic Assumptions

In order to formulate the temperature-rate dependent two-temperature thermoelasticity theory, we postulate that the thermoelastic material is characterized by the response functions U, q_i, σ_{ij}, S and T that are dependent on the thermoelastic functions of Φ, e_{ij}, $\phi_{,i}$, $\phi_{,ij}$, i.e., we have

$$\begin{gathered} U = \hat{U}(\Phi, e_{ij}, \phi_{,i}, \phi_{,ij}),\ q_i = \hat{q}_i(\Phi, e_{ij}, \phi_{,i}, \phi_{,ij}) \\ \sigma_{ij} = \hat{\sigma}_{ij}(\Phi, e_{ij}, \phi_{,i}, \phi_{,ij}),\ S = \hat{S}(\Phi, e_{ij}, \phi_{,i}, \phi_{,ij}), \\ T = \hat{T}(\Phi, e_{ij}, \phi_{,i}, \phi_{,ij}). \end{gathered} \tag{4.1.7}$$

For convenience, we consider the homogeneous material of the body. Also, we assume that the partial derivative of $\hat{T}$ with respect to Φ will never vanish. Then an inverse function of $\hat{T}$ can be assumed as

$$\Phi = \check{\Phi}(T, e_{ij}, \phi_{,i}, \phi_{,ij}) \tag{4.1.8}$$

Therefore, we may assume that

$$\begin{aligned} \hat{U}(\Phi, e_{ij}, \phi_{,i}, \phi_{,ij}) &= \check{U}(T, e_{ij}, \phi_{,i}, \phi_{,ij}), \\ \hat{q}_i(\Phi, e_{ij}, \phi_{,i}, \phi_{,ij}) &= \check{q}_i(T, e_{ij}, \phi_{,i}, \phi_{,ij}), \\ \hat{\sigma}_{ij}(\Phi, e_{ij}, \phi_{,i}, \phi_{,ij}) &= \check{\sigma}_{ij}(T, e_{ij}, \phi_{,i}, \phi_{,ij}), \\ \hat{S}(\Phi, e_{ij}, \phi_{,i}, \phi_{,ij}) &= \check{S}(T, e_{ij}, \phi_{,i}, \phi_{,ij}\phi_{,ij}). \end{aligned} \tag{4.1.9}$$

Here, we have used the super scripted $\hat{f}$ notation to indicate that f is considered as the function of Φ, e_{ij}, $\phi_{,i}$, $\phi_{,ij}$ whereas the superscripted $\check{f}$ notation is used for expressing f as the function of T, e_{ij}, $\phi_{,i}$, $\phi_{,ij}$.

Next, we consider the generalized free energy function as

$$\psi = U - TS \tag{4.1.10}$$

Then, we have

$$\psi = \check{U}(T, e_{ij}, \phi_{,i}, \phi_{,ij}) - T\check{S}(T, e_{ij}, \phi_{,i}, \phi_{,ij}) \tag{4.1.11}$$

4.1.4 Formulation of the Governing Equations and Constitutive Relations

For the formulation of basic equations in the context of the two-temperature generalization of the GL model, i.e., the temperature-rate dependent two-temperature thermoelasticity theory, we postulate that T and Φ are the functions of θ, $\dot{\theta}$ and ϕ, $\dot{\phi}$, respectively.

Therefore, from the Eqs. (4.1.3,4.1.6,4.1.9) and (4.1.11) we find

$$\rho\left(\frac{\partial\breve{\psi}}{\partial\theta}+S\frac{\partial T}{\partial\theta}\right)\dot{\theta}+\rho\left(\frac{\partial\breve{\psi}}{\partial\dot{\theta}}+S\frac{\partial T}{\partial\dot{\theta}}\right)\ddot{\theta}+\left(\rho\frac{\partial\breve{\psi}}{\partial e_{ij}}-\sigma_{ij}\right)\dot{e}_{ij}$$
$$+\rho\left(\frac{\partial\breve{\psi}}{\partial\phi_{,i}}\right)\dot{\phi}_{,i}+\rho\left(\frac{\partial\breve{\psi}}{\partial\phi_{,ij}}\right)\dot{\phi}_{,ij}+T\dot{S}-\rho q_{i,i}-\rho Q=0 \tag{4.1.12}$$

$$\rho\left(\frac{\partial\breve{\psi}}{\partial\theta}+S\frac{\partial T}{\partial\theta}\right)\dot{\theta}+\rho\left(\frac{\partial\breve{\psi}}{\partial\dot{\theta}}+S\frac{\partial T}{\partial\dot{\theta}}\right)\ddot{\theta}+\left(\rho\frac{\partial\breve{\psi}}{\partial e_{ij}}-\sigma_{ij}\right)\dot{e}_{ij}$$
$$+\rho\left(\frac{\partial\breve{\psi}}{\partial\phi_{,i}}+\frac{T}{\Phi^2}q_i\frac{\partial\Phi}{\partial\dot{\phi}}\right)\dot{\phi}_{,i}+\rho\frac{\partial\breve{\psi}}{\phi_{,ij}}\dot{\phi}_{,ij}+T\dot{S}+$$
$$\rho\left(1-\frac{T}{\Phi}\right)q_{i,i}+\frac{T}{\Phi^2}q_i\frac{\partial\Phi}{\partial\phi}\phi_{,i}\leq 0 \tag{4.1.13}$$

Based on the present assumptions that the theory is independent of $\ddot{\theta}$, $\dot{e}_{ij}$, $\dot{\phi}_{,i}$ and $\dot{\phi}_{,ij}$. Therefore, from the inequality (4.1.13), we find that $\frac{\partial\breve{\psi}}{\phi_{,ij}}=0$ and also, we have

$$\frac{\partial\breve{\psi}}{\partial\dot{\theta}}+S\frac{\partial T}{\partial\dot{\theta}}=0 \tag{4.1.14}$$

$$\rho\frac{\partial\breve{\psi}}{\partial e_{ij}}-\sigma_{ij}=0 \tag{4.1.15}$$

$$\frac{\partial\breve{\psi}}{\partial\phi_{,i}}+\frac{T}{\Phi^2}q_i\frac{\partial\Phi}{\partial\dot{\phi}}=0 \tag{4.1.16}$$

$$\left(\frac{\partial\breve{\psi}}{\partial\theta}+S\frac{\partial T}{\partial\theta}\right)\dot{\theta}+\left(1-\frac{T}{\Phi}\right)q_{i,i}+\frac{T}{\Phi^2}q_i\frac{\partial\Phi}{\partial\phi}\phi_{,i}\leq 0 \tag{4.1.17}$$

Further, from Eqs. (4.1.14-4.1.15) and Eq. (4.1.12), we obtain

$$\left(\frac{\partial\breve{\psi}}{\partial\theta}+S\frac{\partial T}{\partial\theta}\right)\dot{\theta}+\left(\frac{\partial\breve{\psi}}{\partial\phi_{,i}}\right)\dot{\phi}_{,i}+\dot{S}T+\frac{1}{\rho}q_{i,i}+\rho Q=0 \tag{4.1.18}$$

Also for the linear theory, we consider that $\frac{1}{\Phi}\simeq\frac{1}{T_0}$, and $\frac{T}{\Phi^2}\simeq\frac{1}{T_0}$. Here, it is interesting to note that if we take $T=\theta$ and $\Phi=\phi$, then the Eqs. (4.1.12-4.1.16) will reduce to the equations under two-temperature theory discussed by Chen et al. (1969).

Further, we use the Maclaurin's series expansion about the reference state. Hence, the expansion of $T(\theta,\dot{\theta})$ about $(T_0,0)$ up to second order by taking the constitutive assumption that $T(\theta,0)=\theta$, we obtain

$$T(\theta, \dot{\theta}) = T(\theta, 0) + \frac{\partial T(T_0, 0)}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial^2 T(T_0, 0)}{\partial \theta \partial \dot{\theta}}(\theta - T_0)\dot{\theta} + \frac{\partial^2 T(T_0, 0)}{\partial \dot{\theta}^2}\dot{\theta}^2 \tag{4.1.19}$$

Similarly, from the expansion of $\Phi(\phi, \dot{\phi})$ with the condition $\Phi(\phi, 0) = \phi$, we have

$$\Phi(\phi, \dot{\phi}) = \Phi(\phi, 0) + \frac{\partial \Phi(T_0, 0)}{\partial \dot{\phi}}\dot{\phi} + \frac{\partial^2 \Phi(T_0, 0)}{\partial \phi \partial \dot{\phi}}(\phi - T_0)\dot{\phi} + \frac{\partial^2 \Phi(T_0, 0)}{\partial \dot{\theta}^2}\dot{\phi}^2 \tag{4.1.20}$$

The two-temperature relation for thermoelasticity theory has been introduced by Chen et al. (1969). Here, from Eq. (4.1.7) and following Chen et al. (1969), we obtain this relation in terms of generalized conductive and thermodynamic temperatures as

$$\Phi - T = A_{ij}\phi_{,ij} \tag{4.1.21}$$

where A_{ij} is the temperature discrepancy tensor.

We define that both of the temperatures (thermodynamic and conductive temperatures) are initially at the same reference temperature (T_0). We further assume that at this reference temperature $\frac{\partial T(T_0,0)}{\partial \dot{\theta}} = \frac{\partial \Phi(T_0,0)}{\partial \dot{\phi}} = t_1$(say).

Then by using Eqs. (4.1.19-4.1.20), we can rewrite the linearized form Eq. (4.1.21) as

$$\left(1 + t_1\frac{\partial}{\partial t}\right)\phi - \left(1 + t_1\frac{\partial}{\partial t}\right)\theta = A_{ij}\phi_{,ij} \tag{4.1.22}$$

It can also be noted from Eq. (4.1.21) that in the special case when we assume $t_1 = 0$, the Eq. (4.1.22) will reduce to the form

$$\phi - \theta = A_{ij}\phi_{,ij} \tag{4.1.23}$$

It must be mentioned here that the relation (4.1.23) coincides with the two-temperature relation introduced by Chen et al. (1969) for his classical theory. Youssef (2006b) also referred the same relation (4.1.23) while introducing the basic equations of the generalized two-temperature theory. Hence, the two-temperature relation given by

Eq. (4.1.22) is the most general relation for the temperature-rate dependent two-temperature thermoelasticity theory.

Next, we expand $\check{\psi}(\theta, \dot{\theta}, e_{ij}, \phi_{,i})$ by retaining the terms up to the second order to formulate the linearized temperature-rate dependent two-temperature thermoelasticity theory and obtain

$$\begin{aligned}\check{\psi} = {} & \psi|_{\mathbf{0}} + (\theta - T_0)\, \frac{\partial \check{\psi}}{\partial \theta}\bigg|_{\mathbf{0}} + \dot{\theta}\, \frac{\partial \check{\psi}}{\partial \dot{\theta}}\bigg|_{\mathbf{0}} + e_{ij}\, \frac{\partial \check{\psi}}{\partial e_{ij}}\bigg|_{\mathbf{0}} + \phi_{,i}\, \frac{\partial \check{\psi}}{\partial \phi_{,i}}\bigg|_{\mathbf{0}} \\ & + \frac{1}{2}\bigg[\theta^2\, \frac{\partial^2 \check{\psi}}{\partial \theta^2}\bigg|_{\mathbf{0}} + 2(\theta - T_0)\dot{\theta}\, \frac{\partial^2 \check{\psi}}{\partial \theta \partial \dot{\theta}}\bigg|_{\mathbf{0}} + \dot{\theta}^2\, \frac{\partial^2 \check{\psi}}{\partial \dot{\theta}^2}\bigg|_{\mathbf{0}} \\ & + 2e_{ij}(\theta - T_0)\, \frac{\partial^2 \check{\psi}}{\partial e_{ij} \partial \theta}\bigg|_{\mathbf{0}} + 2e_{ij}\phi_{,k}\, \frac{\partial^2 \check{\psi}}{\partial e_{ij} \partial \phi_{,k}}\bigg|_{\mathbf{0}} \\ & + e_{ij}e_{kl}\, \frac{\partial^2 \check{\psi}}{\partial e_{ij} \partial e_{kl}}\bigg|_{\mathbf{0}} + \phi_{,i}\phi_{,j}\, \frac{\partial^2 \check{\psi}}{\partial \phi_{,i} \partial \phi_{,j}}\bigg|_{\mathbf{0}} + \theta\phi_{,i}\, \frac{\partial^2 \check{\psi}}{\partial T \partial \phi_{,i}}\bigg|_{\mathbf{0}} \\ & + \dot{\theta}\phi_{,i}\, \frac{\partial^2 \check{\psi}}{\partial \dot{\theta} \partial \phi_{,i}}\bigg|_{\mathbf{0}}\bigg]\end{aligned} \tag{4.1.24}$$

where $f|_{\mathbf{0}}$ denotes the the value of f at natural state $(T_0, 0, 0, 0)$. Now, by following we assume $\frac{\partial^2 \check{\psi}}{\partial e_{ij} \partial e_{kl}}\Big|_{\mathbf{0}} = C_{ijkl}$ (the elasticity tensor), $\frac{\partial^2 \check{\psi}}{\partial e_{ij} \partial T}\Big|_{\mathbf{0}} = -\beta_{ij}$ (the thermoelasticity tensor), $\frac{\partial^2 \check{\psi}}{\partial T \partial \phi_{,i}}\Big|_{\mathbf{0}} = C_i$ (a material parameter), $\frac{\partial^2 \check{\psi}}{\partial \theta^2}\Big|_{\mathbf{0}} = -\frac{\rho c_E}{T_0}$, $\frac{\partial^2 \check{\psi}}{\partial \theta \partial \dot{\theta}}\Big|_{\mathbf{0}} = -\frac{t_1 \rho c_E}{T_0}$, $\frac{\partial^2 \check{\psi}}{\partial \dot{\theta}^2}\Big|_{\mathbf{0}} = -\frac{t_1 t_2 \rho c_E}{T_0}$, $\frac{\partial^2 \check{\psi}}{\partial \phi_{,i} \partial \phi_{,j}}\Big|_{\mathbf{0}} = -K_{ij}$ (the conductivity tensor) with the natural initial conditions for the present formulation. Here, t_1 (same as Eq. 4.1.22) and t_2 can be referred to as thermal relaxation parameters. Therefore, in view of the inequality (4.1.17), Eq. (4.1.24) reduces to the form

$$\begin{aligned}\check{\psi} = {} & \frac{1}{2}C_{ijkl}e_{ij}e_{kl} - \beta_{ij}e_{ij}(\theta - T_0 + t_1\dot{\theta}) \\ & - \frac{1}{2}\rho c_E \frac{((\theta - T_0)^2 + t_1 t_2 \dot{\theta}^2 + 2t_1(\theta - T_0)\dot{\theta})}{T_0} \\ & + C_i\phi_{,i}\dot{\theta} - \frac{1}{2}K_{ij}\phi_{,i}\phi_{,j}\end{aligned} \tag{4.1.25}$$

Hence, by using the expression of $\check{\psi}$ from Eq. (4.1.25) in Eqs. (4.1.14-4.1.18), we obtain the constitutive equations for the present linear theory as

$$\sigma_{ij} = C_{ijkl}e_{kl} - \beta_{ij}(\theta - T_0 + t_1\dot{\theta}) \tag{4.1.26}$$

$$\rho S = \frac{\rho c_E}{T_0}(\theta - T_0 + t_2\dot{\theta}) - \frac{C_i}{T_0}\phi_{,i} + \beta_{ij}e_{ij} \tag{4.1.27}$$

$$q_i = -(C_i\dot{\theta} + K_{ij}\phi_{,j}) \tag{4.1.28}$$

$$q_{i,i} = \rho(Q - \dot{S}T_0) \tag{4.1.29}$$

$$\rho c_E(t_1 - t_2)\dot{\theta}^2 + 2C_i\dot{\theta}\phi_{,i} + (\Phi - T)q_{i,i} + K_{ij}\phi_{,i}\phi_{,j} \geq 0 \tag{4.1.30}$$

Further, by combining the Eqs. (4.1.27-4.1.29) we have the heat conduction equation as

$$K_{ij}\phi_{,ij} = \rho c_E\dot{\theta} + \rho c_E t_2\ddot{\theta} + \beta_{ij}T_0\dot{e}_{ij} + \rho Q - C_i\dot{\theta}_{,i} \tag{4.1.31}$$

Equations (4.1.1), (4.1.31) and the relations given by (4.1.26-4.1.29) along with the two-temperature relation (4.1.22) constitute the basic governing equations and constitutive relations of the present linearized temperature-rate dependent two-temperature thermoelasticity theory for homogeneous and anisotropic body.

Particular case: If we consider that the theory is dependent on rate of thermodynamic temperature but not on the rate of conductive temperature, then Eq. (4.1.15) yields

$$\frac{\partial\check{\psi}}{\partial\phi_{,i}} = 0 \tag{4.1.32}$$

Hence, Eq. (4.1.28) reduces to the form as

$$q_i = -K_{ij}\phi_{,j} \tag{4.1.33}$$

Further from Eq. (4.1.22), we find the different form of the two-temperature relation as

$$\phi - \left(1 + t_1\frac{\partial}{\partial t}\right)\theta = A_{ij}\phi_{,ij} \tag{4.1.34}$$

Hence, in this case, Eqs. (4.1.1), (4.1.26-4.1.27), (4.1.29), (4.1.33) and (4.1.34) will constitute the basic equations for the thermoelasticity theory.

4.1.5 Uniqueness theorem

In order to establish a uniqueness theorem, we consider a homogeneous anisotropic solid body of volume V and surface area A. Further, we assume that the body is regular and simply connected in the volume V and the material of body has the center of symmetry property (i.e., $C_i = 0$). Then, the field Eqs. (4.1.27,4.1.28) and (4.1.30) reduce to the forms

$$\rho S = \frac{\rho c_E}{T_0}(\theta - T_0 + t_2\dot{\theta}) + \beta_{ij}e_{ij} \tag{4.1.35}$$

$$q_i = -K_{ij}\phi_{,j} \tag{4.1.36}$$

$$\rho c_E(t_1 - t_2)\dot{\theta}^2 + (\Phi - T)q_{i,i} + K_{ij}\phi_{,i}\phi_{,j} \geq 0 \tag{4.1.37}$$

Again by using the the center of symmetry property, we find the reduced form of the heat conduction Eq. (4.1.31) as

$$K_{ij}\phi_{,ij} = \rho c_E\dot{\theta} + \rho c_E t_2\ddot{\theta} + \beta_{ij}T_0\dot{e}_{ij} \tag{4.1.38}$$

The material parameters and relaxation times are assumed to be as follows:

$$\rho > 0,\ c_E > 0,\ A_{ij} = \lambda K_{ij},\ \lambda > 0,\ K_{ij}\phi_{,i}\phi_{,j} \geq 0,$$
$$A_{ij}\phi_{,i}\phi_{,j} \geq 0,\ C_{ijkl}e_{ij}e_{kl} \geq 0, t_1 \geq t_2 > 0.$$

Initial and boundary conditions:

Along with the set of Eqs. (4.1.1, 4.1.22, 4.1.26, 4.1.35-4.1.38) we assume the initial conditions as

$$\begin{aligned} u_i(x,0) &= u_{i0}, \;\; \dot{u}_i(x,0) = \dot{u}_{i0}, \qquad \text{for all } x \in V \\ \phi(x,0) &= \phi_{i0}, \;\; \dot{\phi}(x,0) = \dot{\phi}_{i0}, \qquad \text{for all } x \in V \end{aligned} \tag{4.1.39}$$

and the boundary conditions as:

$$\begin{aligned} u_i(x,t) &= \tilde{u}_i, \quad \text{for all } x \in A \text{ and } t \geq 0 \\ \phi(x,t) &= \tilde{\phi}, \quad \text{for all } x \in A \text{ and } t \geq 0 \end{aligned} \tag{4.1.40}$$

Then the uniqueness results are given in following theorem:

Theorem: Let us consider a regular region of anisotropic thermoelastic material occupying space V and surface area A. We assume that $\sigma_{ij}(x,t)$, $e_{ij}(x,t)$, and $q_i(x,t)$ are of class C^1, and $u_i(x,t)$, $\theta(x,t)$, and $\phi(x,t)$ are of class C^2 (where, C^i denotes the class of all functions having continuous i^{th} order space derivatives). Then, there exists at most one set of single valued functions $\{\sigma_{ij}, e_{ij}, u_i, \theta, \phi, q_i\}$ such that Eqs. (4.1.1,4.1.22,4.1.26,4.1.35-4.1.38) with the initial condition (4.1.39) and boundary condition (4.1.40) are satisfied.

Proof: In order to show that there is one set of single valued functions $\{\sigma_{ij}, e_{ij}, u_i, \theta, \phi, q_i\}$ satisfying Eqs. (4.1.1,4.1.22,4.1.26,4.1.35-4.1.38), we take a contradictory argument that $\{\sigma_{ij}^{(1)}, e_{ij}^{(1)}, u_i^{(1)}, \theta^{(1)}, \phi^{(1)}, q_i^{(1)}\}$ and $\{\sigma_{ij}^{(2)}, e_{ij}^{(2)}, u_i^{(2)}, \theta^{(2)}, \phi^{(2)}, q_i^{(2)}\}$ are two different sets of solutions to Eqs. (4.1.1,4.1.22,4.1.26,4.1.35-4.1.38) with the initial and boundary conditions (4.1.39) and (4.1.40), respectively. Then by using the linearity of the problem we may assume that in the absence of external heat source (Q) and body force (b_i), $\{\mathring{\sigma}_{ij}, \mathring{e}_{ij}, \mathring{u}_i, \mathring{\theta}, \mathring{\phi}, \mathring{q}_i\} = \{\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}, e_{ij}^{(1)} - e_{ij}^{(2)}, u_i^{(1)} - u_i^{(2)}, \theta^{(1)} - \theta^{(2)}, \phi^{(1)} - \phi^{(2)}, q_i^{(1)} - q_i^{(2)}\}$ will form another solution to the Eqs. (4.1.1,3.2.22,3.3.4,3.4.4-4.1.38) and homogeneous counterpart of initial and boundary conditions (4.1.39-4.1.40).

Next, we consider the integral,

$$\int_V \mathring{\sigma}_{ij} \dot{\mathring{e}}_{ij} dV = \int_V (\mathring{\sigma}_{ij} \dot{\mathring{u}}_i)_{,j} dV - \int_V \mathring{\sigma}_{ij,j} \dot{\mathring{u}}_i dV \tag{4.1.41}$$

Applying divergence theorem in the first term of R.H.S. of Eq. (4.1.41) and using homogeneous boundary conditions, we find that

$$\int_V (\mathring{\sigma}_{ij}\dot{\mathring{e}}_{ij} - \mathring{\sigma}_{ij,j}\dot{\mathring{u}}_i)dV = 0 \tag{4.1.42}$$

Substituting $\mathring{\sigma}_{ij}$ and $\mathring{\sigma}_{ij,j}$ from Eqs. (4.1.26) and (4.1.1), respectively into Eq. (4.1.42), we have

$$\int_V \left[\left[C_{ijkl}\mathring{e}_{kl} - \beta_{ij}(\mathring{\theta} + t_1\dot{\mathring{\theta}})\right]\dot{\mathring{e}}_{ij} - \rho\ddot{\mathring{u}}_i\dot{\mathring{u}}_i\right] dV = 0 \tag{4.1.43}$$

Further, Eqs. (4.1.38) and (4.1.43) yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_V \left(C_{ijkl}\mathring{e}_{ij}\mathring{e}_{kl} + \rho\dot{\mathring{u}}_i\dot{\mathring{u}}_i\right) dV \\ + \int_V (\mathring{\theta} + t_1\dot{\mathring{\theta}})\left\{\frac{\rho c_E\dot{\mathring{\theta}} + \rho c_E t_2\ddot{\mathring{\theta}} - K_{ij}\mathring{\phi}_{,ij}}{T_0}\right\} dV = 0\end{aligned} \tag{4.1.44}$$

which implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_V \left[\left(C_{ijkl}\mathring{e}_{ij}\mathring{e}_{kl} + \rho\dot{\mathring{u}}_i\dot{\mathring{u}}_i + \frac{\rho c_E}{T_0}(\mathring{\theta} + t_2\dot{\mathring{\theta}})^2\right.\right. \\ \left.\left. + \frac{(t_1 - t_2)t_2}{T_0}\dot{\mathring{\theta}}^2\right) + \frac{(t_1 - t_2)}{T_0}\dot{\mathring{\theta}}^2 - (\mathring{\theta} + t_1\dot{\mathring{\theta}})K_{ij}\mathring{\phi}_{,ij}\right] dV = 0\end{aligned} \tag{4.1.45}$$

Now, we substitute the two-temperature relation (4.1.22) in Eq. (4.1.45) to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_V \left[\left(C_{ijkl}\mathring{e}_{ij}\mathring{e}_{kl} + \rho\dot{\mathring{u}}_i\dot{\mathring{u}}_i + \frac{\rho c_E}{T_0}(\mathring{\theta} + t_2\dot{\mathring{\theta}})^2 + \frac{(t_1 - t_2)t_2}{T_0}\dot{\mathring{\theta}}^2\right)\right. \\ \left.\left(\frac{(t_1 - t_2)}{T_0}\dot{\mathring{\theta}}^2 - K_{ij}(\mathring{\phi} + t_1\dot{\mathring{\phi}})\mathring{\phi}_{,ij} + A_{ij}K_{kl}\mathring{\phi}_{,ij}\mathring{\phi}_{,kl}\right)\right] dV = 0\end{aligned}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\int_V \Bigg[\Bigg(C_{ijkl}\mathring{e}_{ij}\mathring{e}_{kl} + \rho\dot{\mathring{u}}_i\dot{\mathring{u}}_i + \frac{\rho c_E}{T_0}(\mathring{\theta} + t_2\dot{\mathring{\theta}})^2 + \frac{(t_1 - t_2)t_2}{T_0}\dot{\mathring{\theta}}^2$$
$$+t_1 K_{ij}\dot{\mathring{\phi}}_{,i}\dot{\mathring{\phi}}_{,j}\Bigg) + \frac{(t_1 - t_2)}{T_0}\dot{\mathring{\theta}}^2 + K_{ij}\mathring{\phi}_{,i}\mathring{\phi}_{,j} + A_{ij}K_{kl}\mathring{\phi}_{,ij}\mathring{\phi}_{,kl}\Bigg] dV = 0 \qquad (4.1.46)$$

Now, from Eq. (4.1.46), we observe that each term of Eq. (4.1.46) vanishes separately. Therefore, we obtain that each element of the set $\{\mathring{\sigma}_{ij}, \mathring{e}_{ij}, \mathring{u}_i, \mathring{\theta}, \mathring{\phi}, \mathring{q}_i\}$ is separately zero, which contradicts our assumption that there exist two sets of solutions. Hence, the proof of uniqueness of solutions is completed.

4.1.6 Application: A One Dimensional (half space) Problem

In this section, we illustrate the effect of the new two-temperature theory. For this, we consider a one dimensional half space problem of homogeneous and isotropic medium ($x \geq 0$) in respect of four different thermoelasticity theories namely, Biot's theory, GL theory, two-temperature GL theory (TTGL) (Youssef (2006b)) and the present modified thermoelasticity theory (MTRDTT). Here, all field variables are assumed to be functions of x and t only. Therefore, for this problem, in the absence of external body force and heat source, the Eqs. (4.1.26) and (4.1.31) will reduce to the following forms:

$$\sigma_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \beta\left(\theta - T_0 + t_1\frac{\partial \theta}{\partial t}\right) \qquad (4.1.47)$$

$$K\frac{\partial^2 \phi}{\partial x^2} = \rho c_E\left(\frac{\partial \theta}{\partial t} + t_2\frac{\partial^2 \theta}{\partial t^2}\right) + \beta T_0\frac{\partial^2 u}{\partial t \partial x} \qquad (4.1.48)$$

where λ and μ are the Lame's constants and K is the thermal conductivity of the material.

Further, we assume the one dimensional equation of motion as

$$\frac{\partial \sigma_{xx}}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \qquad (4.1.49)$$

Similarly, the two-temperature relation (4.1.22) reduces to

$$\left(1+t_1p_1\frac{\partial}{\partial t}\right)\phi-\left(1+t_1p_1\frac{\partial}{\partial t}\right)\theta=a\frac{\partial^2\phi}{\partial x^2} \tag{4.1.50}$$

where p_1 is a dimensionless parameter used to write the unified form of two-temperature relation proposed by Youssef (2006b) and the two-temperature relation in the present context and a is the two-temperature parameter. From the above set of Eqs. (4.1.47-4.1.50), one can extract the corresponding equations under four different theories by assuming the values of different parameters as follows:

- Biot's theory: $t_1=t_2=a=0$
- GL theory: $a=p_1=0$
- TTGL theory: $p_1=0$, $t_1\neq 0$, $t_2\neq 0$, $a\neq 0$
- MTRDTT theory: $p_1=1$, $t_1\neq 0$, $t_2\neq 0$, $a\neq 0$.

Substituting Eq. (4.1.47) into Eq. (4.1.49), we have

$$(\lambda+2\mu)\frac{\partial u}{\partial x}-\beta\left(\frac{\partial\theta}{\partial x}+t_1\frac{\partial}{\partial x}\frac{\partial\theta}{\partial t}\right)=\rho\frac{\partial^2 u}{\partial t^2} \tag{4.1.51}$$

Next, we make the variables dimensionless by taking the following assumptions of dimensionless quantities

$(x',u')=c_0\eta(x,u)$, $(t',t_1',t_2')=c_0^2\eta(t,t_1,t_2)$, $(\theta',\phi')=\frac{1}{T_0}(\theta-T_0,\phi-T_0)$, and $\sigma_{xx}'=\frac{\sigma_{xx}}{(\lambda+2\mu)}$, where $c_0^2=\frac{(\lambda+2\mu)}{\rho}$ and $\eta=\frac{\rho c_E}{K}$.

Therefore, by using the above relations, we obtain the dimensionless forms of Eqs. (4.1.47-4.1.48) and Eq. (4.1.50) as

$$\sigma_{xx}=\frac{\partial u}{\partial x}-\gamma\left(1+t_1\frac{\partial}{\partial t}\right)\theta \tag{4.1.52}$$

$$\frac{\partial^2\phi}{\partial x^2}=\left(\frac{\partial\theta}{\partial t}+t_2\frac{\partial^2\theta}{\partial t^2}\right)+\varepsilon\frac{\partial^2 u}{\partial t\partial x} \tag{4.1.53}$$

$$\frac{\partial\sigma_{xx}}{\partial x}=\frac{\partial^2 u}{\partial x^2}-\gamma\left(1+t_1\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial x} \tag{4.1.54}$$

$$\left(1+t_1p_1\frac{\partial}{\partial t}\right)\phi-\left(1+t_1p_1\frac{\partial}{\partial t}\right)\theta=a_1\frac{\partial^2\phi}{\partial x^2} \tag{4.1.55}$$

where $\gamma=\frac{\beta T_0}{(\lambda+2\mu)}$, $\varepsilon=\frac{\beta}{\rho c_E}$ and $a_1=ac_0^2\eta^2$.

For the present half space problem, the homogeneous initial conditions are assumed with the dimensionless boundary conditions as follows:

Mechanical boundary condition:

$$\sigma_{xx}(0,t)=0,\ \text{ for } t>0,\, t\in\mathbb{R}^+ \tag{4.1.56}$$

Thermal boundary condition:

$$\phi(0,t)=1\ \text{ for } t>0,\, t\in\mathbb{R}^+ \tag{4.1.57}$$

Now, we apply Laplace transform to Eqs. (4.1.52-4.1.57) to obtain

$$\overline{\sigma}_{xx}=\frac{\partial\overline{u}}{\partial x}-\gamma(1+t_1s)\overline{\theta} \tag{4.1.58}$$

$$\frac{\partial^2\overline{\phi}}{\partial x^2}=(s\overline{\theta}+t_2s^2\overline{\theta})+s\varepsilon\frac{\partial\overline{u}}{\partial x} \tag{4.1.59}$$

$$\frac{\partial\overline{\sigma}_{xx}}{\partial x}=\frac{\partial^2\overline{u}}{\partial x^2}-\gamma(1+t_1s)\frac{\partial\overline{\theta}}{\partial x} \tag{4.1.60}$$

$$(1+t_1sp_1)\,\overline{\phi}-(1+t_1sp_1)\,\overline{\theta}=a_1\frac{\partial^2\overline{\phi}}{\partial x^2} \tag{4.1.61}$$

$$\overline{\sigma}_{xx}(0,s)=0, \tag{4.1.62}$$

$$\overline{\phi}(0,s)=1/s \tag{4.1.63}$$

where for any function $f(t)$, $\overline{f}(s)=\int_0^\infty e^{-st}f(t)dt$ and s is the Laplace transform parameter.

Now, by doing detailed manipulations on Eqs. (4.1.58-4.1.61), the simultaneous linear differential equations are obtained in terms of $\overline{\sigma}_{xx}$ and $\overline{\phi}$ as

$$\frac{\partial^2\overline{\phi}}{\partial x^2}=L_1\overline{\phi}+L_2\overline{\sigma}_{xx} \tag{4.1.64}$$

$$\frac{\partial^2\overline{\sigma}_{xx}}{\partial x^2}=M_1\overline{\phi}+M_2\overline{\sigma}_{xx} \tag{4.1.65}$$

where

$$L_1 = \frac{[(s+t_2s^2)+s\varepsilon\gamma(1+t_1s)](1+st_1p_1)}{[1+st_1p_1+a_1(s+t_2s^2)+s\varepsilon a_1\gamma(1+t_1s)]},$$
$$L_2 = \frac{s\varepsilon(1+st_1p_1)}{[1+st_1p_1+a_1(s+t_2s^2)+s\varepsilon a_1\gamma(1+t_1s)]}$$

and $M_1 = \frac{s^2\gamma(1+t_1s)[(1+st_1p_1)-L_1a_1]}{(1+st_1p_1)}$, $M_2 = \frac{s^2(1+t_1s)[(1+st_1p_1)-L_1a_1]}{(1+st_1p_1)}$.

Therefore, Eqs. (4.1.64-4.1.65) can be written in the matrix form as

$$\frac{d^2\overline{F}(x,s)}{dx^2} = P(s)\overline{F}(x,s), \tag{4.1.66}$$

where $\overline{F}(x,s) = \begin{bmatrix} \overline{\phi}(x,s) \\ \overline{\sigma}_{xx}(x,s) \end{bmatrix}$ and $P = \begin{bmatrix} L_1 & L_2 \\ M_1 & M_2 \end{bmatrix}$.

Hence, the solution of the Eq. (4.1.66), assuming that all field variables are bounded and tends to 0 as $x \to \infty$, can be obtained as

$$\overline{F}(x,s) = e^{-\sqrt{P(s)}x}\overline{F}(0,s) \tag{4.1.67}$$

where $\overline{F}(0,s) = \begin{bmatrix} \overline{\phi}(0,s) \\ \overline{\sigma}_{xx}(0,s) \end{bmatrix} = \begin{bmatrix} 1/s \\ 0 \end{bmatrix}$ (from Eqs. (4.1.62) and (4.1.63)).

Now, to find the value of $e^{-\sqrt{P(s)}x}$, we use the Caley-Hamilton theorem. The characteristic equation of the matrix, P can be written as

$$q^2 - q(L_1 + M_2) + (L_1M_2 - L_2M_1) = 0 \tag{4.1.68}$$

where q is characteristic variable and, q_1 and q_2 are the roots of Eq. (4.1.68). Therefore, we have $q_1 + q_2 = (L_1 + M_2)$ and $q_1q_2 = (L_1M_2 - L_2M_1)$. Now, in view of Eq. (4.1.68) and the expansion $e^{-\sqrt{P(s)}x} = \sum_{n=0}^{\infty}\left(\frac{-\sqrt{P(s)}x}{n!}\right)^n$, it can be assumed that the roots of Eq. (4.1.68) will satisfy the relations

$$e^{-\sqrt{q_1}x} = m_0 + q_1m_1 \tag{4.1.69}$$

$$e^{-\sqrt{q_2}x} = m_0 + q_2m_1 \tag{4.1.70}$$

Further, any higher power of P can be written in terms of I and P, where I is the identity matrix of order 2×2.

Hence, by solving the Eqs. (4.1.69-4.1.70), we obtain

$$m_0 = \frac{q_1 e^{-\sqrt{q_2}x} - q_2 e^{-\sqrt{q_1}x}}{q_1 - q_2},\ m_1 = \frac{e^{-\sqrt{q_1}x} - e^{-\sqrt{q_2}x}}{q_1 - q_2} \tag{4.1.71}$$

Therefore, $e^{-\sqrt{P(s)}x}$ can be represented as

$$e^{-\sqrt{P(s)}x} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \tag{4.1.72}$$

where $V_{11} = \frac{e^{-\sqrt{q_2}x}(q_1-L_1)-e^{-\sqrt{q_1}x}(q_2-L_1)}{q_1-q_2}$, $V_{12} = \frac{L_2(e^{-\sqrt{q_1}x}-e^{-\sqrt{q_2}x})}{q_1-q_2}$
$V_{21} = \frac{e^{-\sqrt{q_1}x}(q_2-M_2)-e^{-\sqrt{q_2}x}(q_1-M_2)}{q_1-q_2}$, $V_{22} = \frac{M_1(e^{-\sqrt{q_1}x}-e^{-\sqrt{q_2}x})}{q_1-q_2}$.

Further, the Eq. (4.1.67) can be written as

$$\overline{F}(x,s) = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \overline{F}(0,s) \tag{4.1.73}$$

Therefore, the solutions for $\overline{\phi}$ and $\overline{\sigma}_{xx}$ can be obtained from Eq. (4.1.73) and boundary conditions Eqs. (4.1.62-4.1.63) as

$$\overline{\phi} = \frac{(q_1 - L_1)e^{-\sqrt{q_2}x} - (q_2 - L_1)e^{-\sqrt{q_1}x}}{(q_1 - q_2)s} \tag{4.1.74}$$

$$\overline{\sigma}_{xx} = \frac{-M_1 e^{-\sqrt{q_2}x} + M_1 e^{-\sqrt{q_1}x}}{(q_1 - q_2)s} \tag{4.1.75}$$

Now, by using the values of $\overline{\phi}$ and $\overline{\sigma}_{xx}$ from Eqs. (4.1.74-4.1.75) in Eqs. (4.1.60) and (4.1.61), we obtain

$$\begin{aligned} \overline{\theta} = &\left(\frac{1 + st_1 p_1}{1 + st_1}\right) \frac{(q_1 - L_1)e^{-\sqrt{q_2}x} - (q_2 - L_1)e^{-\sqrt{q_1}x}}{(q_1 - q_2)s} \\ &- \frac{a_1}{1 + st_1}\left(\frac{q_2(q_1 - L_1)e^{-\sqrt{q_2}x} - q_1(q_2 - L_1)e^{-\sqrt{q_1}x}}{(q_1 - q_2)s}\right) \end{aligned} \tag{4.1.76}$$

$$\overline{u} = \frac{1}{s^2}\left[\frac{\sqrt{q_2}M_1 e^{-\sqrt{q_2}x} - \sqrt{q_1}M_1 e^{-\sqrt{q_1}x}}{(q_1 - q_2)s}\right] \tag{4.1.77}$$

This completes the solution of the problem in Laplace transform domain.

4.1.7 Numerical Results and Discussion

The exact expressions of the field variables in the space-time domain can be obtained by taking the inverse Laplace transform of Eqs. (4.1.74-4.1.77). Next, in order to find the solution in real time domain we apply the numerical inversion of Laplace transform as discussed by Stehfest (1970) and compute the numerical solutions of $\overline{\sigma}_{xx}$, $\overline{\phi}$, $\overline{\theta}$ and $\overline{u}$. For the numerical computation, MATLAB software is used. The material parameters are considered for Aluminium metal as given below (see article Youssef (2007)):

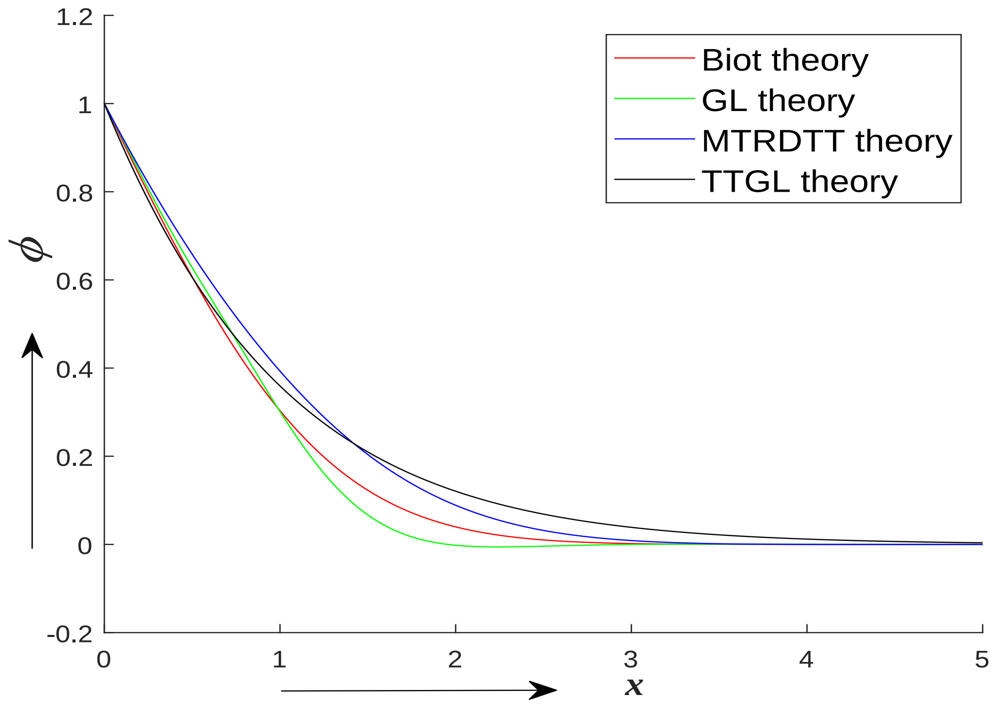
$$\lambda = 7.76 \times 10^{10}\text{Kg cm}^{-1}\text{s}^{-2},\ \mu = 3.86 \times 10^{10}\text{Kg cm}^{-1}\text{s}^{-2},\ \rho = 8854 \text{ Kg cm}^{-3},$$

$$c_E = 383.1\text{cm}^2\text{K}^{-1}\text{s}^{-2}\ K = 386 \text{ Kg cm K}^{-1}\text{ s}^{-3},\ \alpha_t = 1.78 \times 10^{-5}\text{K}^{-1}.$$

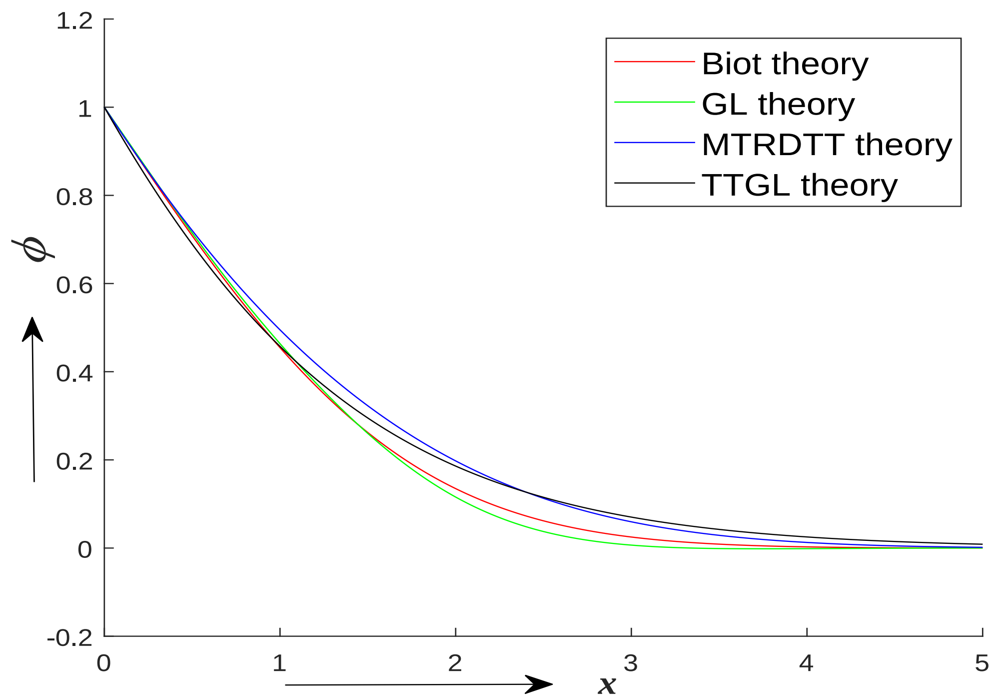
We assume the dimensionless value of $t_1 = 0.5$ and $t_2 = 0.01$.

The computation is carried out for a wide range of values of non-dimensional time, t. We represent the distribution of the field variables graphically at two different time step, $t = 0.25$ and $t = 0.45$. Figs. 4.1.1 (a,b), 4.1.2 (a,b), 4.1.3 (a,b) and 4.1.4 (a,b) show the variations of non-dimensional conductive temperature, thermodynamic temperature, displacement and stress, respectively, for $a_1 = 0.5$. From the Figs. 4.1.1 (a,b)-4.1.4 (a,b), we observed that there is a significant difference in predictions of four theories (Biot, GL, TTGL, MTRDTT) for all the field variables. However, the differences are more prominent between the theories with two-temperature and the theories with one temperature. It is also noted that the effective domain of influence is larger for the two-temperature theories as compared to Biot and GL theories. The effect of time is significant for all field variables as the domain of influence is increasing with time.

Figures 4.1.5 (a)-4.1.8 (a) reveal the effect of non-dimensional two-temperature parameter (a_1) on the field variables at $t = 0.30$. From these figures, it is found that effect of parameter a_1 is prominent and the effective domain of influence increases with the increase in the values of a_1.

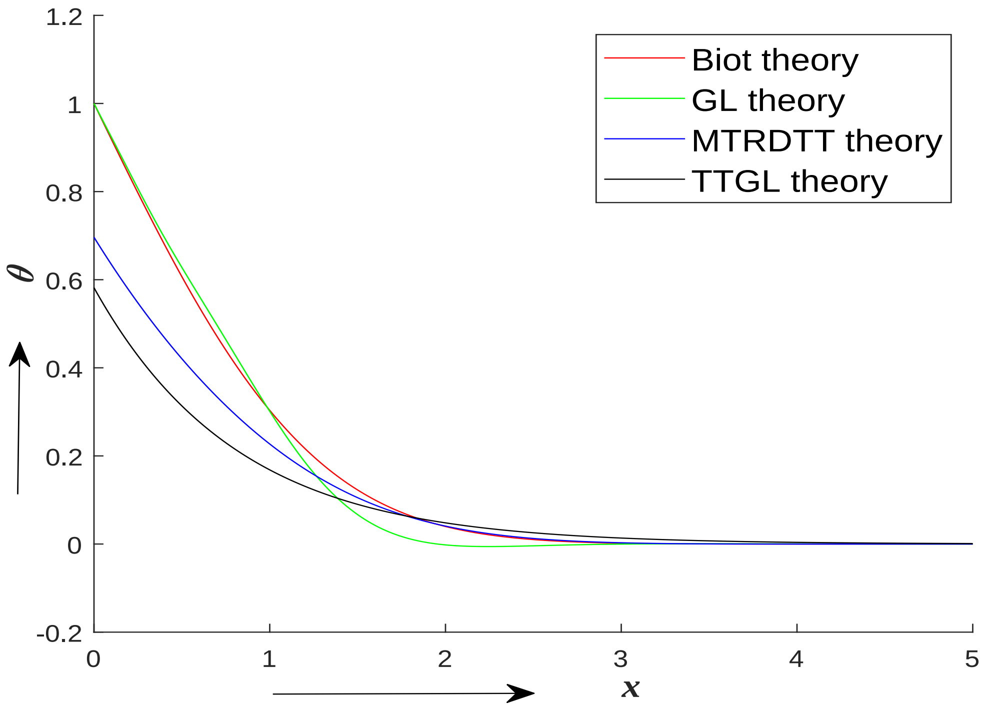


(a) Variation of ϕ along x at $t = 0.25$

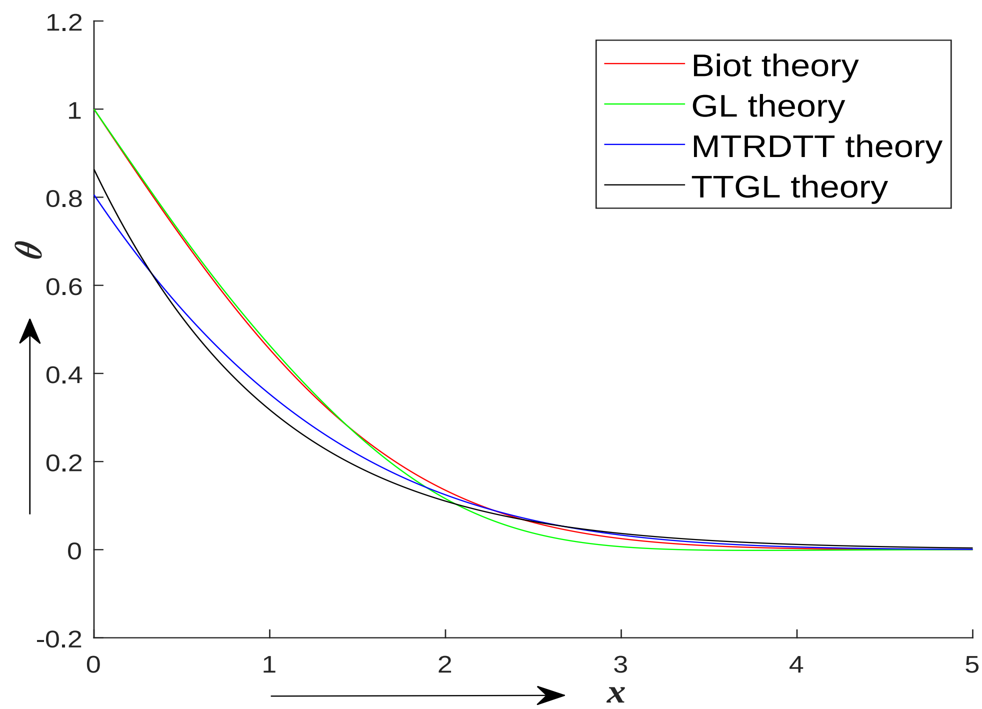


(b) Variation of ϕ along x at $t = 0.45$

Figure 4.1.1: Variation of conductive temperature (ϕ) at $t_1 = 0.5$ and $a_1 = 0.5$

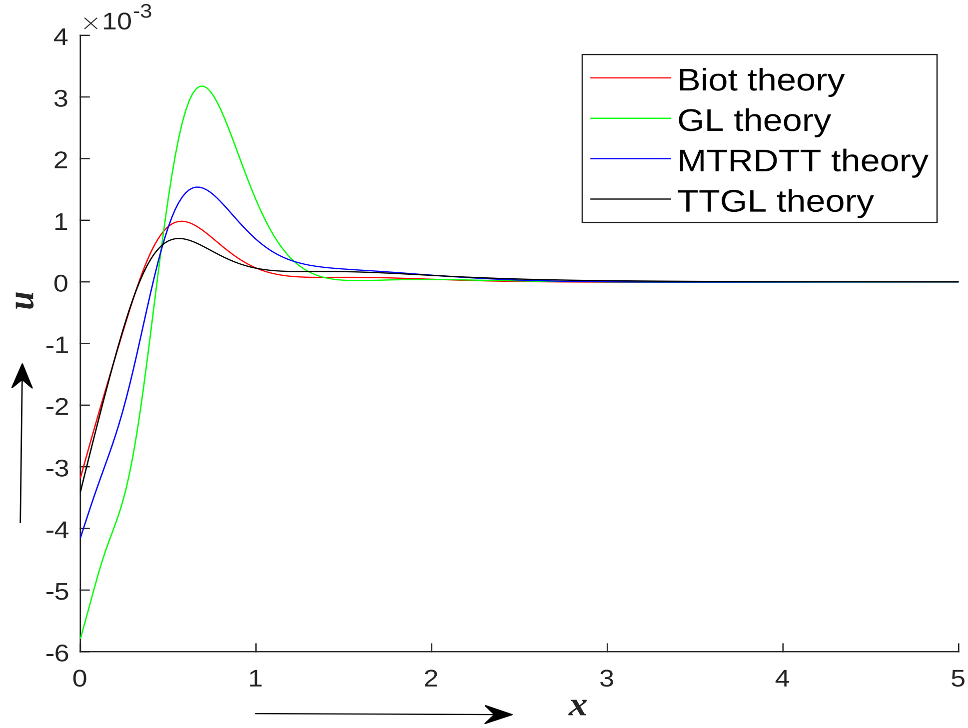


(a) Variation of θ along x at $t = 0.25$

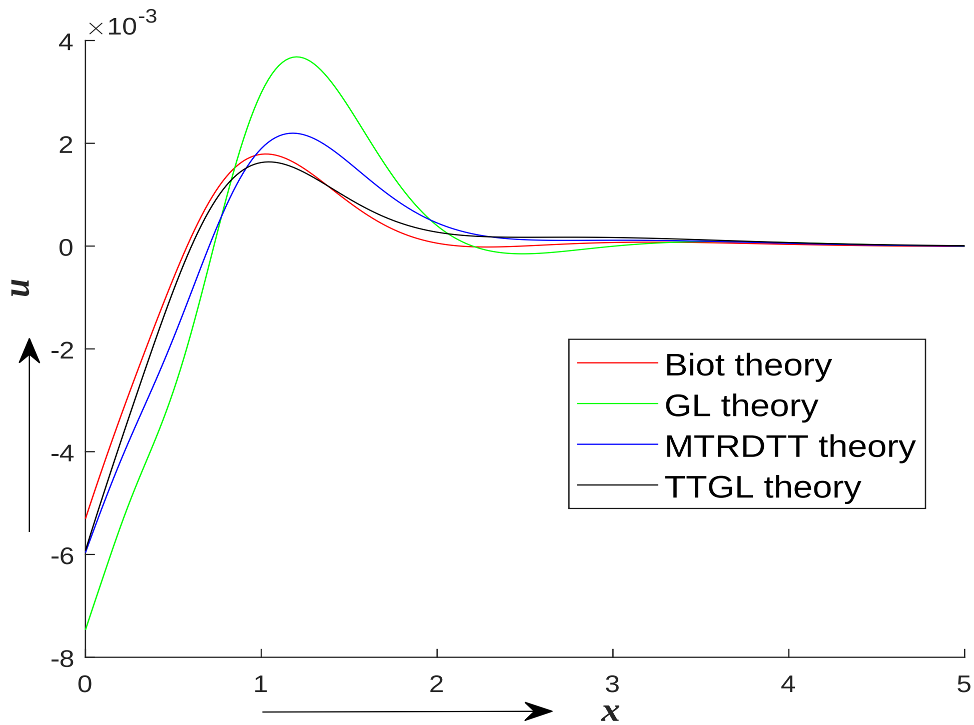


(b) Variation of θ along x at $t = 0.45$

Figure 4.1.2: Variation of thermodynamic temperature (θ) at $t_1 = 0.5$ and $a_1 = 0.5$

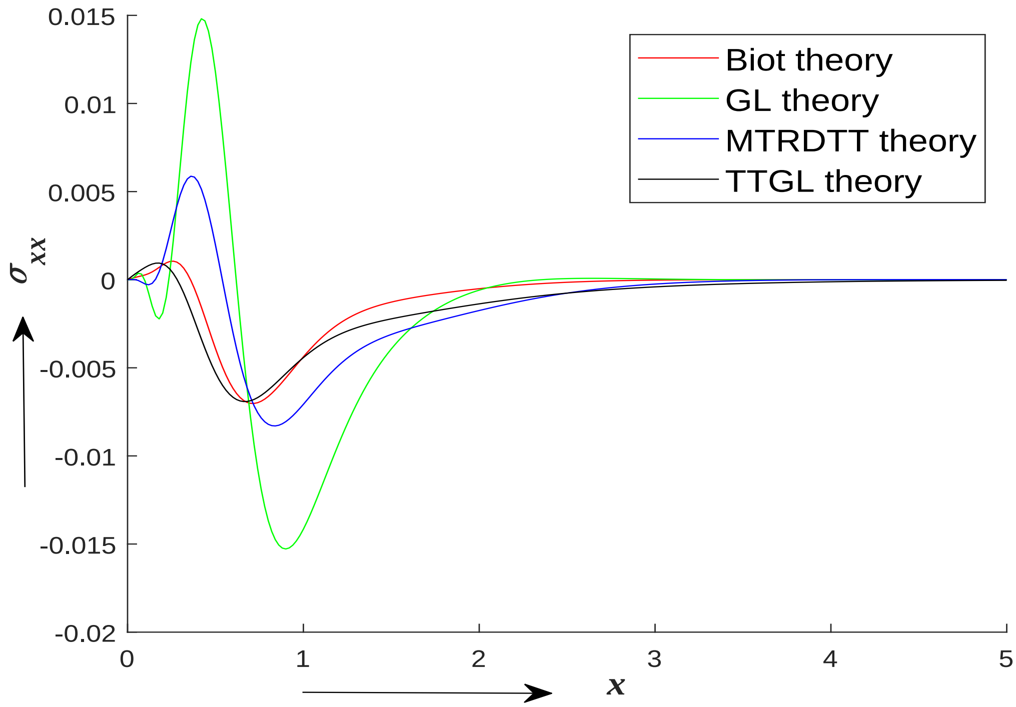


(a) Variation of u along x at $t = 0.25$

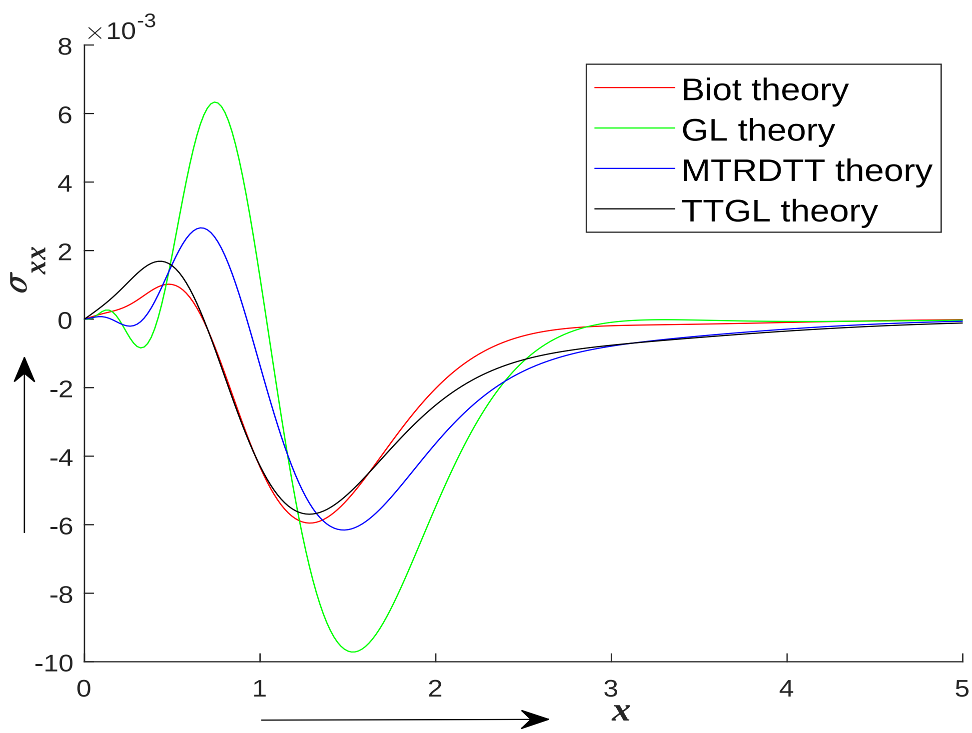


(b) Variation of u along x at $t = 0.45$

Figure 4.1.3: Variation of displacement (u) at $t_1 = 0.5$ and $a_1 = 0.5$

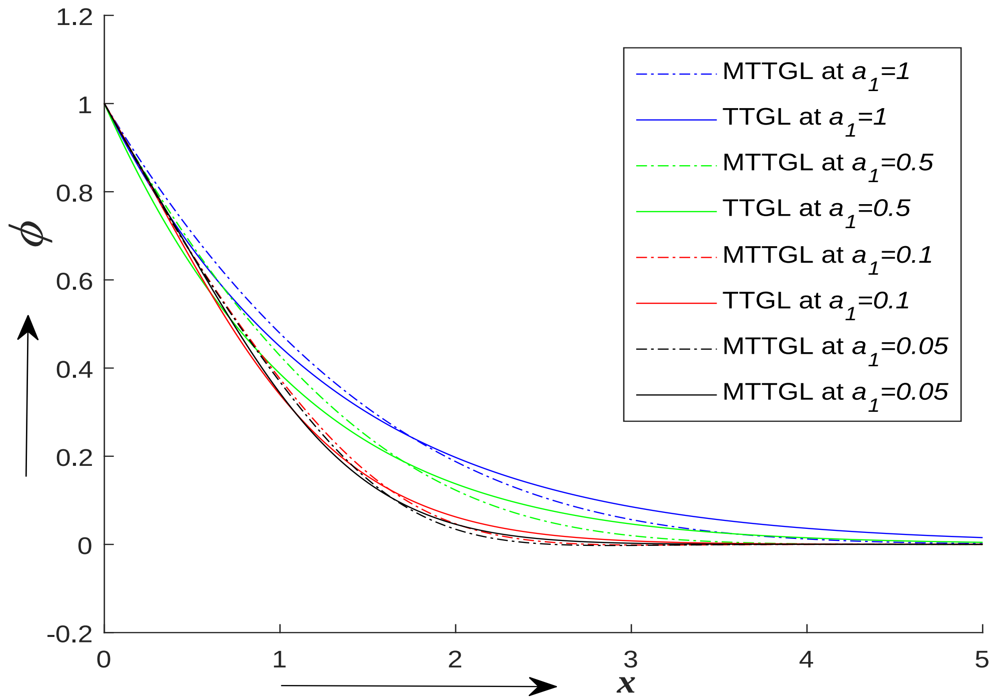


(a) Variation of σ_{xx} along x at $t = 0.25$

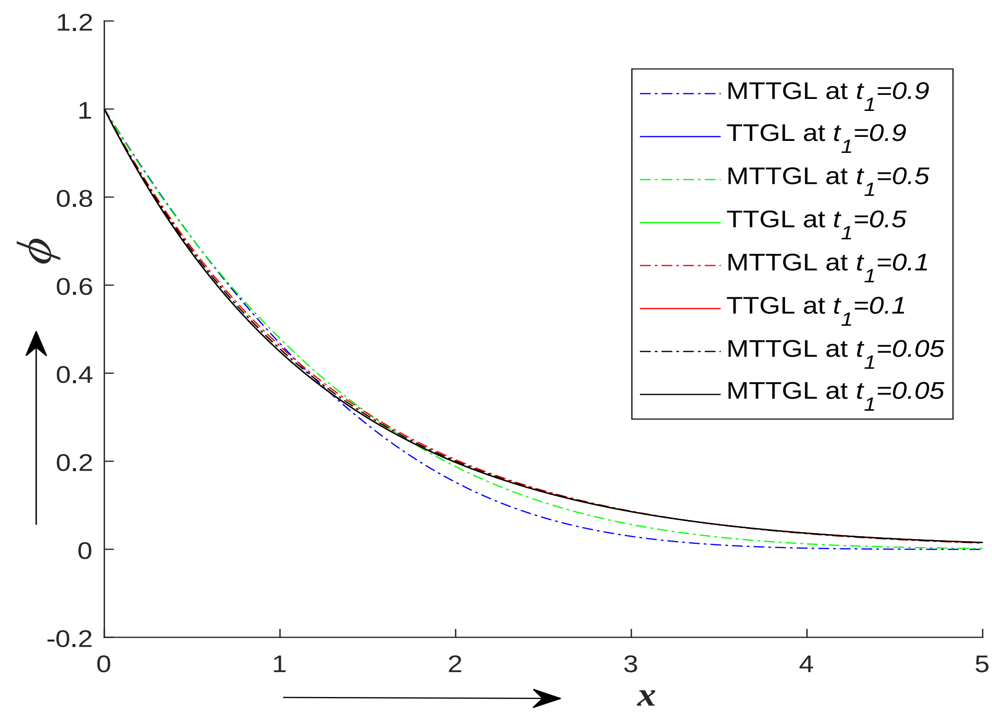


(b) Variation of σ_{xx} along x at $t = 0.45$

Figure 4.1.4: Variation of stress (σ_{xx}) at $t_1 = 0.5$ and $a_1 = 0.5$

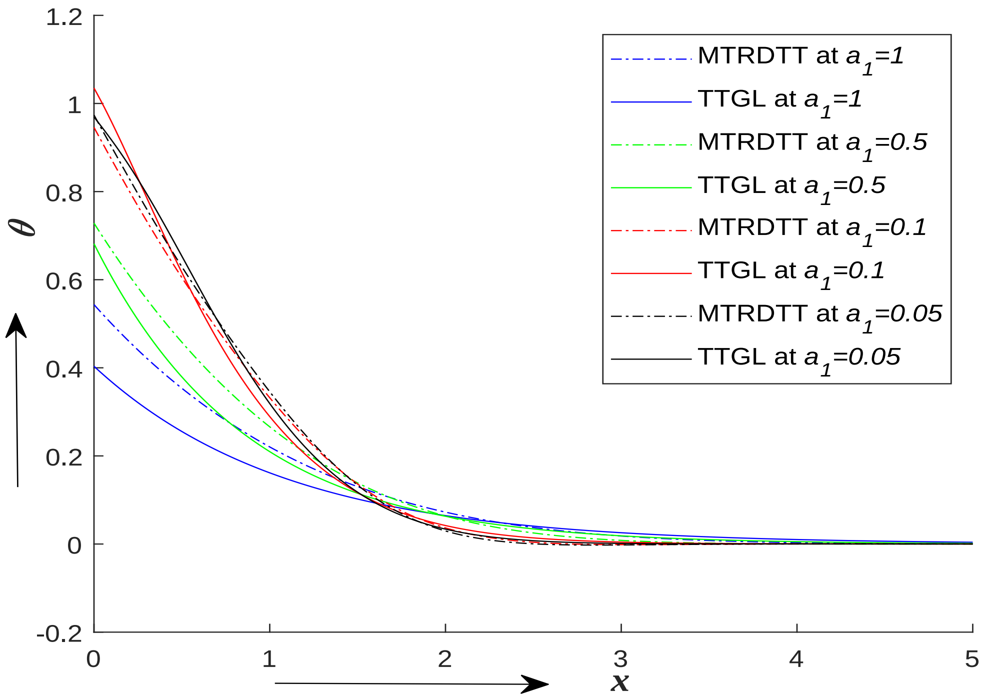


(a) Variation of ϕ along x at $t = 0.30$, $a_1 = 1$

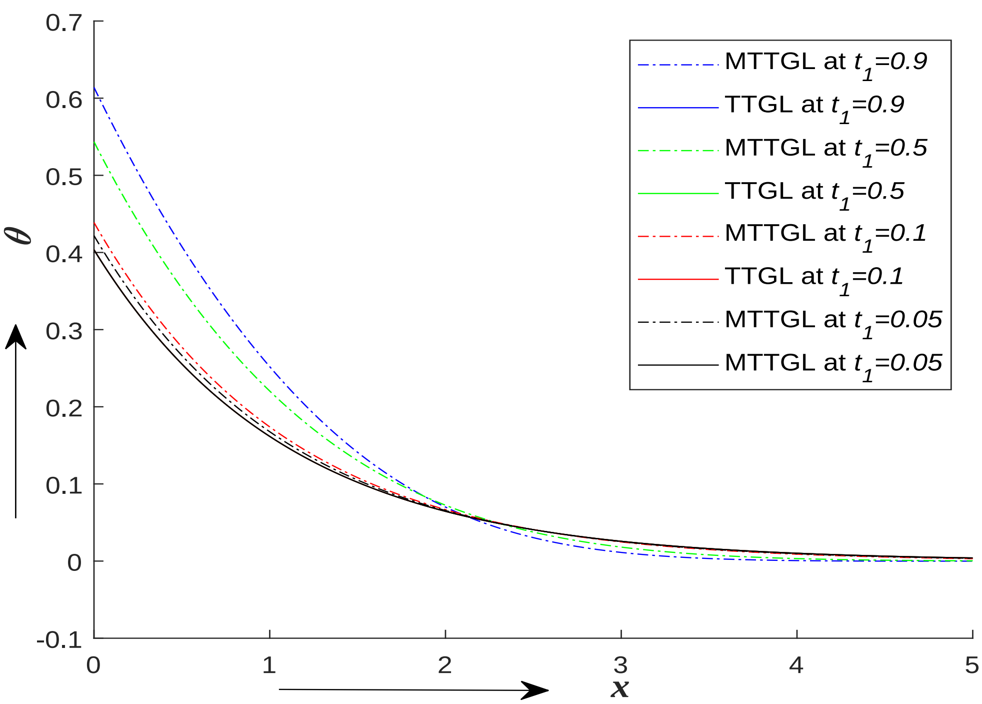


(b) Variation of ϕ along x at $t = 0.30$, $t_1 = 0.5$

Figure 4.1.5: Distribution of conductive temperature (ϕ) at different t_1 and a_1

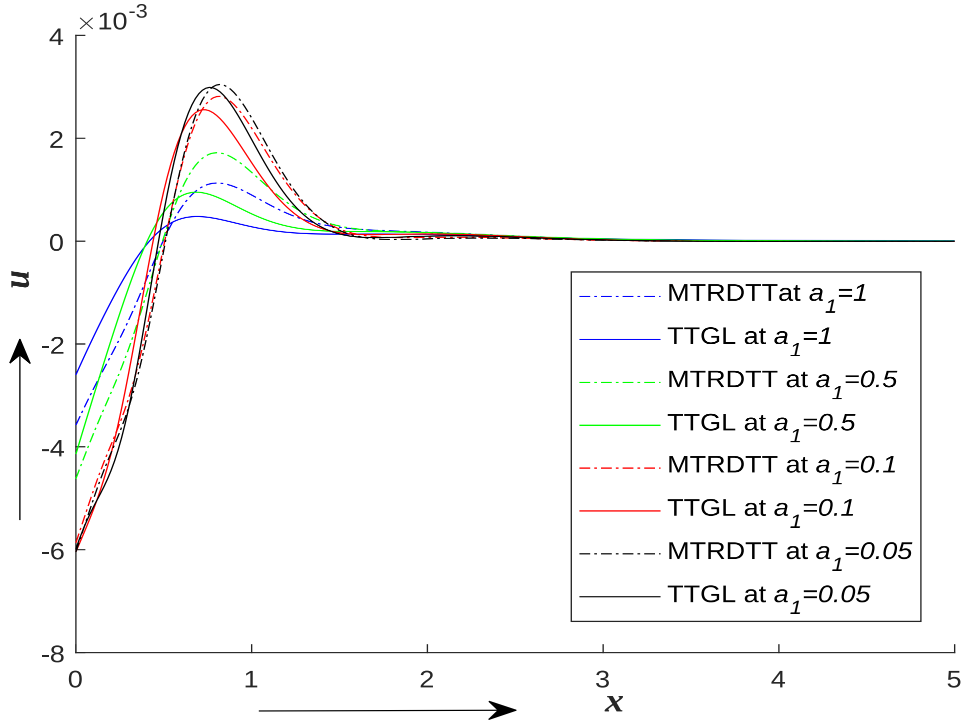


(a) Variation of θ along x at $t = 0.30$, $a_1 = 1$

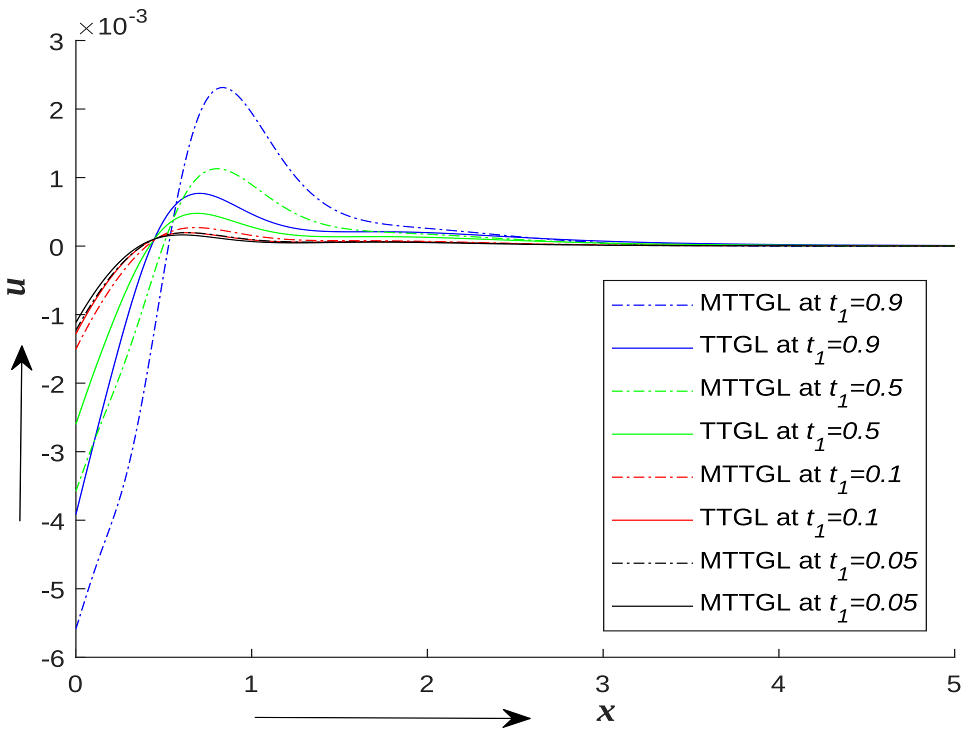


(b) Variation of θ along x at $t = 0.30$, $t_1 = 0.5$

Figure 4.1.6: Distribution of thermodynamic temperature (θ) at different t_1 and a_1

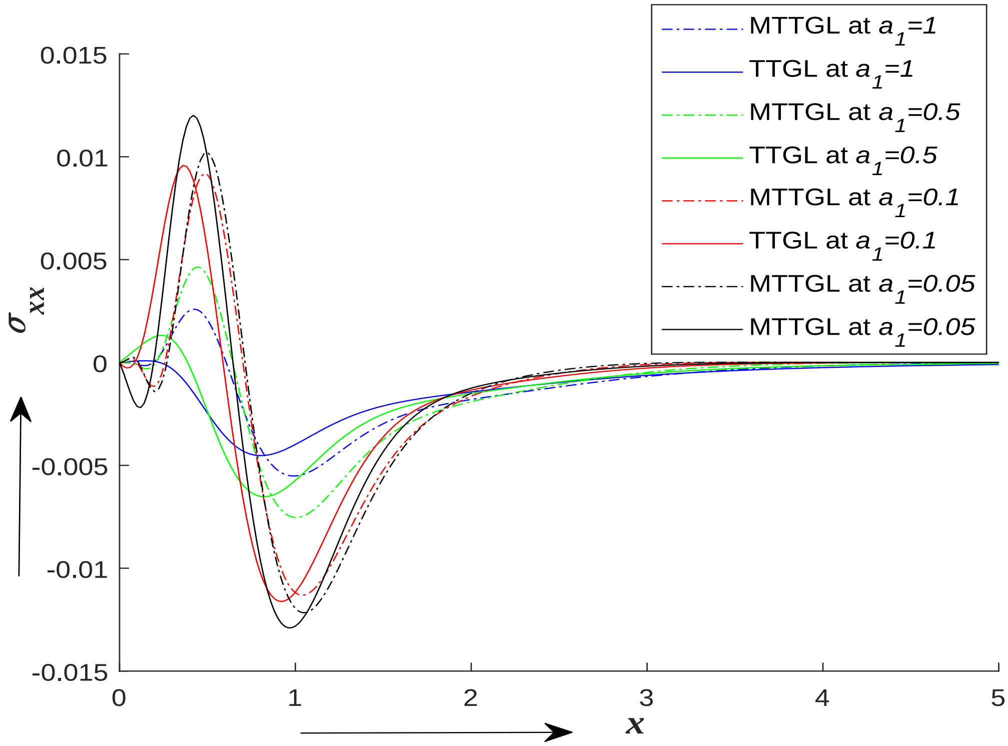


(a) Variation of u along x at $t = 0.30$, $t_1 = 0.5$

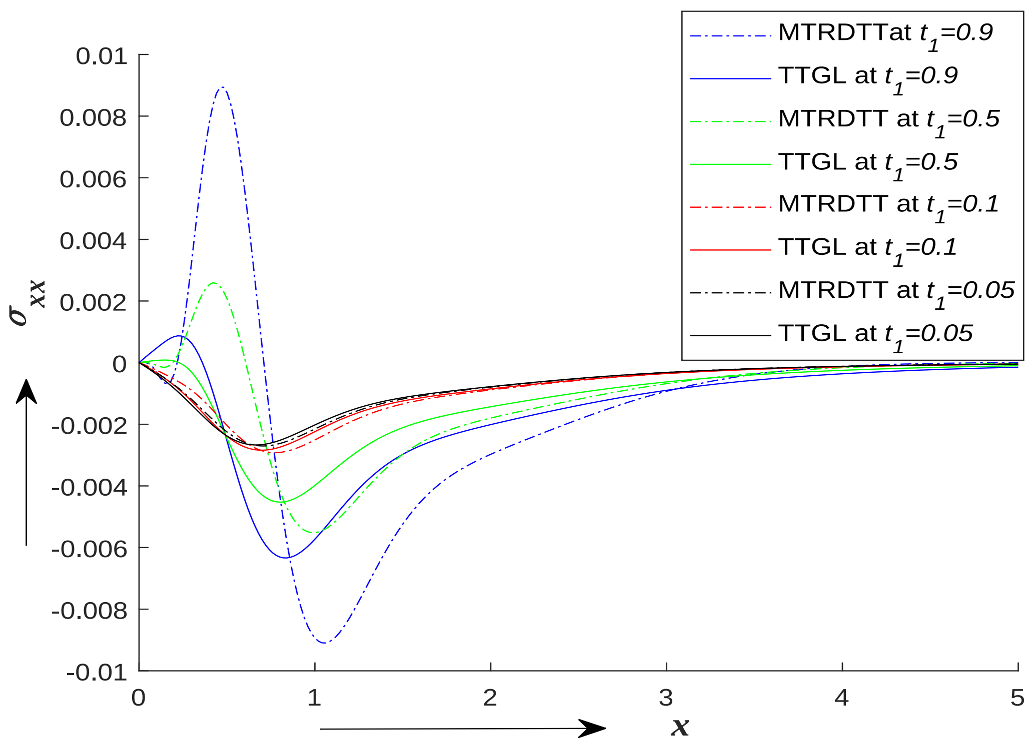


(b) Variation of u along x at $t = 0.30$, $a_1 = 1$

Figure 4.1.7: Distribution of displacement (u) at different t_1 and a_1



(a) Variation of σ_{xx} along x at $t = 0.30$, $t_1 = 0.5$



(b) Variation of σ_{xx} along x at $t = 0.30$, $a_1 = 1$

Figure 4.1.8: Distribution of stress (σ_{xx}) at different t_1 and a_1

Further, we analyze the MTRDTT theory and observe that for the larger values of a_1, the differences between MTRDTT and TTGL theories are much significant. However, with decreasing values of a_1, the plots under these two two-temperature theories get closer and coincide with the plots under GL theory at $a_1 = 0$. It is also noted that at any particular time and for any particular value of a_1, the domain of influence is larger for MTRDTT theory as compared to TTGL theory, and with the decrease in the values of a_1, the domain of influence for both theories are decreasing, with a higher rate for TTGL theory and slightly lower rate for MTRDTT theory.

Figures 4.1.5 (b)-4.1.8 (b) are plotted to show the effect of thermal relaxation time (t_1) on the two-temperature theories and it is discovered that for the small values of t_1, the difference between MTRDTT theory, and TTGL theory is not much significant but, in case of the materials having higher values of t_1, the difference is prominent.

4.1.8 Conclusion

In this work, we have formulated a modified temperature-rate dependent two-temperature (MTRDTT) theory of thermoelasticity based on the generalized first and second laws of thermodynamics. Further, we derive a more general two-temperature relation that involves the temperature-rate terms of both the temperatures. We observe that this relation is different from the two-temperature relation reported earlier in the literature. We have established a uniqueness theorem for a general mixed initial boundary value problem based on the present formulation. We have also discussed two particular cases and obtained the modified sets of constitutive equations. Further, the effect of this modification is examined for a half space problem. It has been highlighted that the effect of the present modified theory indicates a clear difference in numerical results for different field variables, and the modified two-temperature relation needs to be incorporated in the two-temperature theory that is existing in the literature. It has been observed that the modified theory shows significantly different results for the materials having larger values of relaxation parameters. It is believed that the current work may be useful to the researchers working in the concerned area.

4.2 Thermomechanical Interactions Due to Mode-I Crack under Modified Temperature-Rate Dependent Two-Temperature Thermoelasticity Theory [2]

4.2.1 Introduction

The aim of this subchapter is to investigate the thermomechanical interactions due to mode-I crack under modified TRDTT theory. In the previous subchapter, the modified TRDTT theory has been successfully established from firm grounds of irreversible thermodynamics. The implementation of this theory is also shown for the half space problem. Now, it is worth studying study another thermoelastic problem under this modified TRDTT theory.

The crack and failure of solids have a wide range of applications to the various structural engineering fields, like geophysics and earthquake engineering. Most industrial manufacturing may involve cracks in structure due to thermal or mechanical loading. Furthermore, under the frequently applied loadings, the crack may grow in fatigue to a final fracture. There are three basic types of crack problems based on three different modes of displacement (i.e. mode-I, II and III), which are mostly used in the applications as discussed above. For the first time, Griffith (1921) has investigated the crack in a 2D thermoelastic body. Irwin (1962) have studied a Mode-I crack problem in the symmetric opening in the displacement field normal to the fracture surface of the medium. Sherief and El-Maghraby (2005), Prasad et al. (2013), Kant et al. (2018), Lotfy and Yahia (2013) and Sur and Mondal (2020b) have investigated the mode-I crack problem for different thermoelasticity theories. However, the study of a crack problem under the two-temperature theory has not been reported in the literature.

Therefore, this subchapter aims to analyze the modified TRDTT (MTRDTT) theory for a dynamical problem of a 2D homogeneous and isotropic medium that involves a

[2] The content of this subchapter is communicated to an international journal.

mode-I crack. First, the unified governing equations are derived for the TRDTT and TRD theories corresponding to the present context. Then, the Laplace and Fourier transform techniques are used to derive the linear set of differential equations in the Laplace and Fourier transform domain. Further, a direct approach is used to obtain the solution in the Laplace and Fourier transform domain. The boundary of the crack is assumed to be under the action of time dependent loadings which are used to obtain the two sets of dual integral equations, which are further manipulated to reduce them to one dual integral equation. Then the regularization method is applied to solve this dual integral equation. Lastly, we use numerical inversion of the Laplace transform method to obtain the solution in the real time domain. All the field variables are presented graphically under both MTRDTT and TRD theories. We discuss the behaviour of the physical field variables inside the medium and find significant effects in the area near the crack. The behaviour of stress fields are observed to be most affected due to the loading and presence of the crack.

4.2.2 Basic Governing Equations

From the previous subchapter and following Green and Lindsay (1972), the unified basic governing equations of the temperature-rate dependent two-temperature theory (MTRDTT) and the temperature-rate dependent theory (TRD) for the homogeneous and isotropic medium can be presented as

The equation of motion:

$$\sigma_{ji,j} + \rho b_i = \rho \ddot{u}_i. \tag{4.2.1}$$

Stress-strain temperature relation:

$$\sigma_{ij} = (\lambda + \mu)e_{ij} + \mu e_{kk}\delta_{ij} - \beta(\theta + t_1\dot{\theta}). \tag{4.2.2}$$

Energy equation:

$$q_{i,i} = -\rho T_0 \dot{S} + \rho Q. \tag{4.2.3}$$

Entropy equation:

$$\rho T_0 S = \rho c_E(\theta + t_2\dot{\theta}) + \beta e_{kk}. \tag{4.2.4}$$

The heat conduction law:

$$q_i = K\phi_{,i}. \tag{4.2.5}$$

The two-temperature relation:

$$(\theta + p^* t_1\dot{\theta}) - (\phi + p^* t_1\dot{\phi}) = a\phi_{,ii}. \tag{4.2.6}$$

where θ and ϕ are denoting thermodynamic temperature and conductive temperature above reference temperature (T_0), respectively.

Further, the geometrical relation can be given as

$$e_{ij} = \frac{u_{i,j} + u_{j,i}}{2}.$$

The parameter p^* is used as a unifying parameter. By setting the different values of this parameter, above set of Eqs. (4.2.1-4.2.6) reduce to the corresponding equations under different theories as following:

- TRD theory: $p^* = a = 0$

- MTRDTT theory: $p^* = 1, a \neq 0$.

Now, by eliminating q_i and S from the Eqs. (4.2.3-4.2.5), we obtain the unified heat conduction equation as

$$K\phi_{,ii} = \rho c_E(\dot{\theta} + t_2\ddot{\theta}) + \beta T_0 \dot{e}_{kk}. \tag{4.2.7}$$

Also, from the Eqs. (4.2.1-4.2.2), the equation of motion can be obtained in terms of displacement and temperature as

$$(\lambda + \mu)u_{i,jj} + \mu u_{j,ij} - \beta(\theta_{,i} + t_1\dot{\theta}_{,i}) + \rho b_i = \rho\ddot{u}_i. \tag{4.2.8}$$

4.2.3 Formulation of the Problem

Let us consider a two-dimensional infinite thermoelastic body placed on XY plane (i.e., $-\infty < x < \infty$, $-\infty < y < \infty$) with a opening mode crack (mode-I crack) on x axis, at $|x| \leq r, y = 0$ due to applied thermal and mechanical loading on the surface as described in Fig. 4.2.1.

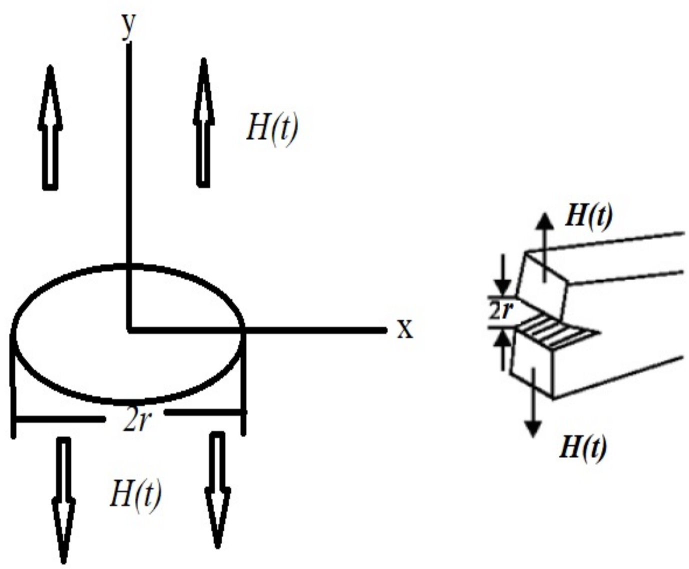


Figure 4.2.1: Geometry of the crack

Therefore, from the Eqs. (4.2.7-4.2.8), we derive the heat conduction equation and the equation of motion in the present context as

$$K\Delta\phi = \rho c_E(\dot{\theta} + t_2\ddot{\theta}) + \beta T_0 \dot{e}, \tag{4.2.9}$$

$$\rho\ddot{u} = (\lambda + \mu)\frac{\partial e}{\partial x} + \mu\Delta u - \beta\frac{\partial}{\partial x}(\theta + t_1\dot{\theta}), \tag{4.2.10}$$

$$\rho\ddot{v} = (\lambda + \mu)\frac{\partial e}{\partial y} + \mu\Delta v - \beta\frac{\partial}{\partial y}(\theta + t_1\dot{\theta}). \tag{4.2.11}$$

Here Δ denotes the Laplacian operator, u and v are the components of displacement along the x and y axis, respectively.

Also, form the Eqs. (4.2.2) and (4.2.6), we obtain the nonzero stress components and two-temperature relation as

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda e - \beta(\theta + t_1\dot{\theta}), \tag{4.2.12}$$

$$\sigma_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda e - \beta(\theta + t_1\dot{\theta}), \tag{4.2.13}$$

$$\sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{4.2.14}$$

$$\left(1 + t_1\frac{\partial}{\partial t}\right)(\theta - \phi) = a\Delta\phi. \tag{4.2.15}$$

Further, we can write the geometrical relation as

$$e = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{4.2.16}$$

Now, we derive non-dimensional form of Eqs. (4.2.9-4.2.15) by considering the following non-dimensional parameters

$(x', y', u', v') = c_1\eta(x, y, u, v)$, $(t', t_1', t_2') = c_1^2\eta(t, t_1, t_2)$, $(\sigma_{xx}', \sigma_{yy}', \sigma_{xy}') = \frac{1}{\mu}(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$, and $(\theta', \phi') = \frac{1}{T_0}(\theta, \phi)$ where $\eta = \frac{\rho c_E}{K}$ and $c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}$.

Therefore, in view of Eqs. (4.2.9-4.2.15) along with the above non-dimensional parameters, we have the corresponding system of equations in terms of non-dimensional variables in the forms

$$\Delta\phi = \left(1 + t_2\frac{\partial}{\partial t}\right)\dot{\theta} + b_1\dot{e}, \tag{4.2.17}$$

$$p^2\ddot{u} = (p^2 - 1)\frac{\partial e}{\partial x} + \Delta u - b_2\frac{\partial}{\partial x}(\theta + t_1\dot{\theta}), \tag{4.2.18}$$

$$p^2\ddot{v} = (p^2 - 1)\frac{\partial e}{\partial y} + \Delta v - b_2\frac{\partial}{\partial y}(\theta + t_1\dot{\theta}), \tag{4.2.19}$$

$$\sigma_{xx} = 2\frac{\partial u}{\partial x} + (p^2 - 1)e - b_2(\theta + t_1\dot{\theta}), \tag{4.2.20}$$

$$\sigma_{yy} = 2\frac{\partial v}{\partial y} + (p^2 - 1)e - b_2(\theta + t_1\dot{\theta}), \tag{4.2.21}$$

$$\sigma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{4.2.22}$$

$$\left(1 + p^*t_1\frac{\partial}{\partial t}\right)(\theta - \phi) = a_1\Delta\phi, \tag{4.2.23}$$

where $b_1 = \frac{\beta}{K\eta}$, $b_2 = \frac{\beta T_0}{\mu}$, $p^2 = \frac{\lambda+2\mu}{\mu}$ and $a_1 = ac_1^2\eta^2$.

Initial and boundary conditions

For present system of partial differential equations, we consider initial conditions as

$$\begin{aligned} u(x,y,0) &= \dot{u}(x,y,0) = 0 \\ v(x,y,0) &= \dot{v}(x,y,0) = 0 \\ \theta(x,y,0) &= \dot{\theta}(x,y,0) = 0 \\ \phi(x,y,0) &= \dot{\phi}(x,y,0) = 0 \end{aligned}, \tag{4.2.24}$$

and the boundary conditions at $y = 0$ are taken as

$$\frac{\partial \phi(x,0,t)}{\partial y} = 0, \qquad |x| > r,\ t > 0, \tag{4.2.25}$$

$$v(x,0,t) = 0, \qquad |x| > r,\ t > 0, \tag{4.2.26}$$

$$\phi(x,0,t) = (x^2 - r^2/2)H(t), \qquad |x| < r,\ t > 0, \tag{4.2.27}$$

$$\sigma_{yy}(x,0,t) = H(t), \qquad |x| < r,\ t > 0, \tag{4.2.28}$$

$$\sigma_{xy}(x,0,t) = 0, \qquad -\infty < x < \infty,\ t > 0. \tag{4.2.29}$$

4.2.4 Solution of the Problem in the Laplace and Fourier Transform Domain

Now, in order to solve the present problem, we apply firstly the Laplace transform on the system of partial differential Eqs. (4.2.17-4.2.23) by using the homogeneous initial conditions (4.2.24). Therefore, we have

$$\Delta\bar{\phi} = s\,(1 + t_2 s)\,\bar{\theta} + b_1 s\bar{e}, \tag{4.2.30}$$

$$s^2p^2\bar{u} = (p^2 - 1)\frac{\partial \bar{e}}{\partial x} + \Delta\bar{u} - b_2(1 + t_1 s)\frac{\partial \bar{\theta}}{\partial x}, \tag{4.2.31}$$

$$s^2p^2\bar{v} = (p^2 - 1)\frac{\partial e}{\partial y} + \Delta v - b_2(1 + t_1 s)\frac{\partial \bar{\theta}}{\partial y}, \tag{4.2.32}$$

$$\bar{\sigma}_{xx} = 2\frac{\partial \bar{u}}{\partial x} + (p^2 - 1)\bar{e} - b_2(1 + t_1 s)\bar{\theta}, \tag{4.2.33}$$

$$\bar{\sigma}_{yy} = 2\frac{\partial \bar{v}}{\partial y} + (p^2 - 1)\bar{e} - b_2(1 + t_1 s)\bar{\theta}, \tag{4.2.34}$$

$$\bar{\sigma}_{xy} = \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right), \tag{4.2.35}$$

$$(\bar{\phi} - \bar{\theta}) = a_2 \Delta \bar{\phi}, \tag{4.2.36}$$

where $a_2 = a_1/(1 + t_1 s p^*)$, s being the Laplace transform parameter.

Eliminating $\bar{u}$ and $\bar{v}$ from the Eqs. (4.2.31-4.2.32), we find that

$$(\Delta - p^2)\bar{e} = b_3 \Delta \bar{\theta}, \tag{4.2.37}$$

where $b_3 = \frac{b_2(1+t_1 s)}{p^2}$.

Next, by using the Eqs. (4.2.36) in Eqs. (4.2.30) and (4.2.37) , we obtain

$$[(1 + a_2(s + t_1 s^2))\Delta - (s + t_2 s^2)]\bar{\phi} = b_1 s \bar{e}, \tag{4.2.38}$$

$$(\Delta - s^2)\bar{e} = b_3(1 - a_2\Delta)\Delta\bar{\phi}. \tag{4.2.39}$$

Further, elimination of $\bar{e}$ from the Eqs. (4.2.38-4.2.39) yields the bi-quadratic partial differential equation satisfied by $\bar{\phi}$ as

$$\left[\Delta^2 - \frac{s^2 + s^2(1 + a_2)(s + t_2 s^2) + b_1 b_3 s}{1 + a_2(s + t_2 s^2) + b_1 b_3 s a_2}\Delta + \frac{s^3(1 + t_2 s)}{1 + a_2(s + t_2 s^2) + b_1 b_3 s a_2}\right]\bar{\phi} = 0. \tag{4.2.40}$$

Let k_1^2 and k_2^2 be the roots of the auxiliary equation corresponding to Eq. (4.2.40), then Eq. (4.2.40) can be factorized as

$$(\Delta - k_1^2)(\Delta - k_2^2)\bar{\phi} = 0. \tag{4.2.41}$$

Therefore, we may assume that

$$\bar{\phi} = \bar{\phi}_1 + \bar{\phi}_2,$$

such that $\bar{\phi}_i$ is the root of equation

$$(\Delta - k_1^2)\bar{\phi}_j = 0, \quad \text{for } j = 1, 2. \tag{4.2.42}$$

Now, to solve the Eq. (4.2.42), we apply the exponential Fourier transform, where the Fourier transform of a function $g(x, y, s)$ can be defined as

$$F[g(x, y, s)] = \mathring{g}(w, y, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x, y, s) \exp(-iwx) dx, \tag{4.2.43}$$

where w is the Fourier transform parameter.

Also, the inverse Fourier transform of the function $\mathring{g}(w, y, s)$ can be given as

$$F^{-1}[\mathring{g}(x, y, s)] = g(x, y, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(w, y, s) \exp(-iwx) dw. \tag{4.2.44}$$

Thus, by applying the Fourier transform on Eq. (4.2.42) we have

$$[d^2 - (k_j^2 + w^2)]\mathring{\bar{\phi}}_j = 0, \quad j = 1, 2, \tag{4.2.45}$$

where $d = \frac{\partial}{\partial y}$.

Assuming that the field variables are bounded at infinity, the solution of the Eq. (4.2.45) can be written as

$$\mathring{\bar{\phi}}_j = A_j(w, s)(k_j^2 - s^2) \exp(-w_j|y|), \quad j = 1, 2,$$

where $w_j = \sqrt{(w^2 + k_j^2)}$ and $A_j(w, s)$ is the integration parameter independent from y. Due to symmetry of the problem about x axis, we consider the region $y > 0$. In this case

$$\mathring{\bar{\phi}}_j = A_j(w, s)(k_j^2 - s^2) \exp(-w_j y), \quad j = 1, 2, \tag{4.2.46}$$

Similarly, if we consider $\bar{e} = \bar{e}_1 + \bar{e}_2$, then the value of $\mathring{\bar{e}}_j$ can be obtained as

$$\mathring{\bar{e}}_j = A'_j(w, s) \exp(-w_j y), \quad j = 1, 2, \tag{4.2.47}$$

where $A'_j(w, s)$ is also independent from y.

Now, substituting the values from Eqs. (4.2.46-4.2.47) in Eq. (4.2.39), we have

$$A'_j(w, s) = b_3 k_j^2(1 - a_2 k_j^2) A_j(w, s), \quad j = 1, 2. \tag{4.2.48}$$

Using Eq. (4.2.48), in Eq. (4.2.47), we get

$$\mathring{\bar{e}}_j = b_3 k_j^2(1 - a_2 k_j^2) A_j(w, s) \exp(-w_j y), \quad j = 1, 2. \tag{4.2.49}$$

Now, applying the Fourier transform on Eq. (4.2.36) and using Eq. (4.2.48), we obtain

$$\mathring{\bar{\theta}} = \sum_{j=1}^{2} A_j(k_j^2 - s^2)(1 - a_2 k_j^2)\exp(-w_j y). \tag{4.2.50}$$

Further, applying the Fourier transform on Eqs. (4.2.31, 4.2.16) results

$$(d^2 - w^2 - p^2 s^2)\mathring{\bar{u}} = (1 - p^2)iw\mathring{\bar{e}} + b_3 iwp^2(1 - a_2(d^2 - w^2))\mathring{\bar{\phi}}, \tag{4.2.51}$$

$$\frac{\partial \mathring{\bar{v}}}{\partial y} = \mathring{\bar{e}} - iw\mathring{\bar{u}}. \tag{4.2.52}$$

Substituting the values of $\mathring{\bar{e}}$ and $\mathring{\bar{\phi}}$ in Eq. (4.2.51), we find that

$$(d^2 - w^2 - p^2 s^2)\mathring{\bar{u}} = iwb_3 \sum_{j=1}^{2}(1 - a_2 k_j^2)(k_i^2 - p^2 s^2)A_i \exp(-w_j y). \tag{4.2.53}$$

Therefore, the solution $\mathring{\bar{u}}$ of Eq. (4.2.53) takes the form

$$\mathring{\bar{u}} = iwb_3 \left(\sum_{j=1}^{2}(1 - a_2 k_j^2)A_j \exp(-w_j y) + L\exp(-\sqrt{w^2 + p^2 s^2}y) \right), \tag{4.2.54}$$

where $L = L(w, s)$ is the parameter independent from y.

Now, from the Eqs. (4.2.49,4.2.52) and (4.2.54), the value of $\mathring{\bar{v}}$ can be obtained as

$$\mathring{\bar{v}} = -b_3 \left[\sum_{j=1}^{2}(1 - a_2 k_j^2)w_j A_j \exp(-w_j y) + \frac{w^2 L}{\sqrt{w^2 + p^2 s^2}}\exp(-\sqrt{w^2 + p^2 s^2}y) \right]. \tag{4.2.55}$$

Taking the Fourier transform of Eqs. (4.2.33-4.2.35) and substituting the values of $\mathring{\bar{u}}, \mathring{\bar{v}}$ and $\mathring{\bar{\theta}}$, we have

$$\mathring{\bar{\sigma}}_{xx} = b_3 \left[\sum_{j=1}^{2}(1 - a_2 k_j^2)(p^2 s^2 - 2w_j^2)A_j \exp(-w_j y) - 2Lw^2 \exp(-\sqrt{w^2 + p^2 s^2}y) \right]. \tag{4.2.56}$$

$$\mathring{\bar{\sigma}}_{yy} = b_3 \left[\sum_{j=1}^{2}(1 - a_2 k_j^2)(p^2 s^2 + 2w_j^2)A_j \exp(-w_j y) + 2Lw^2 \exp(-\sqrt{w^2 + p^2 s^2}y) \right]. \tag{4.2.57}$$

$$\mathring{\bar{\sigma}}_{xy} = -iwb_3 \left[\sum_{j=1}^{2} 2A_j(1 - a_2 k_j^2)w_j \exp(-w_j y) + \frac{2w^2 + p^2 s^2}{\sqrt{w^2 + p^2 s^2}}L\exp(-\sqrt{w^2 + p^2 s^2}y) \right]. \tag{4.2.58}$$

Hence, the Eqs. (4.2.46,4.2.49-4.2.50,4.2.54-4.2.58) represent the solution of the present problem in the Laplace and Fourier transform domain where A_1, A_2 and L are the unknown parameters.

4.2.5 Formulation of the Dual Integral Equation

In this section, we firstly apply the boundary conditions on solutions obtained in the transformed domain to determine the values of unknown parameters A_1, A_2 and L. For this, we use the boundary conditions (4.2.25) and (4.2.27) in the Eq.(4.2.49) to have

$$\int_{-\infty}^{\infty}[A_1(k_1^2-s^2)+A_2(k_2^2-s^2)]\exp(iwx)dw=\frac{\sqrt{2\pi}}{s}\left(x^2-\frac{r^2}{2}\right),\quad |x|<r,\quad (4.2.59)$$

$$\int_{-\infty}^{\infty}[A_1w_1(k_1^2-s^2)+A_2w_2(k_2^2-s^2)]\exp(iwx)dw=0,\quad |x|>r.\qquad (4.2.60)$$

Further, using the boundary conditions (4.2.26,4.2.28) and (4.2.29) in the Eqs. (4.2.55, 4.2.57) and (4.2.58), we find that

$$\int_{-\infty}^{\infty}\left[A_1(1-a_2k_1^2)w_1+A_2(1-a_2k_2^2)w_2+\frac{Lw^2}{\sqrt{w^2+p^2s^2}}\right]\exp(iwx)dw=0,\quad |x|>r,\qquad (4.2.61)$$

$$\int_{-\infty}^{\infty}[A_1(1-a_2k_1^2)(p^2s^2+2w_1^2)+A_2(1-a_2k_2^2)(p^2s^2+2w_2^2)\qquad (4.2.62)$$
$$+2Lw^2]\exp(iwx)dw=-\frac{\sqrt{2\pi}}{b_3s},\quad |x|<r,$$

$$\int_{-\infty}^{\infty}\big[2A_1(1-a_2k_1^2)w_1+2A_2(1-a_2k_2^2)w_2\qquad (4.2.63)$$
$$+\frac{2w^2+p^2s^2}{\sqrt{w^2+p^2s^2}}L\Bigg]\exp(iwx)dw=0,\quad -\infty<x<\infty.$$

From the Eq. (4.2.63), we get

$$L(w,s)=-\frac{2\left[A_1(1-a_2k_1^2)w_1+A_2(1-a_2k_2^2)w_2\right]\sqrt{w^2+p^2s^2}}{2w^2+p^2s^2}.\qquad (4.2.64)$$

Again, due to symmetry of the problem we may consider the region $x>0$. Therefore, in view of Eq. (4.2.64), the Eqs. (4.2.59-4.2.62) can be written as

$$\sum_{j=1}^{2}\int_0^{\infty}A_j(k_j^2-s^2)\cos(wx)dw=\sqrt{\frac{\pi}{2}}\frac{1}{s}\left(x^2-\frac{r^2}{2}\right),\quad 0<x<r,\qquad (4.2.65)$$

$$\sum_{j=1}^{2}\int_0^\infty A_j(k_j^2-s^2)\cos(wx)dw=0,\quad x>r, \tag{4.2.66}$$

$$\sum_{j=1}^{2}\int_0^\infty A_j\frac{(1-a_2k_j^2)\left((p^2s^2+2w_j^2)^2-4w^2w_j\sqrt{w^2+p^2s^2}\right)}{2w^2+p^2s^2}\cos(wx)dw \tag{4.2.67}$$

$$=-\sqrt{\frac{2}{\pi}}\frac{1}{sb_3},\ \ 0<x<r,$$

$$\sum_{j=1}^{2}\int_0^\infty A_j\frac{w_j(1-a_2k_j^2)}{2w^2+p^2s^2}\cos(wx)dw=0,\quad x>r. \tag{4.2.68}$$

Thus, the Eqs. (4.2.65-4.2.68) represent the two sets of dual integral equations and the solution of these integral equations are the unknown parameters A_1 and A_2. Now, in order to solve these dual integral equations, we take the suitable assumption on A_i in the interval $(0,r)$ as

$$A_j=\int_0^r z_j(c,s)J_0(wc)dc,\quad x<r \text{ and } j=1,2, \tag{4.2.69}$$

where $z_j(c,s)$, $j=1,2$ are the functions of c and s only and $J_n(x)$ denotes the Bessel function of the first kind of order n.

Substituting the values of A_j in Eq. (4.2.65) and using the integral relation (see refs. Watson (1996); Mandal (1999))

$$\int_0^\infty \cos(wx)J_0(wv)dw=\begin{cases}\frac{1}{\sqrt{c^2-x^2}}, & \text{if } x<v,\\ 0, & \text{otherwise}\end{cases},$$

we find that

$$\sum_{j=1}^{2}(k_j^2-s^2)\int_0^r\frac{z_j(f,s)}{\sqrt{f^2-x^2}}df=\sqrt{\frac{\pi}{2}}\frac{1}{s}\left(x^2-\frac{r^2}{2}\right),\quad 0<x<r.$$

Now, multiplying by $\frac{x}{\sqrt{x^2-c^2}}$ on the both sides of above equation and integrating w.r.t x from c to r by changing order of integration and further differentiating w.r.t. c, we obtain

$$(k_1^2 - s^2)z_1(c,s) + (k_2^2 - s^2)z_2(c,s) = -\frac{1}{\sqrt{2\pi}}\frac{c(3r^2-4c^2)}{s\sqrt{r^2-c^2}}, \quad 0 < x < r. \qquad (4.2.70)$$

Again, multiplying by $J_0(wc)$ on the both sides of Eq. (4.2.70) and integrating from 0 to r, we find the relations between A_1 and A_2 as

$$A_2 = -\frac{1}{k_2^2 - s^2}\left[\frac{R(w)}{s} + (k_1^2 - s^2)A_1\right], \quad 0 < x < r, \qquad (4.2.71)$$

where $R(w) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{c(3r^2-4c^2)}{s\sqrt{r^2-c^2}}J_0(cw)dc$.

Now, in order to obtain the relation between A_1 and A_2 for the interval (r,∞), we consider

$$A_j = \frac{1}{w_i}\int_0^\infty z_j(c,s)J_0(wc)dc, \quad x > r \text{ and } j = 1,2. \qquad (4.2.72)$$

By using the same argument in the Eqs. (4.2.66) and (4.2.72), we find the relation

$$A_2 = -\frac{(k_1^2 - s^2)w_1}{(k_2^2 - s^2)w_2}A_1, \quad x > r. \qquad (4.2.73)$$

Next, we substitute the values of A_2 from Eqs. (4.2.71) and (4.2.73) in the Eqs. (4.2.67-4.2.68) to get

$$\int_0^\infty \frac{A_1 w_1 T(w,s)}{p^2s^2 + 2w^2}\cos(wx)dw = R_1(x,s), \quad 0 < x < r, \qquad (4.2.74)$$

$$\int_0^\infty \frac{A_1 w_1}{p^2s^2 + 2w^2}\cos(wx)dw = 0, \quad x > r, \qquad (4.2.75)$$

where

$$T(w,s) = [(k_2^2 - k_1^2)(1 - a_2s^2)(p^2s^2 + 2w^2) - 4w^2\sqrt{w^2 + p^2s^2}\{w_1(k_2^2 - s^2)(1 - a_2k_1^2), \qquad (4.2.76)$$

$$- w_2(k_1^2 - s^2)(1 - a_2k_2^2)\}]/w_1$$

$$R_1(x,s) = -\sqrt{\frac{2}{\pi}}\frac{(k_2^2 - s^2)}{sb_3} \qquad (4.2.77)$$

$$+ \frac{1}{s}\int_0^\infty R(w)\frac{(1 - a_2k_2^2)[(p^2s^2 + 2w^2)^2 - 4w^2w_2\sqrt{w^2 + p^2s^2}]}{(p^2s^2 + 2w^2)}\cos(wx)dw.$$

Hence, we have simplified the two sets of dual integral equations and transformed them into a single set of dual integral Eqs. (4.2.74-4.2.75). In the next section, we will discuss the method of solution of this single set of dual integral equations.

4.2.6 Solution of the Dual Integral Equations

In order to simplify the Eqs. (4.2.74-4.2.75), we substitute

$$A_1(w,s) = \frac{(p^2s^2+2w^2)}{q_1}\gamma(w,s). \tag{4.2.78}$$

Therefore, the dual integral Eqs.(4.2.74-4.2.75) take the form

$$\int_0^\infty \gamma(w,s)T(w,s)\cos(wx)dw = R_1(x,s), \quad 0<x<r, \tag{4.2.79}$$

$$\int_0^\infty \gamma(w,s)\cos(wx)dw = 0, \qquad x>r. \tag{4.2.80}$$

Now, to solve this dual integral Eq. (4.2.79-4.2.80), we combine these two equations into a single integral equation of the first kind. Therefore, we extent the definition of the integral in the left hand side of Eq. (4.2.80) for all $x>0$ such that

$$\int_0^\infty \gamma(w,s)\cos(wx)dw = \begin{cases} \sqrt{2\pi}\frac{d}{dx}\left(x\int_x^r \frac{\psi(h,s)}{\sqrt{h^2-s^2}}dh\right), & 0<x<r \\ 0, & x>r \end{cases}, \tag{4.2.81}$$

where $\psi(h,s)$ is a unknown function.

Clearly, the L.H.S. of Eq. (4.2.81) can be taken as the Fourier cosine transform of the function $\gamma(w,s)$. Therefore, by using the inverse cosine transform on both sides of Eq. (4.2.81), we obtain

$$\gamma(w,s) = \int_0^r \frac{d}{dx}\left(x\int_x^r \frac{\psi(h,s)dh}{\sqrt{h^2-s^2}}\cos(wx)dx\right). \tag{4.2.82}$$

By changing the order of integration in Eq. (4.2.82), we get

$$\gamma(w,s) = w\int_0^r \psi(h,s)\left(\int_0^h \frac{x\sin(wx)}{\sqrt{h^2-x^2}}dx\right)dh, \tag{4.2.83}$$

which implies that

$$\gamma(w,s) = \frac{\pi w}{2}\int_0^r h\psi(h,s)J_1(wx)dh. \tag{4.2.84}$$

Now, to determine the value of $\psi(h, s)$, we substitute the value of $\gamma(w, s)$ in Eq. (4.2.79) and find that

$$\int_0^r T_1(h, x, s)\psi(h, s)dh = R_1(x, s), \quad x < r, \tag{4.2.85}$$

where

$$T_1(h, x, s) = \int_0^\infty wT(w, s)J_1(wh)\cos(wx)dw.$$

Hence, we obtain the Fredholm integral equation of the first kind given by Eq. (4.2.83) in the unknown function $\psi(h, s)$. The function $\psi(h, s)$ will be obtained by solving the integral Eq. (4.2.85) numerically, and therefore from Eq. (4.2.84), we find the numerical values of γ by applying the numerical integration technique. Thus by substituting these values in Eq. (4.2.78), $A_1(w, s)$ is obtained. Thereafter the values of A_2 can be obtained from the Eqs. (4.2.71) and (4.2.73) in the intervals (0,r) and (r, ∞), respectively. This completes the solution of problem in the Laplace transform domain.

4.2.7 Numerical Results and Discussion

In the previous section, we have derived the solution of the present crack problem in the Laplace transform domain in terms of the parameter A_1. Further, the values of A_1 can be obtained by solving the Fredholm integral Eq. (4.2.85). Here, we have followed a numerical method for solving a Fredholm integral equation as outlined in the *Appendix* (see also Delves and Mohamed (1985) and Sherief and El-Maghraby (2005)). Now, to obtain the solution in the real space-time domain, the inverse Laplace and Fourier transformations are to be applied on the solution obtained as (4.2.46,4.2.50,4.2.54-4.2.58) in the transformed domain. For, the Laplace inversion, we apply the numerical inversion of Laplace transformation technique as introduced by Bellman et al. (1966). To perform the numerical computation, we have developed a MATLAB code and formulate the numerical solution of the present crack problem. The numerical values of the material parameters are considered for the copper metal by following Sherief and

El-Maghraby (2005) as

$p = 2$, $\alpha_t = 1.78 \times 10^{-5}\,\mathrm{K}^{-1}$, $c_1 = 4.158 \times 10^3\,\mathrm{m\,s}^{-1}$, $c = 0.001$, $\rho = 8954\,\mathrm{Kg\,m}^{-3}$, $\lambda = 7.76{\times}10^{10}\,\mathrm{N\,m}^{-2}$, $\mu = 3.36{\times}10^{10}\,\mathrm{N\,m}^{-2}$, $c_E = 383.1\,\mathrm{J\,Kg}^{-1}\,\mathrm{K}^{-1}$, $T_0 = 293\,\mathrm{K}$,
$K = 386\,\mathrm{Kg\,m\,K}^{-1}\,\mathrm{s}^{-3}$,$b_1 = 0.042$.

The dimensionless values of two-temperature parameter and relaxation time parameters are taken as, $a_1 = 1$, $t_1 = 0.05$, $t_2 = 0.04$.

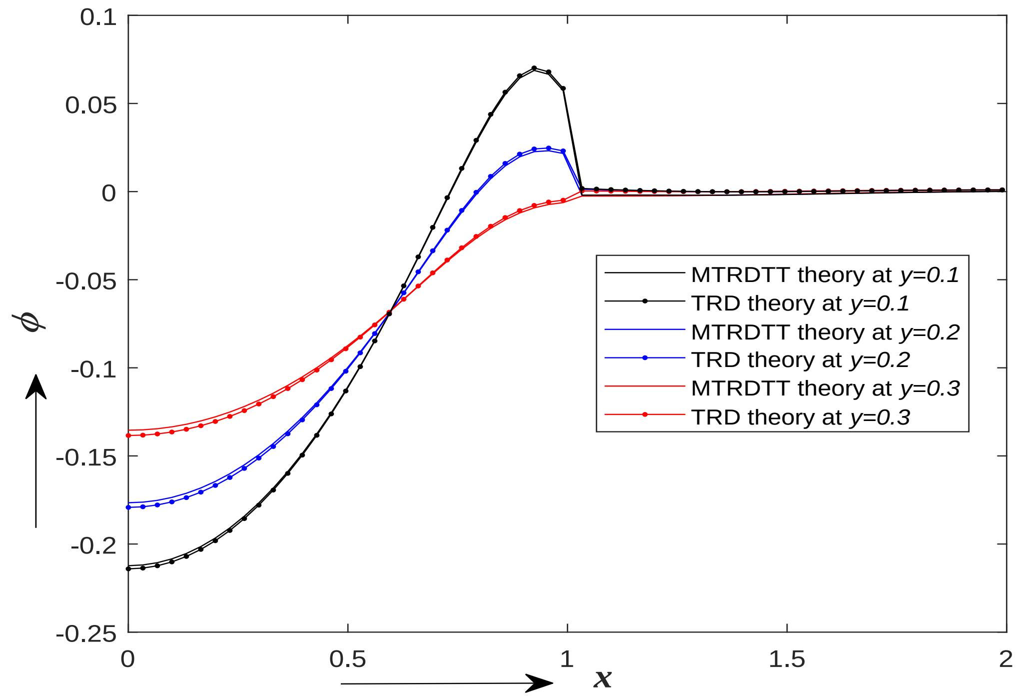
The numerical values of all the nonzero field variables are computed as mentioned above for different values of y and t. We have shown the results by plotting the distributions of the field variables under two different theories (MTRDTT, TRD) along x direction for three different values of y $(0.1, 0.2, 0.3)$ at $t = 0.35$. Due to symmetry of the problem about the y axis, the behaviour of the field variables are identical along with the positive and negative direction of the x axis. Therefore, we have shown the plots in region $x \geq 0$ only. The Figs 4.2.2 show the distributions of the conductive temperature and thermodynamic temperature, Figs. 4.2.3 show the horizontal and vertical displacement distributions, Figs. 4.2.4 and 4.2.5 depict the distributions of the stress components σ_{xx}, σ_{yy} and σ_{xy}, respectively.

From the Figs. 4.2.2 and 4.2.3 representing the distributions of the temperature and displacement fields, it is observed that due to the presence of the crack, the effects of loading are prominent within the region around crack length (i.e. between 0 to 1), and there is a sudden change in the behaviour of both the fields at the crack edge $x = 1$. However, there is a difference in the stress distributions as displayed in Figs. 4.2.4 and Fig 4.2.5. It is observed that the effect of the thermomechanical loading is significant even outside the crack length ($x > 1$), and the stress fields show oscillatory behaviour near the crack edge ($x = 1$). Peak values of stress components are attained near the crack edge.

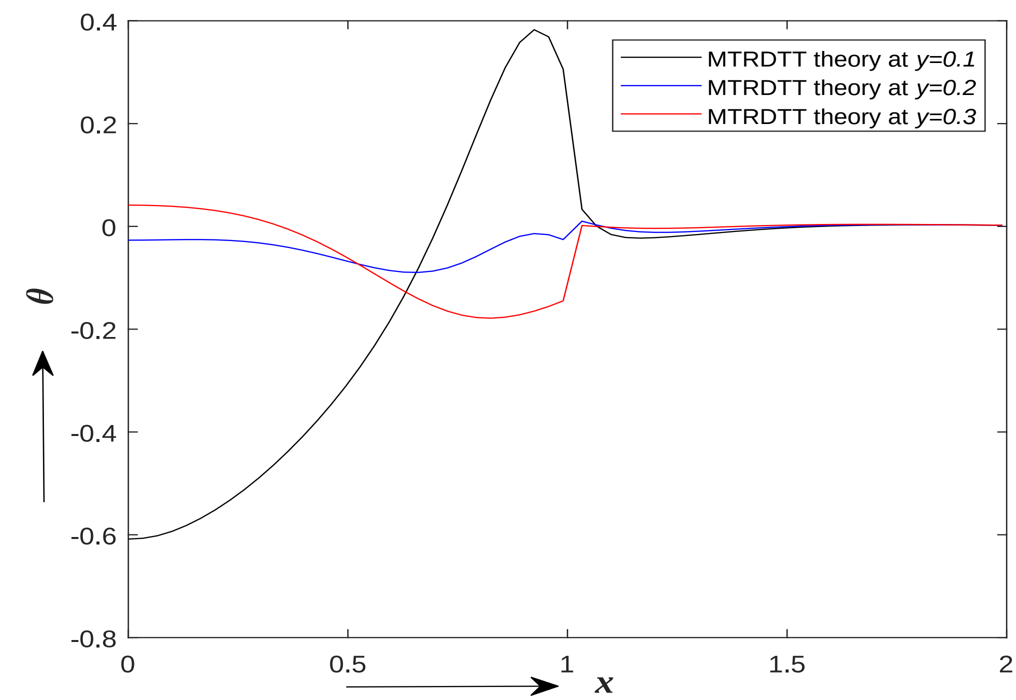
Furthermore, we find from Figs. 4.2.2-4.2.5, the distinction between the behaviour of field variables under two different theories (MTRDTT and TRD). In these Figures, the solid curves are used to represent the MTRDTT theory, and the dotted plots are used for the TRD theory. All the field variables have the same pattern of behaviour under both the thermoelasticity theories. However, it is revealed that except for the temperature field, there is a significant difference in the predictions by these two thermoelasticity theories for all field variables. The horizontal displacement shows the most prominent difference between the predictions by two different theories. Here, it can be observed that the effective domain of influence under the MTRDTT theory is larger as compared to the TRD theory, which is happening due to the infinite speed behaviour of thermal waves under MTRDTT theory.

Figures 4.2.2-4.2.5 also show the distribution of the field variables for three different values of y, ($y = 0.1, 0.2, 0.3$). From these figures, it is highlighted that the behaviour of all the physical fields is significantly affected due to the change in the values of y. It can be concluded from this behaviour that under both theories, the effects of thermal and mechanical loadings are more prominent near x axis due to the presence of crack, and it is decreasing in nature for the larger distance from x axis, i.e. the effective domain of influence decreases along the y direction. The variation along the y direction is more prominent under TRDTT than TRD theory, which is even more significant for the displacement and stress components.

In order to observe the effects due to change in time, we have shown the plots of conductive temperature in Fig. 4.2.6, at two different time steps, $t = 0.35$ and $t = 0.69$ under MTRDTT and TRD theories, and we find that as time progresses, there is a remarkable effect on the behaviour of field variables and under both theories the effective domain of influence increases with the increase of time. A similar effect is observed for other fields.However, the effect is more prominent in the case of stress distributions.

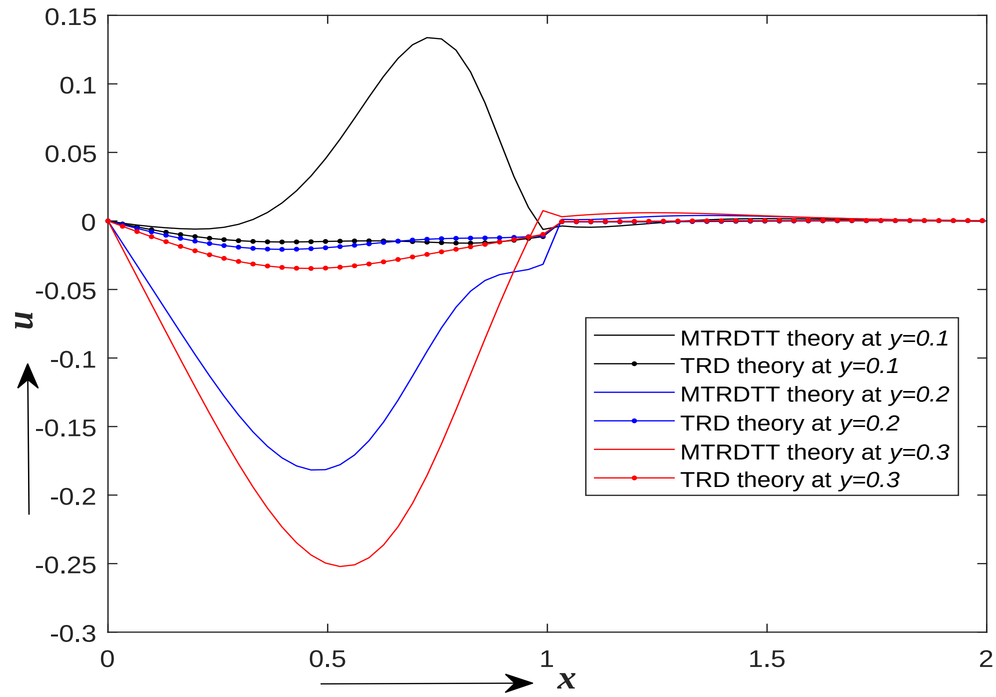


(a) Distribution of conductive temperature (ϕ) along x at $t = 0.35$

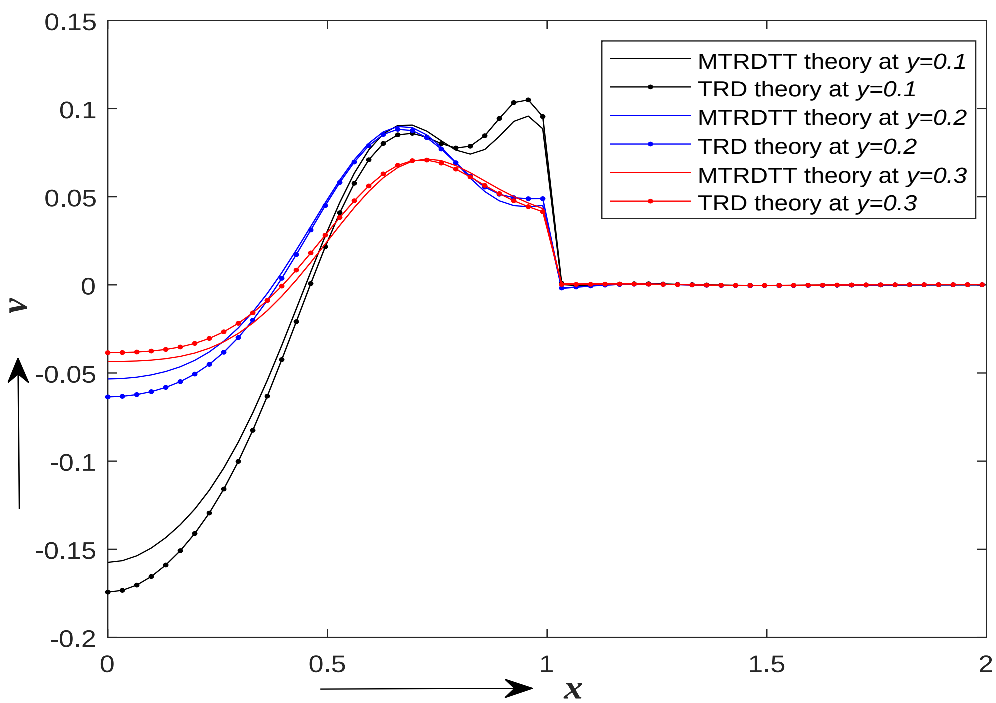


(b) Distribution of thermodynamic temperature (θ) along x at $t = 0.35$

Figure 4.2.2: Variation of temperature (a) Conductive temperature (ϕ) and (b) Thermodynamic temperature (θ)

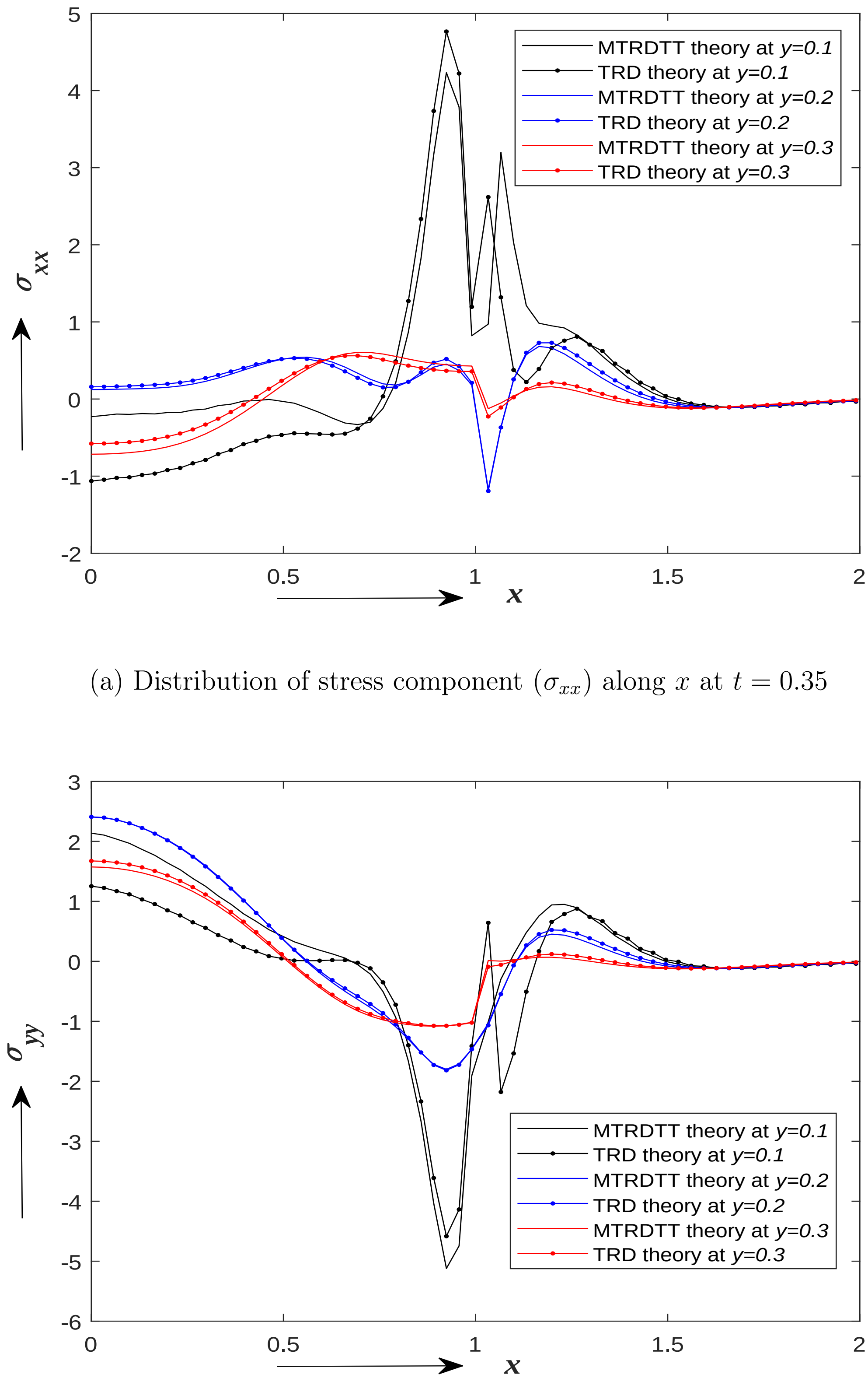


(a) Distribution of horizontal displacement (u) along x at $t = 0.35$



(b) Distribution of vertical displacement (v) along x at $t = 0.35$

Figure 4.2.3: Variation of displacement components (a) Horizontal displacement (u) and (b) Vertical displacement (v)



(a) Distribution of stress component (σ_{xx}) along x at $t = 0.35$

(b) Distribution of stress component (σ_{yy}) along x at $t = 0.35$

Figure 4.2.4: Variation of stress components (a) σ_{xx} and (b) σ_{yy}

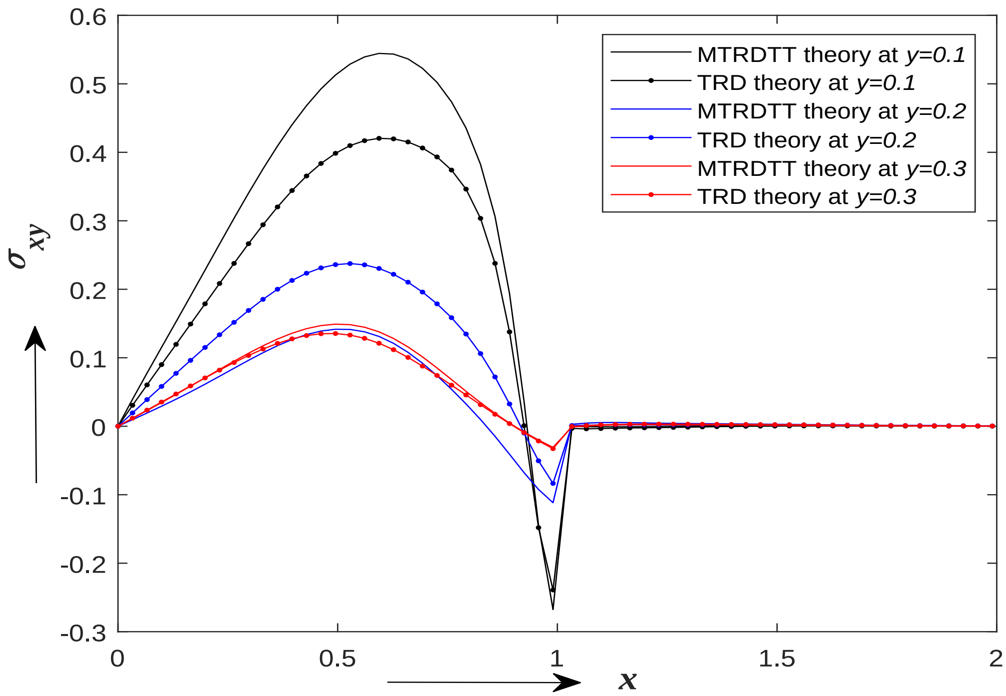


Figure 4.2.5: Distribution of shear stress (σ_{xy}) along x at $t = 0.35$

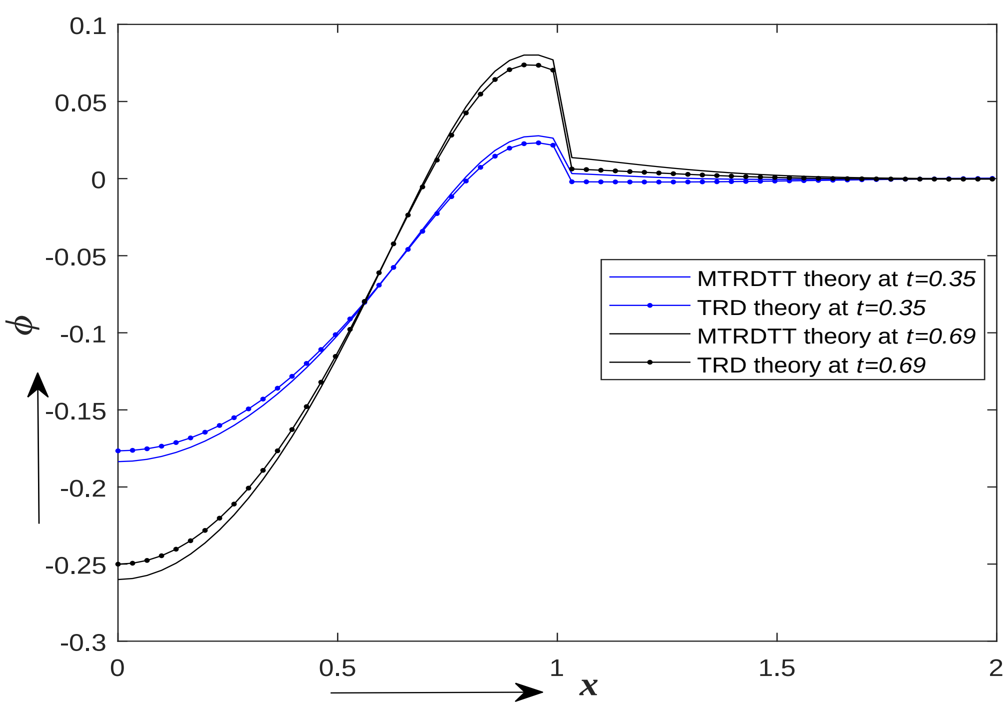


Figure 4.2.6: Distribution of conductive temperature (ϕ) along x at $y = 0.2$

4.2.8 Conclusion

This subchapter has investigated a dynamical problem of 2D infinite homogeneous and isotropic medium weakened by mode-I crack under temperature-rate dependent two-temperature thermoelasticity theory. The integral transformation techniques and their inverse transformations, along with the solution of the integral equation approach, are used to derive the numerical solution of the present problem. The impact of thermal and mechanical loadings on the field variables is investigated in the weakened medium. It is noted that the effect of boundary loadings on the field variables are significant in the vicinity of the crack. However, the effect is not prominent for the field variables outside the crack length except for horizontal and vertical stress components. The stress components exhibit the effect of boundary loadings even outside the crack length and show oscillatory nature in the vicinity of the crack. The behaviour of the field variables are also compared under MTRDTT and TRD thermoelasticity theory, and it is found the effect of the modified two-temperature theory is prominent. Hence, it is believed that the present investigation of the two-temperature theory is important and might be useful for future research in the concerned area.