CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Thermoelasticity: Perspectives and Applications

Thermoelasticity is a branch of science that is developed by considering the mutual interactions of thermal and mechanical fields in an elastic body. It is concerned with predicting the thermomechanical behaviour of the medium more precisely. From the experimental results, it is well known that the time varying mechanical loading results not only in the variation of displacement field but also affects the temperature distribution of the system. The converse of this phenomenon, i.e., a change in the deformation along with the change in temperature field due to applied heating on the body is also validated from the experimental point of view. These experimental results lead to an argument for taking into account of the coupling between the thermal and elastic fields. Therefore, the concept of thermoelasticity admits the existence of the temperature term appearing in the equation of motion with a deformation term in the heat conduction equation. Thermoelasticity theory is therefore based on two different fundamental theories: theory of elasticity and theory of heat conduction.

In various disciplines of science and technology, thermoelasticity theory has gained considerable interest from engineers and researchers due to its innumerous applications to multiple fields. Thermal stress analysis is significant in a variety of structural issues, such as high-speed plane manufacturing, designing of space vehicle, rocket and jet engine etc, nuclear reactor design, and so on. Thermomechanical processes may be described using the governing equations of continuum mechanics and thermodynamics, which can be utilised to address such problems. The thermoelasticity theory is also finding increasing use in a variety of engineering issues, such as developing material parts that can withstand abrupt thermal and mechanical loads and function at high temperatures. For example, behaviour of carbon steel can be well explained by the classical theory of elasticity at normal temperature and moderate stress field, however at the temperature higher than 450^oC, it behaves quite differently. The design and development of turbine also requires the knowledge of the material behaviour at high temperature. Hence, the study of this branch of science is essential for the designing and development of structures under various engineering fields, like nuclear, chemical, metallurgy etc. Porothermoelasticity, visco-thermoelasticity, pizo-thermoelasticity, magneto-thermoelasticity, and many other sub-branches of research are all built on the foundation of thermoelasticity.

1.2 Development of the Conventional (Classical) Thermoelasticity Theory

The concept of the coupling between thermal and mechanical fields has been originated from the work reported by Duhamel (1837), where he has formulated a boundary value problem and derived the equations involving a temperature gradient term for the strain fields. According to this theory, the conduction of heat in a material is only dependent on the temperature gradient and ignores the effect of other mechanical causes such as the effect of elastic property of the material on the heat transfer. Therefore, this theory is now termed as the uncoupled thermoelasticity theory. For the large time scale or small heat flux values, this theory behaves well. However, it fails to explain the accurate physical phenomena completely for the short time scale or high heat flux values. Duhamel (1837) has taken a simplified assumption that effect of the deformation on the thermal field can be neglected and under this assumption, he has derived the governing equations of the thermoelasticity involving the effect of temperature field on the deformation only. It is worth mentioning here that Neumann (1841) has independently developed the similar stress-strain and temperature relation like Duhamel (1837). Hence, this relation is also called as Duhamel-Neumann equation.

In view of the formulation of the thermal stresses introduced by Duhamel (1837; 1836), the theory starts gaining attention by the researchers. Thompson (1853) established some theoretical results on this theory. Joule and Thomson (1857) has described the thermal effects inside a thermoelastic body due to longitudinal compression and tensile stresses. Todhunter (1886) has reported historical efforts and development in this direction. Duhamel-Neumann equation has been used to deal with various thermoelastic problems, however, this equation was not supported by the thermodynamic principles. Therefore, Voigt (2014) and Jeffreys (1930) have made attempt to justify this relation from the thermodynamic sense. Later on, the most pioneering work in this respect is reported by Biot (1956b), who has developed a fully justified theory of thermoelasticity and derived the basic governing equations of coupled thermoelasticity. He has also presented a method for obtaining the general solution of thermoelastic problem for homogeneous and isotropic medium. This theory is known as the classical coupled dynamical thermoelasticity theory. Unlike the uncoupled thermoelasticity, the coupled thermoelasticity theory takes into account the elastic effects on the heat conduction along with the influence of strain on the temperature field. Hence, the coupled theory overcomes the deficiency of the uncoupled thermoelasticity theory by considering the effect of elastic changes on the thermal fields. The classical dynamical theory of thermoelasticity is based on the Fourier's law and among other constitutive relations including Duhamel-Neumann relation.

1.2.1 Formulation of the Classical Coupled Thermoelasticity Theory

Biot's theory is based on firm grounds of irreversible thermodynamics. On recalling the work by Biot (1956b), the outlines of the formulation of this coupled thermoelasticity theory from the thermodynamic principles can be presented as follows:

Consider a homogeneous and anisotropic thermoelastic body of volume V bounded by the surface A. Then, the first law of thermodynamics can be writen as

$$\frac{d}{dt}\int_V \frac{1}{2}\rho\dot{u}_i\dot{u}_i dV + \frac{d}{dt}\int_V \rho I dV = \int_V \rho b_i\dot{u}_i dV + \int_A \sigma_{ij}n_j\dot{u}_i dA + \int_V \rho Q dV - \int_A q_i n_i dA, \tag{1.2.1}$$

where I is the intrinsic energy per unit volume, b_i is the body force per unit volume, $\sigma_{ij}n_j$ is the traction force, Q is the heat produced per unit time and unit volume, and q_i is the heat flux through the surface of the body being positive outward in positive direction of the unit outward normal vector n_i. Here, the terms on the L.H.S. of Eq. (1.2.1) are the rate of change of kinetic and intrinsic energies. Therefore, the first two terms in the R.H.S. of this equation give the rates of the work done by all the body and external traction forces, the third term is the heat produced per unit time inside the body, and the fourth term denotes the heat transported into the body from an external source.

Further, from the principle of virtual work done, it is known that the rate of change of kinetic energy is equal to sum of all forces including internal and external forces, i.e.

$$\frac{d}{dt}\int_V \frac{1}{2}\dot{u}_i\dot{u}_i dV = \int_V \rho b_i\dot{u}_i dV + \int_A \sigma_{ij}n_j\dot{u}_i dA - \int_V \sigma_{ij}\dot{e}_{ij} dV. \tag{1.2.2}$$

Now, the equations (1.2.1) and (1.2.2) yield

$$\int_V \rho\dot{I} dV = \int_V \rho Q dV - \int_A q_i n_i dA + \int_V \sigma_{ij}\dot{e}_{ij} dV. \tag{1.2.3}$$

Equation (1.2.3) holds for any arbitrary volume V, therefore, this equation can be simplified as

$$\rho\dot{I} = \rho Q - q_{i,i} + \sigma_{ij}\dot{e}_{ij}. \tag{1.2.4}$$

Now, the second law of the thermodynamics which requires the positive entropy production, can be given in terms of the Clausius inequality for the present thermoelastic system as

$$\frac{d}{dt}\int_V \rho S \geq \int_V \rho \frac{Q}{T} dV - \int_A \frac{q_i n_i}{T} dA, \tag{1.2.5}$$

where S denotes the entropy of the system and T is the temperature.

Further, Eq. (1.2.5) can be simplified as

$$\rho(T\dot{S} - Q) + q_{i,i} - \frac{q_i}{T} T_{,i} \geq 0. \tag{1.2.6}$$

The Helmholtz free energy function (f) is introduced as

$$f = I - TS. \tag{1.2.7}$$

For the derivation of the governing equations, the response functions I and f are assumed to be the functions of T, $T_{,i}$ and e_{ij}. Therefore, the time rate of internal energy can be written as

$$\dot{I} = \frac{\partial f}{\partial T}\dot{T} + \frac{\partial f}{\partial e_{ij}}\dot{e}_{ij} + \frac{\partial f}{\partial T_{,i}}\dot{T}_{,i} - \dot{T}S - T\dot{S}. \tag{1.2.8}$$

Now, in view of Eq. (1.2.8), the Eqs. (1.2.4) and (1.2.6) can be obtained in terms of free energy function f as

$$\left(\rho\frac{\partial f}{\partial e_{ij}} - \sigma_{ij}\right)\dot{e}_{ij} + \left(\rho\frac{\partial f}{\partial T} + S\right)\dot{T} + \rho\frac{\partial f}{\partial T_{,i}}\dot{T}_{,i} + \rho\left(T\dot{S} - Q\right) + q_{i,i} = 0, \tag{1.2.9}$$

$$\left(\rho\frac{\partial f}{\partial e_{ij}} - \sigma_{ij}\right)\dot{e}_{ij} + \left(\rho\frac{\partial f}{\partial T} + S\right)\dot{T} + \rho\frac{\partial f}{\partial T_{,i}}\dot{T}_{,i} + \rho\left(T\dot{S} - Q\right) + q_{i,i} - q_i T_{,i} \geq 0. \tag{1.2.10}$$

Based on the assumption, the present system is independent from the functions $\dot{T}$, $\dot{T}_{,i}$ and $\dot{e}_{ij}$, and therefore, in view of the linearity assumptions, the equations (1.2.9) and (1.2.10) result as following:

$$\sigma_{ij} = \rho\frac{\partial f}{\partial e_{ij}}, \tag{1.2.11}$$

$$S = -\rho\frac{\partial f}{\partial T}, \tag{1.2.12}$$

$$\frac{\partial f}{\partial T_{,i}} = 0, \tag{1.2.13}$$

$$q_{i,i} = -\rho\left(T_0\dot{S} - Q\right), \tag{1.2.14}$$

and

$$q_i T_{,i} \leq 0. \tag{1.2.15}$$

Eqs. (1.2.11-1.2.15) represent the nonlinear theory of coupled thermoelasticity.

Now, Taylor series expansion of the free energy function about its natural state $\mathbf{0}(T_0, 0, 0)$ (where T_0 is the reference temperature) can be given as

$$\begin{aligned} f(T, e_{ij}) = f|_{\mathbf{0}} + (T - T_0)\left.\frac{\partial f}{\partial T}\right|_{\mathbf{0}} + e_{ij}\left.\frac{\partial f}{\partial e_{ij}}\right|_{\mathbf{0}} + \frac{1}{2}\left[(T - T_0)^2\left.\frac{\partial^2 f}{\partial \theta^2}\right|_{\mathbf{0}} + \right. \\ \left. + (T - T_0)e_{ij}\left.\frac{\partial^2 f}{\partial T \partial e_{ij}}\right|_{\mathbf{0}} + e_{ij}e_{kl}\left.\frac{\partial^2 f}{\partial e_{ij}\partial e_{kl}}\right|_{\mathbf{0}}\right]. \end{aligned} \tag{1.2.16}$$

In view of Eq. (1.2.16) and by substituting $\theta = T - T_0$, which denotes the temperature above reference temperature T_0 along with the natural initial assumptions $\left.\frac{\partial f}{\partial T}\right|_{\mathbf{0}} = 0$ and $\left.\frac{\partial f}{\partial e_{ij}}\right|_{\mathbf{0}} = 0$, the equations (1.2.11-1.2.12) are reduced to the forms

$$\sigma_{ij} = C_{ijkl}e_{kl} - \beta_{ij}\theta, \tag{1.2.17}$$

$$\rho T_0 S = \rho c_E \theta + T_0 \beta_{ij} e_{ij}, \tag{1.2.18}$$

where $C_{ijkl} = \rho\left.\frac{\partial^2 f}{\partial e_{ij}\partial e_{kl}}\right|_{\mathbf{0}}$ is the elasticity tensor, $\beta_{ij} = -\rho\left.\frac{\partial^2 f}{\partial e_{ij}\partial e_{kl}}\right|_{\mathbf{0}}$ is the thermoelasticity tensor, and $c_E = T_0\left.\frac{\partial^2 f}{\partial T^2}\right|_{\mathbf{0}}$ is the specific heat.

Again, by taking the Taylor's expansion of q_i about its natural state as

$$q_i(T, T_{,i}, e_{ij}) = q_i|_{\mathbf{0}} + \theta\left.\frac{\partial q_i}{\partial T}\right|_{\mathbf{0}} + \theta_{,j}\left.\frac{\partial q_i}{\partial T_{,j}}\right|_{\mathbf{0}} + e_{ij}\left.\frac{\partial q_i}{\partial e_{ij}}\right|_{\mathbf{0}}, \tag{1.2.19}$$

and taking $q_i|_{\mathbf{0}} = \left.\frac{\partial q_i}{\partial T}\right|_{\mathbf{0}} = \left.\frac{\partial q_i}{\partial e_{ij}}\right|_{\mathbf{0}} = 0$ and $K_{ij} = -\left.\frac{\partial q_i}{\partial T_{,j}}\right|_{\mathbf{0}}$ at natural state, the Eq. (1.2.19) can be simplified as

$$q_i = -K_{ij}\theta_{,j}. \tag{1.2.20}$$

Hence, the linearized basic governing equations of the coupled thermoelasticity are derived as

Equation of motion:

$$\sigma_{ij} + \rho b_i = \rho \ddot{u}_i. \tag{1.2.21}$$

Energy equation:

$$q_{i,i} = \rho(Q - \theta \dot{S}). \tag{1.2.22}$$

Constitutive relations:

$$q_i = -K_{ij}\theta_j, \tag{1.2.23}$$

$$\rho\theta_0 S = \rho c_E \theta - T_0 \beta_{ij} e_{ij}, \tag{1.2.24}$$

$$\sigma_{ij} = C_{ijkl} e_{kl} - \beta_{ij}\theta. \tag{1.2.25}$$

Now, from the above set of the Eqs. (1.2.21-1.2.25), the field equations can be given in terms of displacement (u_i) and temperature (θ) as

$$K_{ij}\theta_{ij} = \rho c_E \dot{\theta} + T_0 \beta_{ij} \dot{e}_{ij} - \rho Q, \tag{1.2.26}$$

$$C_{ijkl} e_{kl,j} - \beta_{ij}\theta_{,j} + \rho b_i = \rho \ddot{u}_i. \tag{1.2.27}$$

Eqs. (1.2.21-1.2.27) represent the complete set governing equations of the classical coupled thermoelasticity (1956b) for homogeneous and anisotropic medium. For the isotropic medium, the tensors β_{ij}, C_{ijkl} and K_{ij} are reduced as

$$\begin{gathered} \beta_{ij} = \beta\delta_{ij}, \\ C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mu, \\ K_{ij} = K\delta_{ij}. \end{gathered} \tag{1.2.28}$$

Hence, the constitutive relations for the isotropic medium can be given as

$$\begin{gathered} q_i = -K\theta_{,i}, \\ \rho T_0 S = \rho c_E(\theta) - \beta T_0 e_{kk}, \\ \sigma_{ij} = \lambda\delta_{ij} e_{kk} + 2\mu e_{ij} - \beta\theta. \end{gathered} \tag{1.2.29}$$

Also, the Eqs. (1.2.26-1.2.27) can be simplified as

$$K\theta_{,ii} = \rho c_E \dot{\theta} + T_0 \beta \dot{e}_{kk} - \rho Q, \tag{1.2.30}$$

$$(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} - \beta\theta_{,i} + \rho b_i = \rho\ddot{u}_i. \tag{1.2.31}$$

Equations (1.2.21-1.2.22, 1.2.29-1.2.31) represent the linear system of equations under coupled thermoelasticity theory for homogeneous and isotropic medium.

1.3 Limitations of the Classical Coupled Thermoelasticity

The classical thermoelasticity theory as described above has been widely used to study the thermal and elastic coupling involved in various thermoelastic problems. Clearly, the heat conduction Eq. (1.2.30), which is based on the Fourier's law of heat conduction is a parabolic type partial differential equation (diffusion equation), the Eq. (1.2.31) being hyperbolic type differential equation. Therefore, this equation predicts that the thermal signal propagates with infinite speed, i.e., the effect of any disturbance on a material can be instantaneously felt at infinite distance from the source of disturbance. It has been observed that this theory successfully explains the thermomechanical interactions for the thermoelastic problems involving low heat flux and large time response. However, the theory gives physically unrealistic results and fails to explain the transient behaviour of heat propagation, specially for the short time responses or high heat flux values. Also, the micro and nano-scale devices have demonstrated the distinct behaviour, which is not compatible with the classical thermoelasticity theory. Moreover, several researchers have worked in this direction and have developed theoretical as well as practical evidences and results admitting a finite speed of thermal wave. An exciting research interest has therefore been developed in last few decades to address the shortcomings in Fourier law and also in the classical coupled thermoelasticity theory. It is worth mentioning here the work by Maxwell (1867), who had postulated a modification in the Fourier's law while doing his experiments for gases and commented that "heat propagation is a wave type phenomenon rather than diffusion type". A detailed history in this respect can be found in the review article by Chandrasekharaiah (1986b). The finite speed behaviour of heat wave is now called as 'second sound effect' (see ref. Chandrasekharaiah (1986b)). The 'second sound effect' has also been discussed by several eminent researchers like, Nernst (1917), Landau (1941), Tisza (1947). Landau (1941) has reported the 'second sound effect' for the liquid helium as the propagation of the phonon density disturbance and approximated its speed to be equal to $\frac{v_p}{\sqrt{3}}$ at 0 K temperature, where v_p is the speed of the ordinary sound (first sound). Further, Peshkov (1994) has also detcted the 'second sound effect' experimentally.

The advancement of the micro-scale technologies has also supported the wave type heat conduction. Therefore, a serious attention has been paid by the researchers to overcome apparent drawback of classical thermoelasticity theory in presenting the unconvincing results in the case of short time responses and high temperature gradients (see refs. Lord and Shulman (1967), Green and Lindsay (1972), Francis (1972), Chandrasekharaiah (1986b), Ignaczak and Ostoja-Starzewski (2010) etc.). Some useful modifications in classical thermoelasticity theory have been proposed during the last six decades. These modified thermoelasticity theories are often referred to as generalized thermoelasticity theories. A brief introduction regarding the generalized thermoelasticity theories relevant to the present thesis is given in the next sub-sections.

1.4 Generalized Thermoelasticity Theory

Generalized thermoelasticity theories are the theories developed as the modified or alternative forms of the conventional (classical) coupled thermoelasticity theory to overcome the appearent paradox of infinite speed of thermal wave propagation. The development of these generalized theories are mainly based on the following three different approaches:

- Incorporating the concept of phase-lags/thermal relaxation parameters for constitutive variables in the Fourier law of heat conduction (see Lord-Shulman (LS) theory, dual-phase-lag (DPL) theory, three-phase-lag (TPL) theory).

• Considering the effects of higher order terms of constitutive field variables in the formulation of the governing equations (see temperature-rate dependent theory, strain and temperature-rate dependent theory).

• Developing alternative formulation of the coupled theory by introducing new constitutive field variables in the derivation of governing equations (see Green and Naghdi (GN) theory).

1.4.1 Lord-Shulman (LS) Thermoelasticity Theory or Extended Thermoelasticity Theory (ETE)

Lord and Shulman (1967) have proposed a generalized thermoelastic model suggesting the finite speed of heat propagation. This theory is also known as the extended thermoelasticity theory (ETE). The first modification in the Fourier's heat conduction theory has been suggested by Cattaneo (1958) and Vernotte (1958; 1961). They introduced the heat flux rate term in Fourier's law with a time relaxation parameter and hence, proposed a hyperbolic type heat conduction equation predicting the finite speed of thermal waves. This work was motivated by Onsager (1931), who has suggested that heat conduction requires a delay time to accelerate the heat flow. Lord and Shulman (1967) have applied this modified Fourier's law of heat conduction (Catteneo-Vernotte law) and derived the first generalized coupled theory of thermoelasticity. The heat conduction law based on the LS thermoelasticity theory for the homogeneous and isotropic medium can be given as

$$q_i + t_q \frac{\partial q_i}{\partial t} = -K\theta_{,i} \tag{1.4.1}$$

Here, t_q is the relaxation time required to achieve the steady-state of heat conduction under a suddenly applied temperature gradient. Combining Eq. (1.4.1) with the energy equation (1.4.9) and entropy equation, the heat conduction equation for LS thermoelasticity theory can be obtained as

$$K\theta_{,jj} = \left(1 + t_q \frac{\partial}{\partial t}\right) \left(\rho c_E \dot{\theta} + T_0 \beta \dot{u}_{j,j} - \rho Q\right). \tag{1.4.2}$$

Clearly, Eq. (1.4.2) is a hyperbolic type heat conduction equation and admits the propagation of thermal wave with a finite speed of $\sqrt{\frac{K}{\rho c_E t_q}}$.

The physical values of the time relaxation parameter (t_q) for different materials have been reported in literatute by various researchers while carrying out their experimental work. Chester (1963) has supported the existence of relaxation parameter for heat conduction and estimated the value of the relaxation parameter, t_q in terms of material constants as $t_q = \frac{3K}{\rho c_E v_s^2}$. Here, v_s denotes the velocity of sound. From this relation, it can be observed that in general, the value of t_q is very small and therefore, its effect can be neglected in the study of heat conduction. However, several authors including Baumeister and Hamill (1969; 1971), Chan et al. (1971), Maurer and Thompson (1973), Sadd and Cha (1982) have demonstrated the essence of this modified heat conduction theory for very high heat flux and very short time intervals. Several other authors (see refs. Nettleton (1960), Chester (1963; 1966), Maurer (1969), Mengi and Turhan (1978) etc.) have attempted to determine the values of the relaxation parameter experimentally and reported that the value of relaxation time range from 10^{-14} s to 10^{-10}s for metals. It is also noted that the relaxation time may be as large as 11s for glass and 21s for the sand at laboratory temperature, and for the organic materials and tissues, the relaxation time can range from 10s to 1000s (see Hetnarski et al. (2009), Chandrasekhariah (1998), Tzou (1995b; 1995a)).

1.4.2 Green-Naghdi (GN) Thermoelasticity Theory

In the 1990s, Green and Naghdi (1991; 1992; 1993) have followed a completely different approach to develop an alternative version of thermoelasticity theory. They modified Fourier's law by introducing a new constitutive variable in the theory of heat conduction and developed their new thermoelasticity theory from thermodynamic principles. The concept of this new constitutive variable arosed from limiting case in Cattaneo and Vernotte heat conduction model $q_i + t_q \frac{\partial q_i}{\partial t} = -K\theta_{,i}$, when t_q approaches to infinity such that $\frac{K}{t_q}$ remains finite. In this limiting case, an alternative heat conduction law can be derived as

$$\frac{q_i}{t_q} + \frac{\partial q_i}{\partial t} = -\frac{K}{t_q}\theta_{,i}, \tag{1.4.3}$$

which in the limiting case gives

$$\frac{\partial q_i}{\partial t} = -K^*\theta_{,i}, \tag{1.4.4}$$

where $K^* = K/t_q$ is referred as the conductivity rate which is considered as a new material parameter, characteristic of Green-Naghdi theory.

Following this idea Green and Naghdi (1991) have established the alternative thermoelasticity theory by introducing a new constitutive variable ν in the heat tonduction law. This constitutive variable is related to the thermodynamic temperature as $\frac{\partial \nu}{\partial t} = \theta$. Therefore, ν is termed as the thermal displacement. The general form of heat conduction law suggested by Green and Naghdi (1991; 1992; 1993) for the homogeneous and anisotropic medium is given as

$$q_i = -K\theta_{,i} - K^*\nu_{,i}, \tag{1.4.5}$$

where θ, $\theta_{,i}$, ν and $\nu_{,i}$ are taken as independent variables. Therefore, the heat conduction equation based on this theory can be derived as

$$K\theta_{,ii} + K^*\nu_{,ii} = \rho c\dot{\theta} + T_0\beta\dot{e}_{kk} - \rho Q. \tag{1.4.6}$$

This theory can be categorized into three different cases. The theory based on Eq. (1.4.6) represents the type-III (GN-III) thermoelasticity theory. Further, Green and Naghdi (1991; 1992; 1993) have shown that type-I (GN-I) and type-II (GN-II) theory can be obtained as particular cases of the general theory, i.e., GN-III theory. When $K = 0$, the theory is termed as GN-I theory, which also coincides with Biot thermoelasticity theory. For $K = 0$, this theory is inferred as the GN-II thermoelasticity theory. This theory admits undampped thermal wave and suggests no dissipation in the energy during the heat flow, therefore, GN-II theory is also known as the thermoelasticity theory without energy dissipation. Equations for the anisotropic case for both GN-II and GN-III thermoelasticity theories are further articulated by Quintanilla (1999; 2001; 2002).

1.4.3 Dual-Phase-Lag (DPL) Thermoelasticity Theory

The dual-phase-lag thermoelasticity theory has been developed on the basis of dual-phase-lag heat conduction theory (Tzou (1995b; 1995a)) that incorporates two different phase-lag parameters. Tzou (1993) has introduced the concept of phase-lag in the theory of heat conduction. He has presented a time phase-lag term for heat flux vector in the Fourier law and suggested that the Cattaneo (1958) and Vernotte (1958; 1961) heat conduction equation can be redeemed from this equation by taking the Taylor series expansion of the heat flux vector about phase-lag t_q. The phase-lag based heat conduction law suggested by Tzou (1995b; 1995a) for anisotropic medium can be written as

$$q_i(\boldsymbol{x}, t + t_q) = -K_{ij}\theta_{,j}(\boldsymbol{x}, t). \tag{1.4.7}$$

Further, this idea of phase-lag has been extended by Tzou (1995b; 1995a) who has suggested to include two phase-lag parameters, one in the heat flux vector t_q and another in the temperature gradient vector t_θ, to take into account the microscopic effects in heat transport phenomonon. Due to the presence of two different phase-lag parameters, this theory is referred to as dual-phase-lag or two phase-lag heat conduction theory. The dual-phase-lag (DPL) heat conduction relation proposed by Tzou (1995b; 1995a) is given as follows:

$$q_i(\boldsymbol{x}, t + t_q) = -K_{ij}\theta_{,j}(\boldsymbol{x}, t + t_\theta). \tag{1.4.8}$$

By introducing these phase-lag parameters, Tzou has considered the effect of both macro as well as micro scale interactions due to applied thermoelastic loadings. Further, Tzou has shown that by taking different Taylor series approximations of Eq. (1.4.8), different constitutive relations can be drawn. In particular, Tzou (1995b; 1995a) has demonstrated two different constitutive relations for heat flux and temperature gradient vectors as follows:

Taking the first order Taylor series expansion in Eq. (1.4.8), Tzou considered his first DPL heat conduction model as

$$\left(1 + t_q \frac{\partial}{\partial t}\right) q_i = -K_{ij} \left(1 + t_\theta \frac{\partial}{\partial t}\right) \theta_{,j}. \tag{1.4.9}$$

Further, by taking the second order Taylor's expansion heat flux vector and first-order expansion of temperature gradient term, the second DPL model has been derived as

$$\left(1 + t_q \frac{\partial}{\partial t} + t_q^2 \frac{\partial^2}{\partial t^2}\right) q_i = -K_{ij} \left(1 + t_\theta \frac{\partial}{\partial t}\right) \theta_{,j}. \tag{1.4.10}$$

It can be verified that (1.4.10) exhibits thermal propagation as wave in nature, whereas results of Eq. (1.4.9) depend on the values of t_q and t_θ. More detailed elaboration of these two models and several important findings in this respect are available in the book given by Tzou (1997).

Chandrasekharaiah (1998) has subsequently extended the DPL heat conduction theory to theory of thermoelasticity and developed a DPL thermoelasticity theory based on modified Fourier's heat conduction Eqs.(1.4.9-1.4.10). The author has combined Eq. (1.4.10) with energy and entropy equations to derive heat conduction equation for DPL thermoelasticity theory in the context of isotropic medium as

$$\left(1 + t_\theta \frac{\partial}{\partial t}\right) K_{ij}\theta_{,ij} = \left(1 + t_q \frac{\partial}{\partial t} + t_q^2 \frac{\partial^2}{\partial t^2}\right) \left(\rho c_E \dot{\theta} + T_0 \beta_{ij} \dot{u}_{i,j} - \rho Q\right). \tag{1.4.11}$$

1.4.4 Three-Phase-Lag (TPL) Thermoelasticity Theory

Roychoudhuri (2007a) has further generalized the concept of phase-lag to Green-Naghdi thermoelasticity theory by incorporating three different phase-lag (TPL) parameters in the constitutive relation for heat conduction relation (1.4.5) given by Green and Naghdi. One additional phase-lag parameter τ_ν is incorporated here for the gradient of thermal displacement, along with the incorporation of phase-lag parameters for the heat flux as well as temperature gradient terms. The modified heat conduction law corresponding to the TPL theory for the homogeneous and anisotropic medium can be given as

$$q_i(\boldsymbol{x}, t + t_q) = -K_{ij}\theta_{,i}(\boldsymbol{x}, t + t_\theta) - K^*_{ij}\nu_{,i}(\boldsymbol{x}, t + t_\nu). \tag{1.4.12}$$

Further, by combining energy equation and entropy equation along with the Eq. (1.4.12) heat conduction equation corresponding to the TPL theory can be obtained. Similar to DPL theory, different order Taylor series approximation of Eq. (1.4.12) also results different constitutive relation corresponding TPL theory. Roychoudhuri has derived two different version of the TPL theory in his article. Following Roychoudhuri (2007a) the heat conduction equation for the TPL theory on the context of homogeneous and isotropic medium can be given as

$$\begin{aligned}\left(1 + t_q\frac{\partial}{\partial t} + t_q^2\frac{\partial^2}{\partial t^2}\right)\left(\rho c\dot{\theta} + T_0\beta_{ij}\dot{u}_{j,j} - \rho Q\right) \quad = \quad & K^*\dot{\theta}_{,ii} + \\ & Kt_\theta\ddot{\theta}_{,ii} + K^*\theta_{,ii}.\end{aligned} \tag{1.4.13}$$

Neglecting t_q^2 in Eq. (1.4.13) will result another version of the thermal field equation of three-phase-lag thermoelasticity theory (see Roychoudhuri (2007a)). By setting the different values of the material parameters in the Eq. (1.4.13), all the previously discussed thermoelasticity theories can be derived. Hence, this theory is the most general theory of thermoelasticity.

In 2009, Dreher et al. (2009) have mathematically examined the dual-phase-lag and three-phase-lag theories and reported that when these constitutive equations are combined with classical energy equation $-q_{i,i}(x,t) = c_E\dot{\theta}(x,t)$, there exists a sequence of solutions such that the real part of its eigenvalues tends to infinity. Thus, the mathematical system is not well-posed in the Hadamard sense and it is observed that these theories are not based on a priori thermodynamic formulation. Therefore, it becomes interesting to study various Taylor's approximations of these models rather than studying the models mentioned above. The theories obtained by Taylor series approximation of these models provide mathematical formulations where well-posedness can be established with some conditions on the parameters.

Recently, Quintanilla (2009) has proposed some modifications to the three-phase-lag model and studied the well-posedness and spatial behaviour of this newly proposed models. In his new formulation, the parameters are assumed to be $t_\nu < t_q = t_\theta$ and $t_0 = t_q - t_\nu$. Now, the three-phase-lag model is reduced to a heat conduction model with a single delay term, $t_0 > 0$, and by considering Taylor's approximation up to second order for the delay term, the heat conduction relation is given here in the form

$$q_i = -K\theta_{,ii} - K^*\nu_{,ii} - K^*t_0\dot{\nu}_{,ii} - K^*\frac{t_0^2}{2}\ddot{\nu}_{,ii}. \tag{1.4.14}$$

A detailed discussion on this model and spatial behaviour of its solution has been elaborated by Leseduarte and Quintanilla (2018).

1.4.5 Green-Lindsay (GL) Thermoelasticity Theory

All the previously discussed thermoelasticity theories are based on the modification in Fourier's heat conduction law. In 1972, Green and Lindsay (1972) proposed a completely different generalization of classical thermoelasticity theory, which is developed by applying the effect of temperature-rate along with the temperature in thermoelasticity theory. Therefore, this theory is also known as the temperature-rate dependent (TRD) thermoelasticity theory. Prior to this, in 1967, Muller (1967) has postulated a modified form of entropy inequality (Clausius-Duhem inequality Eq.(3.2.5)) . Later on, Green and Laws (1972) have generalized this entropy production inequality and suggested that for non-equilibrium situations, the temperature associated with heat supply in the entropy inequality should be replaced with a constitutive relation involving temperature, temperature-rate, and temperature gradient. Subsequently, Green and Lindsay (1972) have employed this concept and established his temperature-rate dependent thermoelasticity theory (GL theory). Unlike most of the thermoelasticity theories, this theory describes the finite speed of thermal wave without violating the Fourier law and in spite of that, this theory modifies the classical theory by introducing two thermal relaxation time parameters: one in the stress–strain–temperature relation and another in the entropy equation. Due to the presence of two relaxation parameters, this theory is also referred to as the thermoelasticity theory with two relaxation times.

The constitutive relations for linear generalized thermoelasticity theory given by Green and Lindsay (1972) for the anisotropic homogeneous medium with the center of symmetry are given as follows:

The equation of motion:

$$\sigma_{ij,j} + \rho b_i = \rho \ddot{u}_i. \tag{1.4.15}$$

The energy equation:

$$\rho T_0 \dot{S} = -q_{i,i} + \rho Q. \tag{1.4.16}$$

The constitutive relations:

$$\theta_0 \rho S = \rho c_E(\theta + t_1\dot{\theta}) + \beta_{ij} T_0 e_{ij}, \tag{1.4.17}$$

$$\sigma_{ij} = C_{ijkl} e_{kl} - \beta_{ij}(\theta + t_2\dot{\theta}), \tag{1.4.18}$$

$$q_i = -K_{ij}\theta_{,j}, \tag{1.4.19}$$

where t_1 and t_2 are the two thermal relaxation time parameters such that $t_1 > t_2$.

1.4.6 Modified Green-Lindsay (MGL) Thermoelasticity Theory

Green and Lindsay (1972) thermoelasticity theory has successfully overcome the drawback of the infinite speed of heat propagation in classical thermoelasticity theory and has been applied to analyze the transient response of various thermoelastic problems in a more realistic way. However, critical analysis has been reported on this temperature-rate dependent thermoelasticity theory that this model experiences a drawback of discontinuity in the displacement field at both thermal and elastic wavefronts under the elevated temperature at the boundary surface of an elastic medium (see Chandrasekharaish and Srikantiah (1987); Dhaliwal and Rokne (1989); Chatterjee and Roychoudhuri (1990); Ignaczak and Mr'owka-Matejewska (1990)). Discontinuity in the displacement field suggests that one part of matter penetrates into the other, which disobeys the continuum hypothesis (Chandrasekharaiah (1998)).

Therefore, in order to overcome this drawback in the TRD thermoelasticity, recently, Yu et al. (2018) have proposed a modified Green-Lindsay (MGL) thermoelasticity theory. While developing the MGL theory, authors have incorporated the effects of strain-rate along with the temperature-rate terms in the thermoelasticity theory. Hence, this theory can also be termed as the strain and temperature-rate dependent thermoelasticity theory. Yu et. al. (2018) have also discussed a half space problem based on MGL theory and demonstrated that this theory successfully overcomes the drawback on the discontinuity in the displacement field. While developing this theory, the modification in constitutive relations are resulted as follows:

$$T_0\rho S = \rho c_E(\theta + t_1\dot{\theta}) + \beta_{ij}T_0(e_{ij} + t_1\dot{e}_{ij}), \tag{1.4.20}$$

$$\sigma_{ij} = C_{ijkl}(e_{kl} + t_2\dot{e}_{kl}) - \beta_{ij}(\theta + t_2\dot{\theta}). \tag{1.4.21}$$

Eq. (1.4.20) and Eq. (1.4.21) identify the incorporation of strain-rate term and temperature-rate term in the entropy and stress-strain-temperature relations, respectively.

1.4.7 Two-Temperature Thermoelasticity Theory

Gurtin and Williams (1966) have suggested that the entropy inequality needs a modification based on the distinction between heat flux inside the body and the external heat supply. In view of these two different mechanisms, the entropy flow is shown here to be separated in the second law of thermodynamics, and they have postulated that by the same factor of proportionality, two different temperatures can be assumed. Therefore, Gurtin and Williams (1966) have proposed a modified form of the Clausius inequality involving two temperatures. The volume-relevant temperature is referred to as conductive temperature, while the surface-relevant temperature is referred to as the thermodynamic temperature. Based on this modified second law of thermodynamics, Chen and Gurtin (1968) have formulated the two-temperature theory of heat conduction for nonsimple materials. This modification has suggested the new classification of materials termed as simple and nonsimple materials. For the simple materials, the energy, heat flux, and the thermodynamic temperature (i.e., local states) are dependent on the history up to present time of the strain, conductive temperature, and their first spatial gradients (Gurtin and Williams (1966)), while in the nonsimple materials local state is characterized by the instantaneous values of the deformation gradient, the conductive temperature, and their first two spatial gradients. Thus, simple materials form a subset of the class of nonsimple materials (Chakrabarti (1973)). Chen and Gurtin (1968) have further shown that the difference between these two temperatures is proportional to the heat supply in the time independent situation, and in the absence of an external heat source, the two temperatures will be the same. However, in the time dependent situation, the two temperatures will not be the same, even in the absence of a heat source. Further, Chen et al. (1969) have derived the governing equations of the two-temperature theory of thermoelasticity from the fundamental laws of thermodynamics. The modified heat conduction law and two-temperature relation corresponding to two-temperature thermoelasticity based on Biot theory for isotropic medium are given in the forms as

$$K\phi_{,ii} = \rho c_E \dot{\theta} + T_0 \beta \dot{e}_{kk} - \rho Q, \tag{1.4.22}$$

$$\phi - \theta = a\phi_{,ii}, \tag{1.4.23}$$

where ϕ is the conductive temperature and θ is the thermodynamic temperature. a is called as the two-temperature parameter. Tzou (1995b) has indicated that in situations of microscale heat transfer, the temperature discrepancy can be measured in terms of the mean free path (l) as $a \sim l^2$ and mathematically mean free path is approximated as $l \sim c\tau$, where c is the mean phonon speed and τ is the relaxation parameter. Therefore, temperature discrepancy (two-temperature parameter) can be approximated as $a \sim (c\tau)^2$.

The two-temperature generalization of LS theory has been introduced by Youssef (2006b), who has derived the constitutive relations from the energy equation and the first law of thermodynamics by keeping the two-temperature relation suggested by Chen et al. (1969) unchanged. later on, the two-temperature model of Green-Naghdi theory has been established by Youssef (2011). Sur and Kanoria (2012) have proposed a fractional order theory of two-temperature thermoelasticity.

1.4.8 Other Generalized Thermoelasticity Theories

Apart from the thermoelasticity theories discussed above, other modified theories have also been developed on the basis of the coupling of the thermoelasticity with other branches of physics like magnetic fields, electric fields, viscosity etc. In this context the books by Das (2017), Truesdell (2013), Ignaczak and Ostoja-Starzewski (2010), Irgens (2008) can be studied.

Some generalizations of the thermoelasticity theories have also been established on the basis of material properties of the medium like, nonlocal, micropolar thermoelasticity theories. Eringen (1970) has reported the fundamental concepts and the development of the governing equation of the micropolar thermoelasticity theory. Several other authors have worked on the development of the generalized thermoelasticity for the micropolar theory. In this context, the articles by Boschi and Ieşan (1973), Chandrasekharaiah (1986a), Ciarletta (1999), Sherief et al. (2005), Othman and Singh (2007), Ciarletta et al. (2007) can be consulted. Systematic development of the fractional order thermoelasticity theories that involve the fractional order derivative terms in the heat conduction law has been reported in the books by Povstenko (2015; 2019).

1.5 Porothermoelasticity Theory

Porosity is found in a variety of natural and artificial composites like crustal and reservoir rock in the earth, sound-absorbing materials, polyurethane foam, osseous tissue, etc. Consequently, the problems dealing with fluid flow and heat conduction and the dynamics of a system with porosity have aroused much interest in researchers. It is to be recalled that first time the concept of porosity in the presence of elasticity has been introduced by Biot (1956a). He has considered the general class of solid materials consisting of innumerable interconnected fluid saturated cavities and used the Lagrange's equations to derive the stress-strain relation and the coupled equations of motion in the context of generalized poroelasticity. The study of porous materials includes a large class of engineering problems related to water saturated soil, asphalt concrete pavements etc. Porous solids also exist in nature in the form of crustal and reservoir rocks in the earth and therefore have a wide range of applications in the field of geophysics and related topics. The researchers are devoting increasing attention to study the thermal and mechanical interactions in the porothermoelastic solids. Bear et al. (1992) and Levy et al. (1995) have dealt with the fluid transport phenomenon through a porous medium and derived basic equations for the microscopic dynamics. Further, Biot (1962a; 1962b) has extended the concept of poroelasticity to the acoustic propagation theory. Crown and Nuziato (1983) have developed the linear theory for the elastic materials with voids. Several authors have further worked for the evolution of the theory to various thermomechanical problems (see refs. Fourie and Du Plessis (2003); Wang (2017); Ghassemi and Diek (2002)). While incorporating the thermal effects, most of these authors have followed the Classical theory of heat conduction (Biot (1956b)). The basic governing equations and the constitutive relations of the porothermoelasticity theory corresponding to classical thermoelasticity theory can be given as

Equations of motion:

$$\tau_{ji,j} + \rho^s b_i^s = \rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i, \tag{1.5.1}$$

$$\tau_{,j} + \rho^f b_i^f = \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i. \tag{1.5.2}$$

The energy equations:

$$q_{i,i}^s = -\rho T_0 \dot{S}^s, \tag{1.5.3}$$

$$q_{i,i}^f = -\rho T_0 \dot{S}^f. \tag{1.5.4}$$

The constitutive relations:

$$\tau_{ij} = C_{ijkl}e_{kl} + D_{ij}\varepsilon - \beta_{ij}^s \vartheta^s - \beta_{ij}^f \vartheta^f, \tag{1.5.5}$$

$$\tau = C\varepsilon + D_{ij}e_{ij} - \beta^s \vartheta^s - \beta^f \vartheta^f, \tag{1.5.6}$$

$$\rho\vartheta_0 S^s = -T_0(\beta_{ij}^s e_{ij} + \beta^s \varepsilon) + \rho c_E^s \vartheta^s, \tag{1.5.7}$$

$$\rho\vartheta_0 S^f = -T_0(\beta_{ij}^f e_{ij} + \beta^f \varepsilon) + \rho c_E^f \vartheta^f, \tag{1.5.8}$$

$$q_i^s = -K_{ij}^s \vartheta_{,j}^s, \tag{1.5.9}$$

$$q_i^f = -K_{ij}^f \vartheta_{,j}^f, \tag{1.5.10}$$

where τ_{ij} and τ denote the components of stress tensor and fluid pressure corresponding to the solid and fluid phases, respectively; b_i is the external body force per unit volume;

C_{ijkl} is the elasticity tensor; D_{ij} is poroelastic coupling tensor; C is the bulk modulus of fluid phase; β_{ij}^{ω} is the thermoelasticity tensor; β^{ω} is the thermoelasticity constant due to the presence of porosity; q_i^{ω} is heat flux vector; K_{ij}^{ω} is the conductivity tensor; θ^{ω} is temperature above reference temperature T_0, such that $\frac{|\theta^{\omega}|}{\theta_0} \ll 1$; c_E^{ω} is the specific heat at constant strain; ρ^{ω} is the density of material; S^{ω} is the entropy, where, super scripted ω will be used s to denote the solid phase and f to denote the fluid phase for material parameters and field variables.

In 2007, Youssef (2007) has established a theoretical foundation to the generalized theory of porothermoelasticity admitting finite speed of heat signals.

1.6 Literature Review

After the well postulated thermoelasticity theory came into existence in the mid of 19th century, the theory has been continuously evolving, and various generalized thermoelasticity theories have been developed over time. Several researchers have paid significant attention to the investigation of these thermoelasticity theories for different engineering problems to study the thermomechanical behaviour of the field variables for different (homogeneous and non-homogeneous) mediums under various thermoelastic environments. A broad range of studies on the advancement and analysis of coupled thermoelasticity theories has been reported in the literature. The extensive review articles and books by Chandrasekhariah (1986b; 1998) and Nowacki (1969), Joseph and Prezoski (1989), Straughan (2011), Parkus (2012), Suhubi (1975), Iesan (1994), Hetnarski and Ignaczak (1999), Hetnarski and Eslami (2009) and Ignaczak and Ostoja-Starzewski (2010) are worth to be mentioned in this respect. Detailed analysis on some recently introduced thermoelasticity theories and their applications to specific problems can also be found in the Ph.D. theses by Roushan Kumar (2010), Rajesh Prasad (2012), Shweta Kothari (2013), Rakhi Tiwari (2017), Bharti Kumari (2017), Shashi Kant (2018), and

Anil Kumar (2018). In addition, the present section aims at reporting state of art in the context of the modified thermoelasticity theories as described in previous sections.

Sternberg and McDowell (1957) have employed Green's function method to investigate a semi-infinite elastic medium under Biot's theory and demonstrated the steady state behaviour of stress and displacement fields. Lessen (1957; 1959) has studied the wave nature of the solution due to thermal and elastic effects under the classical theory and shown the propagation of thermal wave along with the elastic wave. Paria (1958) has applied this theory to study a half space thermoelastic problem. Further, Hetnarski (1961; 1964) has discussed a one dimensional thermoelastic problem subjected to thermal shock at boundary and obtained an analytical solution of the problem by using short-time approximation approach. Chadwick (1962) has analyzed the thermoelastic interaction inside thick beam and plate. Parkus (1963) has reported a detailed discussion about various methods for solving different thermoelastic problems under Biot's theory. Goodman (1964) has developed an integral method for deducing the ordinary initial value problem of heat conduction from the nonlinear boundary conditions and hence, approximated the solution of heat conduction based on Biot's theory. Nickell and Wilson (1966) have applied the variational approach of the finite element method to derive the continuous spatial solutions of two dimensional heat conduction problem. Nickell and Sackman (1968) have presented a variational principle for the Biot's thermoelasticity theory for inhomogeneous and isotropic continuum. Further, Nowacki (1968) has generalized some theorems for the coupled thermoelasticity medium, which is characterized by displacement and rotation vectors. Verruijt (1969) has applied Mindlin's theorem of completeness and proved the completeness of the solution under Biot's theory. Several results and applications of the classical thermoelasticity theory have been reported in the books by Boley and Winer (1960), Nowacki (1975) and Nowinski (1978). This theory has been widely applied to study the thermoelastic problem, and it is observed that although the theory explains the coupling effects, however, it still predicts the paradox of infinite speed for heat propagation.

The extended thermoelasticity theory is given by Lord and Shulman (1967), who has generalized the classical thermoelasticity theory by proposing an appropriate modification in Fourier's law of heat conduction. They have further derived an exact solution for a one dimensional half space thermoelastic problem for homogeneous and isotropic medium and demonstrated the elimination of paradox of infinite speed by comparing their results with the results obtained under Biot's thermoelasticity theory. Following Lord and Shulman (1967), Fox (1969) and further Lord and Lopez (1970) have investigated the thermoelastic disturbances and wave propagation in thermoelastic solids at a very low temperature. Chen and Gurtin (1970) have applied the LS theory to analyze the second sound effect for thermoelastic materials with memory. Further, Nayfeh and Nemat-Nasser (1971) have studied the plane harmonic and Rayleigh's surface wave propagation under this modified theory and derived the explicit expressions for various parameters characterizing these waves. They have also verified the phenomenon of the finite speed of heat propagation in their work. Puri (1973) analyzed phase velocity, specific loss, and amplitude ratio of the plane waves and approximated the expressions for very low and high frequency values. He has observed that this generalized theory concern with the wave nature of the thermal disturbance. The thermoelastic responses inside a cylindrical medium have been analyzed by Wadhawan (1973). He has considered a two dimensional infinite circular cylinder of isotropic medium and solved the harmonic problem of thermoelasticity under small vibration. Gonsovskii and Rossikhin (1974) have investigated the propagation of plane harmonic wave in an anisotropic medium and observer four different types of damped and dispersed waves, a quasi-longitudinal, two quasi-transverse corresponding to the elastic field and a heat wave corresponding to the thermal field. Ignaczak (1979) has established the uniqueness results under this generalized thermoelasticity theory. Chandrasekharaiah (1980) has applied the short time approximation to study the half space problem of thermoelasticity and analyzed the exact discontinuities in mechanical and thermal fields due to applied thermal shock on the boundary. Thermoelastic wave propagation inside an infinite medium with a spherical cavity has been investigated by Mukhopadhyay et al. (1991) under the step-rise in temperature and pressure on the boundary surface. Some other investigations and discussion under the LS thermoelasticity theory have been reported in the review articles by Boley (1980), Chandrasekharaiah (1986b) and Joseph and Preziosi (1989; 1990). Several theoretical results and some numerical approaches to solve the thermoelastic problems under the LS theory have been demonstrated in the books by Hetnarski and Eslami (2009) and Ignaczak and Ostoja-Starzewski (2010). Picard (2005) has reported the structural formulation for linear thermoelasticity in nonsmooth media. Subsequently, a structural formulation for linear material laws in classical mathematical physics has been introduced by Picard (2009), who has considered a class of evolutionary problems that covered a number of initial boundary value problems of classical mathematical physics. The corresponding solution theory is also established here, and the well-posedness of classical thermoelasticity and Lord–Shulman theory are shown to be covered by this model. This study has been further extended by Mukhopadhyay et al. (2016). They have studied various models of thermoelasticity theory and have shown that these models can be treated within the common structural framework of evolutionary equations, and considering the flexibility of the structural perspective, they have obtained well-posedness results for a large class of generalized models allowing for more general material properties such as anisotropies, inhomogeneities, and so on.

The thermoelasticity theories by Green and Naghdi (1991; 1992; 1993) have also aroused much interest in researchers over the years to understand these new generalized thermoelasticity theories (GN-I, GN-II and GN-III theories) that involve thermal displacement as a new constitutive variable. An alternative formulation of the GN-II thermoelasticity theory in terms of entropy heat flux has been developed by Chandrasekharaiah (1996a) to establish the uniqueness theorem. Chandrasekharaiah (1996c) has developed an alternative approach to discuss the uniqueness results on the GN thermoelasticity theory, and further Chandrasekharaiah (1996b) has also studied a one dimensional linear thermoelastic problem to investigate the propagation of plane wave under the thermoelasticity theory without energy dissipation (GN-II theory). Chandrasekharaiah and Srinath (1997c) have discussed the thermoelastic interactions under GN-II theory due to point heat source. Chandrasekharaiah and Srinath (1997b) have also investigated wave propagation in a rotating thermoelastic body. Further, Chandrasekharaiah and Srinath (1997a) have employed the GN-II thermoelasticity theory to study the thermoelastic behaviour inside an axisymmetric unbounded medium with a cylindrical cavity. Dhaliwal et al. (1997) have solved the thermoelastic problem under the GN-III theory and noted that this theory also suffers from the drawback of the infinite speed of heat propagation. Ieşan (1998), Quintanilla (1999; 2002) have developed some theoretical results under the GN thermoelasticity theory. Ciarletta (1999) has developed a micropolar thermoelasticity theory based on GN theory and also formulated a Galerkin type solution under this theory. Other relevant works under Green and Naghdi theory have been carried out by Svanadze et al. (2006), Chandrasekharaiah and Srinath (2000), Misra et al. (2000), Wang and Slattery (2002), Sharma et al. (2003), Quintanilla and Straughan (2004), Roychoudhuri and Bandyopadhyay (2005), Bagri and Eslami (2007b), Mallik and Kanoria (2008), Abbas and Othman (2009), Chiriţă and Ciarletta (2010) and many others. Mukhopadhyay and Kumar (2010) have applied a state-space approach to solve a thermoelastic problem under the GN model. Tiwari and Mukhopadhyay (2017) have investigated the wave propagation inside an electro-magneto-thermoelastic medium under GN-II theory. Abbas (2018) studied the free vibration of a nanobeam resonator under the GN thermoelasticity theory. Jahangir et al. (2020) have investigated the diffusion effect on the propagation of a plane harmonic wave under the micro-stretched thermoelastic medium. EL-Attar et al. (2019) have examined the effect of phase-lag on the GN theory for electro-thermoelastic medium. Abouelregal (2020) has developed a new generalization of the GN thermoelasticity theory, including the higher order time differential and phase-lag terms. Sarkar et al. (2020) have considered the non-local effect to study the effect of the laser pulse on the transient wave under GN thermoelasticity theory. Recently, Zenkour (2021), Chirilă et al. (2021) and Hendy et al. (2021) have investigated the GN thermoelasticity theory in different thermomechanical contexts.

The DPL thermoelasticity theory by Chandrasekharaiah (1998) is the generalization of heat conduction theory involving two phase-lag parameters Tzou (1995a; 1995b). Chaandrasekharaiah (1998) has presented a detailed discussion about the DPL theory along with other hyperbolic thermoelasticity theories in his review article. Hetnarski and Ignaczak (1998) have described the analytical approach for the generalized thermoelasticity theories and compared the results under all previously developed thermoelasticity theories with the results under DPL thermoelasticity theory. Quintanilla (2002; 2003) has performed some qualitative analysis on the DPL theory to study the stability of the thermoelastic problem and proved that the DPL heat conduction model is not unconditionally stable. Further, he has also derived the stability condition for the one dimensional problem. Further, Horgan and Quintanilla (2005) have investigated the spatial behaviour of solutions under the DPL heat conduction theory. Al-Huniti (2005) and Al-Nimir (2005) have analyzed the thermoelastic responses inside a composite slab under the DPL thermoelasticity theory. Roychoudhary (2007b) studied the thermoelastic wave inside an elastic half space under DPL theory and observed two different waves in the solution of the problem. Abdullah (2009) has applied the DPL theory to investigate the thermomechanical properties inside semi-infinite medium under the influence of ultrashort laser pulse heating on the boundary. Ghazanfarian and Abbassi (2009) have examined the effects of phonon scattering on boundary under the DPL theory to simulate the micro and nano scale heat conduction. Prasad et al. (2010) have investigated the propagation of the plane harmonic wave under the DPL thermoelasticity theory. Authors have formulated an exact expression for the dispersion relation analytically and also derived the asymptotic expressions for the variables characterizing thermoelastic waves such as phase velocity, penetration depth, specific loss and amplitude ratio. Mukhopadhyay et al. (2011) have established the domain of influence results for DPL theory to prove that there are no thermal and elastic signals outside a suitably defined bounded domain. They also verified the finite speed of thermal and mechanical wave propagation under DPL theory. Several other authors have also worked on the DPL theory to investigate different thermoelastic problems. In this context, the articles Abouelregal (2011), Zenkour et al. (2013), El-Karamany and Ezzat (2014), Sarkar (2017), Megana and Quintanilla (2018), Liu and Quintanilla (2018), Mondal et al. (2019), Biswas (2019), Gupta and Mukhopadhyay (2019b), Mondal (2020) and Campo et al. (2021) can also be referred.

Three-phase-lag (TPL) theory is a more generalized form of the thermoelasticity theory that contains the three-phase-lag parameters in the heat conduction law. Detailed qualitative analysis on this model has been carried out by Quintanilla (2008) to obtain the restriction of the material parameters to ensure the exponential stability of solutions. Further, Quintanilla (2009) has investigated the well-posedness of thermoelastic problems under TPL theory. Kar and Kanoria (2009) have solved a thermoelastic problem of functionally graded hollow sphere under thermal shock to analyze this generalized theory. Authors have applied the Laplace transform technique to simplify the governing equations and then applied an eigenvalue approach to solve the matrix form of the system of equations. Kumar and Mukhopadhyay (2009) have solved a thermoelastic problem of an infinite cylindrical cavity under step input temperature on the boundary and highlighted the significance of phase-lag parameters in the TPL thermoelasticity theory. Subsequently, Mukhopadhyay et al. (2010) have formulated Galerkin's type representation of the solution under this theory. Kothari et al. (2010)

have derived the fundamental solution for the thermoelastic problem of homogeneous and isotropic medium, and by using this solution, the authors have examined the effect of heat source and concentrated load on the unbounded medium. Analysis on the effects of TPL theory for the plane harmonic wave propagation has been carried out by Kumar and Mukhopadhyay (2010a). Kumar and Chawla (2011) have investigated the plane wave propagation in an anisotropic thermoelastic medium under DPL and TPL thermoelasticity theories. El-Karamany and Ezzat (2013) have established the generalization of the TPL theory of inhomogeneous and anisotropic medium in the context of micropolar thermoelasticity theory and also presented some theoretical results based on this theory. Kothari and Mukhopadhyay (2013) have investigated the thermoelastic interactions inside a functionally graded hollow disk by applying the finite element method along with the Laplace transform technique. Abbas (2014) has examined the effect of TPL theory inside the fiber reinforced anisotropic medium. Later on, Othman and Said (2014) have investigated a two dimensional magneto-thermoelastic problem of fiber reinforced medium under the TPL theory. Kumar et al. (2015) have established a domain of influence theorem under the three-phase-lag thermoelasticity theory. Biswas et al. (2017) have investigated the propagation Rayleigh wave under the TPL thermoelasticity theory. The study carried out by Singh et al. (2019), Abouelregal (2019), Kumar and Mukhopadhyaya (2020), Liu and He (2020), Mondal and Kanoria (2020), Prasad and Kumar (2021), Othman and Abbas (2021) are also worth to be mentioned in this context.

In 1972, Green and Lindsay (1972) demonstrated a completely different approach to generalize Biot's thermoelasticity theory, which is based on the modified Clausius inequality (1967) and well established from the firm grounds of the irreversible thermodynamics. This theory has attracted the serious attention of researchers since its development. Green (1972) investigated the propagation of acceleration waves under the linear thermoelasticity based on the GL model. The generalization of the Green and

Lindsay (GL) theory to the micropolar thermoelasticity has been derived by Boschi and Iesan (1973). Further, Boschi and Iesan (1973) considered a plain stress condition in linear thermoelasticity under the GL theory and applied the associated matrix method to demonstrate the Galerkin type representation. Moreover, authors have used this representation to obtain the solution of the vibration problem under the concentrated body forces and heat source. Dost and Tabarrok (1978) determined the condition for the existence of acceleration waves in micropolar thermoelastic solids under GL theory and formulated an implicit expression for the acoustic tensor, which provides the speed of wave propagation. Igznaczak (1978b) derived the domain of influence theorem based on the GL thermoelasticity theory and verified the finite speed of thermal wave propagation theoretically. Further, Igznaczak (1978a) established the Boggio type decomposition theorem for linear thermoelasticity theory under the GL model. Agarwal (1978) considered a homogeneous and isotropic thermoelastic half space to study the surface waves under LS and GL theories. Agarwal (1979) has further investigated the propagation and stability for the time dependent plane harmonic waves under these theories. Mechanical and thermal acceleration wave propagation under the nonlocal thermoelasticity theory of the GL model has been examined by Lindsay and Straughan (1979). A detailed literature review of the thermoelasticity theories predicting the finite speed of thermal waves, including GL theory, has been reported by Igznaczak (1980). Prevost and Tao (1983) considered GL theory to demonstrate the finite element formulation for the transient problems of thermoelasticity. Chandrasekharaiah and Srikantiah (1984) formulated the governing equations for the temperature-rate dependent theory (GL theory) for thermo-pizoelectricity and proved a uniqueness theorem based on the derived theory. Further, Chandrasekharaiah and Srikantiah (1984) discussed a problem of homogeneous and isotropic unbounded thermoelastic body rotating with uniform angular velocity and examined the effect of rotation on the characteristics of wave propagation. Some theoretical results on temperature-rate dependent theory have been established by Chandrasekharaiah and Srikantaiah (1983) and Gladysz (1985). Tao and Prevost (1984) applied the perturbation technique to study the wave propagation under GL theory and analyzed the effects of relaxation time parameters. Ignaczak (1985) investigated thermoelasticity theory with two relaxation times (GL theory) under instantaneous heat source in an infinite medium and discovered a jump discontinuity in displacement function. While studying the thermoelastic plane wave under GL theory, Choudhuri (1985) analyzed the effect of rotation and the relaxation parameters on wave propagation. Chen and Wang (1988) applied the finite element technique along with the Laplace transform method to investigate the thermomechanical effects inside an axisymmetric cylinder under GL theory. Noda et al. (1989) considered a system of unified governing equations for the LS and GL theories and examined the thermoelastic interactions inside a one dimensional infinite solid with a hole. Dhaliwal and Rokne (1989) used the GL theory to solve a half space problem under a sudden applied temperature rise and noted two discontinuities in the temperature and displacement fields along with the infinite discontinuity for the stress fields at the wave fronts. Roychoudhari and Roy (1990) considered the generalization of the GL theory to magneto-thermoelasticity and studied the wave propagation due to thermal effect in a finitely conducting half space. Several other problems of the thermoelasticity in the context of GL theory has been investigated by Chandrasekharaiah and Murthy (1991), Sherief (1992), Hetnarski and Ignaczak (1993), Sherief (1993; 1994), Anwar and Sherief (1994), Chandrasekharaiah and Murthy (1994), Sanderson et al. (1995), Misra et al. (1996), Singh and Kumar (1998), Singh (2000), Ezzat and El-Karamany (2002), Othman (2003; 2004), El-Maghraby (2005), Youssef (2006a; 2006c), Abbas and Abdalla (2008). Othman (2010a) has analyzed the effect of rotation and thermal shock on the magneto-thermoelastic half space under GL theory. Further, Othman (2010b) has extended this work on electro-magneto-thermoelastic medium under GL theory. Darabseh et al. (2012) have investigated a thermoelastic problem of a functionally graded thick hollow cylinder under the GL theory. Sarkar and Lahiri (2012) have discussed the modified Ohm's law, including the effect of thermal gradient and charge density under this temperature-rate dependent theory. Lotfy (2012) have discussed a two dimensional mode-I crack problem of the fiber-reinforced thermoelastic medium under GL theory. Othman et al. (2013) have studied the effect of temperature-rate on the thermoelasticity theory under the influence of the gravitational field. Youssef and EL-Bary (2014) have examined thermomechanical interactions under the GL theory by comparing the behaviour of field variables under four different thermoelasticity theories. Filopoulos et al. (2014) have developed an enhanced GL theory for the linear thermoelastic medium with microstructures. Zenkour (2015) have analyzed the effect of thermal shock inside a three dimensional thermoelastic medium and examined the results under different thermoelasticity theories. Aouadi and Moulahi (2015) studied the optimal decay rate for the unidimensional thermoelastic problem under GL thermoelasticity theory. Abbas (2015) derived an analytical solution to study the free vibration problem of a thermoelastic hollow sphere. Ailawalia et al. (2016) have examined the effect of internal heat source on microelongated thermoelastic half space under GL theory and derived the analytical expressions for the field variables. Kumar et al. (2016) have investigated a problem of the cylindrical cavity to analyze the GL theory in the context of two-temperature theory and highlighted the differences of GL theory with different other thermoelasticity theories. Reflection and refraction of P wave have been examined at the interface of thermoelastic and porothermoelastic medium by Wei et al. (2016). Chyr and Shynkarenko (2017) have established a well-posedness result for the dynamical problem of thermoelasticity under the GL theory. Ezzat and Al-Bary (2017) have discussed the application of the magneto-thermoelasticity theory involving fractional order derivatives for the perfectly conducting cylindrical cavity. Several other thermomechanical problems have also been studied in recent years to explore the application of GL theory (see the refs. Abd-alla et al. (2017), Kumar et al. (2017), Magaña et al. (2018), Ezzat et al. (2018), Aouadi et al. (2019), Guo et al. (2019), Quintanilla et al. (2019), Marin et al. (2020a), Sherief et al. (2020), Mondal and Pal (2021), Sarkar (2021)).

Modified GL (MGL) theory or strain and temperature-rate dependent theory (2018) is a recently proposed thermoelastic model that attempts to overcome the drawback of discontinuity in the displacement field and involving the effect of stain and temperature-rate terms in the governing equations. This theory has gained attention from the researchers who have investigated this theory in the contexts of different thermoelastic problems and reported some interesting observations on this theory. Quintanilla (2018) presented some qualitative results on this theory, including continuous dependence of the solution on initial conditions and Phragman-Lindelof alternative for the spatial behaviour of the solution. Gupta and Mukhopadhyay (2019a) derived the general solution of the thermoelastic system of governing equations in terms of metamorphic functions with the help of the representation theorem of Galerkin type solution under MGL theory. Sarkar et al. (2019) investigated the reflection and wave propagation in an isothermal stress free surface of a thermoelastic medium in this context. Further, Singh and Mukhopahdhyay (2020) have considered a homogeneous and isotropic cylindrical cavity under thermal shock and observed the infinite speed behaviour of the thermal wave under the MGL thermoelasticity theory. Singh et al. (2020) derived the fundamental solution for the distribution of thermal and elastic fields under the strain and temperature-rate dependent theory (MGL theory). Sarkar et al. (2019; 2020) and Sarkar and De (2020) have investigated the reflection and time harmonic wave propagation under different thermoelastic mediums and observed the existence of longitudinal wave and one vertically shear type wave (SV wave). Gupta and Mukhopadhyay (2020) have considered three different theories: LS, GL and MGL to study the harmonic plane wave propagation and highlighted the differences under these thermoelasticity theories. Shakeriaski et al. (2021a) have considered the MGL theory to demonstrate the implementation of a nonlinear numerical method for solving the coupled thermoelastic problems and also validated the numerical results by comparing them with the analytical solution of the corresponding problem. Recently, Shakeriaski et al. (2021b) have reported the advancement of the recent thermoelasticity theories (including MGL theory) and their applications. Mohamed et al. (2021) examined the thermal effect on elastic solid due to absorption of the leaser pulse radiation under MGL theory. Although the MGL has been studied for various thermoelastic problems. However, it is a topic of active interest of researchers, and several thermoelastic problems are yet to be investigated under this theory.

The two-temperature thermoelasticity theory is a generalization of thermoelasticity theory for non-simple materials. Iesan (1970) considered a linear theory of two-temperature thermoelasticity given by Chen et al. (1969) for homogeneous and isotropic medium and presented the uniqueness theorem, variational principle and reciprocity results based on this theory. Warren (1972) has studied the two-temperature theory for the cylindrical and spherical cavities of isotropic materials. Warren and Chen (1973) have studied a wave propagation problem under the two-temperature theory. Nunziato (1975) formulated the general condition for the acceleration wave inside the thermoelastic medium and observed that the wave velocity of the two-temperature theory is always greater than the Classical theory. Colton and Wimps (1979) examined the asymptotic behaviour of the fundamental solution under the two-temperature theory of heat conduction. Here, the authors have also noted that the main effect of the two-temperature theory is to mitigate the maximum compressive stress of the thermoelasticity theory without two-temperature. Quintanilla (2004a) demonstrated the existence, uniqueness, stability, and convergence of solution characteristics under the two-temperature theory. Puri and Jorden (2006) proposed a generalization of the two-temperature theory in the context of LS theory and investigated a wave propagation problem under this theory. However, a more appropriate justification of the two-temperature theory for the isotropic medium has been proposed by Youssef (2006b). Further, Youssef and AL-Harby (2007) have applied a state space approach to solve the thermoelastic problem of an infinite medium under the two-temperature LS theory. Moreover, Youssef (2008) has solved a thermoelastic problem under the ramp type heating to study the two-temperature LS theory. Abbas and Youssef (2009) and Ezzat et.al. (2009) have solved the different thermoelastic problems to analyze the two-temperature theory in the context of magneto-thermoelasticity. Magana and Quintanilla (2009) have considered the two-temperature theory of the GL model and derived the uniqueness and growth of the solution under two-temperature LS and GL theories. Quintanillla and Jordan (2009) have presented an exact solution for the mixed initial boundary value problem of two-temperature thermoelasticity theory in the context of DPL theory. Mukhopadhyay and Kumar (2009) have investigated the thermomechanical interactions inside the infinite cylindrical cavity under the two-temperature LS theory. Ezzat and Awad (2010) have derived the constitutive relations for the micropolar thermoelasticity of two-temperature and established the uniqueness results based on this theory. Youssef (2010) and Youssef and El-Bary (2010) have solved different thermoelastic problems under the two-temperature LS theory. Kaushal et al. (2010) have solved a thermoelastic problem and compared the results under the two-temperature versions of LS and GL theories to highlight the differences between these two theories. Convolutional type variational principle for two-temperature thermoelasticity theory for the homogeneous and isotropic medium has been established by Kumar et al. (2010). Further, Kumar and Mukhopadhyay (2010b) have studied the propagation of plane harmonic waves inside a thermoelastic medium under two-temperature theory. Banik and Kanoria (2011) employed the two-temperature theory to investigate the behaviour of the physical field variables inside an infinite thermoelastic medium with a spherical cavity under the LS and GN models. El-Karamany and Ezzat (2011a) have introduced the fractional order term in the two-temperature relation and derived a fractional thermoelasticity theory of two-temperature model and also proved the uniqueness and reciprocity theorems on this model for the homogeneous and isotropic medium. Further, El-Karamany and Ezzat (2011b) and Youssef and Elsibai (2015) have investigated the two-temperature generalization of the Green and Naghdi (1991; 1992; 1993) theory and shown that this generalization exhibits the dissipation of energy for the nonzero values of the two-temperature parameter. Mukhopadhyay et al. (2011) and Singh and Bijarnia (2012) have studied the two-temperature version of the DPL thermoelasticity theory. Prasad and Mukhopadhyay (2012) have investigated the effect of rotation on the harmonic wave propagation of the two-temperature thermoelasticity theory. Kumar et al. (2017) have performed an in-depth analysis of plane harmonic waves under two-temperature thermoelasticity of GL model and observed that the longitudinal wave is coupled with the temperature effect. However, the transverse wave has no effect on the thermal field. Miranville and Quintanilla (2017) have discussed the spatial behaviour of the solutions under the two-temperature LS and GL thermoelasticity theories. Mukhopadhyay et al. (2017) have applied the Hilbert space framework to investigate the well-posedness of the governing equations of the two-temperature thermoelasticity theory. An alternative two-temperature thermoelastic model has also been proposed here in which one can avoid involving roots of an unbounded operator. An et al. (2017) have presented a generalization to the two-temperature model involving the coupled phonon interactions in the nanosized graphene. Sur and Kanoria (2017) have investigated the effect of temperature dependent thermal loading inside three dimensional thermoelastic medium under the TPL version of two-temperature theory. Youssef and Al-Bary (2018) have proposed new two-temperature relation predicting the finite speed of thermal waves. Zenkour (2018) examined the thermomechanical interactions inside a micro beam under the refined two-temperature multi phase-lag theory of thermoelasticity. Kaur and Lata (2019) have addressed the hall current effect on the plane wave propagation inside a transversely isotropic elastic medium in the context of fractional order two-temperature theory. Mittal and Kulkarni (2019) have investigated a thermoelastic problem under the fractional order two-temperature theory. Deng et al. (2020) have applied the two-temperature theory to study the non-equilibrium transport involving weak phonon coupling. Sur and Mondal (2020a) have applied the two-temperature theory to examine the non-locality effect in the vibration of microbeams. Recently, Abbas et al. (2021), Lotfy (2021), Youssef et al. (2021) have used the two-temperature theory in the context of different thermoelastic models to study the various thermomechanical problems.

The study of porous materials includes a large class of engineering problems related to water saturated soil, sound absorbing materials, asphalt concrete pavements etc. Porous solids also exist in nature in the form of crustal and reservoir rocks in the earth and therefore have a wide range of applications in the field of geophysics and related topics. The researchers are devoting increasing attention to study the thermal and mechanical interactions in the porothermoelastic solids. Treitel (1959), Armstrong (1984), Jacquey et al. (2015), Fu (2012) etc., have successfully verified its geophysical relevance by investigating seismic attenuation, geothermal and hydrocarbon exploration. Bear et al. (1992) and Levy et al. (1995) have dealt with the fluid transport phenomenon through a porous medium and derived basic equations for the microscopic dynamics. It is worth mentioning that Biot (1962a; 1962b; 1964) has extended the concept of poroelasticity to the acoustic wave propagation theory. Biot and Temple (1972), Rice and Cleary (1976) have discussed different problems based on poroelasticity. Pecker and Deresiewicz (1973) have further included the thermal effects in poroelasticity and studied the wave propagation in liquid filled porous media. Crown and Nuziato (1983) have developed the linear theory for elastic materials with voids. Mctigue (1986), Kurshige (1989), Wang and Papamichos (1994) have discussed the problems on fluid saturated solid rocks and explained the heat and fluid flow in a poroelastic medium. Several authors have further worked on the evolution of the theory to various thermomechanical problems (see refs. Fourie and Du Plessis (2003);

Wang (2017); Ghassemi and Diek (2002)). While incorporating the thermal effects, most of these authors have followed Biot's theory of heat conduction (1956b). During the last few years, several developments have been made to the theory of thermoelasticity, which motivated the researchers to establish various generalized theories and further investigate the theoretical results, including uniqueness, stability analysis, existence, reciprocity and domain of influence results. The developments of generalized porothermoelasticity theory, thermoelasticity theory for materials with voids, thermoelasticity with double porosity, the theory of dipolar bodies are worth to be mentioned in this direction. In 2007, Youssef (2007) has established a theoretical foundation to the generalized theory of porothermoelasticity admitting finite speed of heat signals. He derived the governing equations for isotropic medium and also proved the uniqueness theorem based on this theory. Sharma (2008b) has discussed the results of wave propagation under porothermoelasticity. Sherief and Hussein (2012) have also derived the mathematical model for porothermoelasticity theory for the short time filtration cases and investigated the uniqueness and reciprocity results corresponding to the proposed model. Nunziato and Cowin (1979) had established the theory for the behaviour of elastic porous structures. Later on, in 2014, Ishan and Quintanilla (2014) have extended the Nunziato–Cowin theory of materials with voids and derived the thermoelasticity theory for the materials with double porosity. In their work, they have also shown the uniqueness results of a double porosity problem using the logarithmic convexity argument. Emin et al. (2021) have also discussed the uniqueness results for thermoelastic materials with double porosity structures. Rohan et al. (2016) have studied the fluid saturated elastic media with double porosity by deriving effective parameters of the static problems. The fractional thermoelasticity for the porous asphalt materials has been founded by Ezzat and Ezzat (2016). Roubíček (2017) has formulated a geophysical model for the heat and fluid flow in a damageable poroelastic medium. Miller and Penta (2020) have derived quasi-static governing equations for the macroscale behaviour in the poroelastic solids. Iovane and Passarella (2004) have studied Saint-Venant's principle in a dynamical porous elastic medium with a memory of heat flux and obtained the domain of influence theorem accordingly. Iesan and Quintanilla (2014) have used the semi-group approach to show the existence results for the double porosity theory. Marin and Nicaise (2016) have derived the existence and exponential stability results for the dipolar thermoelastic bodies with double porosity. Further, Marin et al. (2015) have established an extension to the theory of double porosity to the micropolar bodies. To mention some more interesting results for the theory of dipolar thermoelastic bodies, we refer to the work carried out very recently by Marin et al. (2020c) and Marin et al. (2020b). Liu and Chen (2017) have investigated the well-posedness and exponential decay results for porothermoelastic systems having a time varying delay term in the internal feedback. Zampoli and Amendola (2019) have shown the spatial behaviour using the domain of influence results for the cylindrical anisotropic and inhomogeneous porothermoelastic solid under dual-phase-lag and three-phase-lag theories. Marin et al. (2020e) have formulated the domain of influence results for the initially stressed thermoelastic solid with voids. Wei and Fu (2020) have derived the fundamental solution for the porothermoelastic solids. Recently, Marin et al. (2020d) have derived the structural stability results for an elastic body with voids. Along with the theoretical development, several thermomechanical problems have also been investigated to study the effects of porosity on the thermoelastic solid under various thermal and mechanical loads. The effect of thermal loading due to laser pulse on thermoelastic porous medium under a thermoelasticity has been discussed by Othman and Marin (2017). Sherief and Hussein (2012) have solved a half space problem to study liquid filled solid medium under applied thermal shock. Ezzat and Ezzat (2016) have solved a porothermoelastic problem under the fractional order theory. Carcione et al. (2019) have carried out a numerical simulation to study the wave propagation in the porothermoelastic solid. Sur (2020) has analyzed the wave propagation in porous asphalts on accounts of a memory response. Alzahrani and Abbas (2020a) have implemented the finite element method to solve a one dimensional problem of porothermoelasticity under Green and Naghdi theory. Saeed et al. (2020) have considered the governing equations for temperature-rate dependent (GL) theory of porothermoelasticity for isotropic medium and solved a one dimensional problem using FEM. Alzahrani and Abbas (2020b) and Guo et al. (2019) have analyzed different thermomechanical problems in this context. Guo and Xiang (2021) have analyzed the effect of the visco-elastic relaxation parameters on the homogeneous and isotropic hydro-thermoelastic medium.

1.7 Objective of the Thesis

The main objective of the present thesis is to study some aspects of temperature-rate dependent (TRD) thermoelasticity theory and analyze its applications to the various thermomechanical problems. It also aims at developing various generalizations of the TRD theory to different thermomechanical contexts and establish some useful theoretical results on these theories. Applications of the theories are further elaborated by solving different thermomechanical problems. Implementation and efficiency of an alternative numerical technique are also dicussed in the thesis for the solution of coupled dynamical thermoelastic problems.

To understand thermomechanical behaviour of TRD theory, a thermoelastic problem of the hollow disk is formulated in the context of LS and TRD thermoelasticity theories and a unified system of governing equations is derived. Along with the investigation of the TRD theory, an implementation of the complete finite element approach is presented as an alternative and more efficient approach as compared to trans-finite element approach for solving the dynamical problems of thermomechanics. The differences in results under the two theories are highlighted. In order to investigate the strain and temperature-rate dependent theory, a thermoelastic problem of the functionally graded hollow disk is further solved in a unified way under TRD and STRD theories, and the effects of the strain-rate term in the theory of thermoelasticity are investigated. The functionally graded material is taken to study the thermoelastic interactions due to variable material configuration.

An attempt is made to derive the basic governing equations of the two-temperature generalization of the TRD theory by following the principles of irreversible thermodynamics. This theory is not yet available in the literature. A more general two-temperature relation involving the temperature-rate term is derived here. Further, this new relation is examined for one dimensional half space problem. Thermoelastic interactions due to applied thermomechanical load in a two dimensional medium with the presence of crack inside the medium are further investigated in this context. The effects of temperature-rate terms are analyzed in detail.

Investigation on thermomechanical interactions inside a porothermoelastic material in the context of generalized thermoelasticity theories is limited in literature, and the porothermoelasticity theory in the context of TRD theory for the anisotropic medium is not yet proposed in the literature. Therefore, to establish TRD porothermoelasticity (TRDPTE), a mathematical formulation from fundamental laws of thermodynamics is presented by using the necessary constitutive assumptions. Some basic fundamental theorems are established in the context of this new theory. An applications of this theory is further elaborated by considering a half space problem to investigate the thermomechanical interactions due to the application of thermal shock at the boundary of the two phase medium. The present analysis illustrates various essential aspects of the theory which serve the purpose of the thesis.