Chapter 5

Solution of Riemann Problem of Conservation laws in van der Waals Gas

"Mathematics reveals its secrets only to those who approach it with pure love, for its own beauty".

-Archimedes

5.1 Introduction

During the recent years, the study of Riemann problem (RP) is frequently used for theoretical and numerical study of the quasilinear hyperbolic system of partial differential equations. It is well known that RP is a specific initial boundary value problem having contact discontinuity for the hyperbolic system of non-linear PDEs which is described by discontinuous initial data. However, solution of RP composed of elementary waves depending upon its characteristics field. The occurrence of discontinuities is a natural phenomenon in several areas such as photoionized gas, space science i.e., supernova explosions, space re-entry vehicles, stellar winds, collision of galaxies etc. Solution of RP is unique under the consideration of discontinuous initial jumps in terms of constant states which are separated into elementary waves such as shock waves, contact discontinuities or rarefaction waves. In the area of engineering and science, many of the researchers deal with mathematical system which composed of quasilinear PDEs. The analytical study of non-linear hyperbolic conservation law is interesting but leads to cumbersome task in Mathematics. In various areas of natural science and physical science, the analytical solution of the quasilinear hyperbolic system of partial differential equations plays a prominent role for the qualitative characterization of many physical processes and phenomena.

The purpose of the present chapter is to provide a detailed analysis of the analytical solution to the RP for 1-D, time dependent Euler's equation for van der Waals gas. The analytical solution of the RP is widely used for the understanding of Euler's equation because all the elementary wave properties such as rarefaction waves, shock waves and contact discontinuities appear in the form of characteristics. The shock-tube problem, a well known physical problem of Gasdynamics and also for the other basic physical problems in conservation form may be well explained through RP. The interested reader is referred to reference Courant and Friedrichs [23] for more details related to the basic physical problem in gasdynamics. Lax [74] determined the solution of RP for the condition when initial states consist of two constant states which are separated by jump in flow variables. The investigation of solution of RP in van der Waals gas is more complex than ideal gas.

A detailed discussion related to the solution of RP in van der Waals gasdynamics model have been presented by several researchers. In past, many attempts have been made to analyze the classical wave properties of solution to the RP in various gasdynamic regimes where the basic equations are system of quasilinear hyperbolic PDEs. The analytical solution of RP in magnetogasdynamic flow was discussed by Singh et al.[100]. The similar technique has been utilized by Ambika et al.[95] and Nath et al.[150] to obtain the solution of RP for nonideal gas flow and ideal polytropic dusty gas flow respectively. By using direct approach, the solution of RP for ideal dusty gas is analyzed by Gupta et al.[101]. The Riemann problem (RP) for magnetogasdynamic flow is solved by Hu et al.[98]. Pooja et al.[151] determined the (exact) closed form solution of the generalized riemann problem (GRP) for the Chaplygin gas equation by using the Method of Differential Constraint (DC). Pooja et al.[152] determined the existence and uniqueness of solution to RP for nonideal magnetogasdynamic flow. Yang et al.[105] and Guo et al.[106] presented the solution of RP with delta initial data for hyperbolic system. Kuila et al.[102, 103] discussed about the Riemann solution for 1-D ideal and nonideal isentropic magnetogasdynamics flow. Z. Shao [153] studied the Riemann problem with the initial data containing the Dirac delta function for the isentropic relativistic Chaplygin Euler equations. Z. Shao [154] obtained the solution of the Riemann problem for the relativistic full Euler system with generalized Chaplygin proper energy density-pressure relation. Z. Shao [155] and Tatsien et al.[156] investigated the nonlinear Riemann problem with nonlinear boundary conditions for hyperbolic system of quasilinear PDE's. By neglecting the intermolecular forces of attraction between the particles in covolume equations of state, Kipgen et al.[157] obtained the analytical solution of Riemann problem for conservation laws of van der Waals reacting gases with dust particles. By utilizing the iterative technique for the explicit solution of RP for classical gasdynamics have been discussed by several researchers such as[20, 108, 22, 109, 21] in different gaseous media.

The present chapter is devoted to determine the classical solution to the RP for van der Waals gasdynamics. The study is arranged as follows:

In section 2, we describe the basic equations for 1-D motion of van der Waals gas and characteristics of the governing system. In section 3, the governing system is reduced into conservation form and corresponding Riemann invariants have been calculated. The classical wave solutions i.e. shock wave, simple wave and contact discontinuity is determined in section 4, section 5 and section 6 respectively. In section 7, the properties of elementary wave curves is discussed. Section 8, consists of results and discussion concerning to the present investigation. Finally, in the last section 9, conclusions of the study is presented.

5.2 Basic Equations

The governing equations for 1-D motion of unsteady planar flow of non-ideal gas modeled by van der Waals EOS is written as

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0, \tag{5.1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{5.2}$$

$$\frac{\partial E}{\partial t} + v\frac{\partial E}{\partial x} - \frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x}\right) = 0, \tag{5.3}$$

where ρ and v is fluid density and particle velocity respectively. p denotes the pressure. Here, x represents spatial coordinate and t represents time. In the above equation (5.3), the internal energy per unit mass of the mixture is denoted by E and defined as

$$E = \frac{(p + a\rho^2)(1 - b\rho) - a(\gamma - 1)\rho^2}{(\gamma - 1)\rho}, \tag{5.4}$$

where $\gamma = \frac{C_p}{C_v}$, C_p and C_v represent the specific heat of gas at constant pressure and constant volume respectively. Here, a and b are constants which represents attractive force between the gas molecules and covolume of the gas respectively.

By using Eq.(5.4) in Eq.(5.3), we obtain

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + c^2\rho\frac{\partial v}{\partial x} = 0, \tag{5.5}$$

where, c is the sound velocity which is written as

$$c = \left(\frac{\gamma p + a\rho^2(\gamma - 2 + 2b\rho)}{\rho(1 - b\rho)}\right)^{1/2}. \tag{5.6}$$

Now, governing Eqs.(5.1) to (5.3) can be rewritten in the following form

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0, \tag{5.7}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right) = 0, \tag{5.8}$$

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + c^2\rho\frac{\partial v}{\partial x} = 0, \tag{5.9}$$

The system of governing Equations(5.7) to (5.9) can be written in the matrix form as

$$\frac{\partial V}{\partial t} + M\frac{\partial V}{\partial x} = 0, \tag{5.10}$$

where

$$V = \begin{bmatrix} \rho \\ v \\ p \end{bmatrix}, \quad M = \begin{bmatrix} v & \rho & 0 \\ 0 & v & \frac{1}{\rho} \\ 0 & \rho c^2 & v \end{bmatrix}.$$

The eigenvalues of matrix M can be obtained as

$$\lambda_1 = v - c, \lambda_2 = v, \lambda_3 = v + c. \tag{5.11}$$

The eigenvectors corresponding to distinct eigenvalues are

$$k_1 = \begin{bmatrix} -\frac{\rho}{c} \\ 1 \\ -\rho c \end{bmatrix}, k_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, k_3 = \begin{bmatrix} \frac{\rho}{c} \\ 1 \\ \rho c \end{bmatrix}. \tag{5.12}$$

Since all the $\lambda_i' s, i = 1, 2, 3$ are real and distinct and eigenvectors $k_i' s, i = 1, 2, 3$ are linearly independent. Hence, we deduce that the modified system (5.10) is strictly hyperbolic.

5.3 Riemann problem(RP) and Generalized Riemann Invariants(GRI)

The system of equations (5.10) can be written in the following conservation form as

$$\frac{\partial V^*}{\partial t} + \frac{\partial F(V^*)}{\partial x} = 0, \tag{5.13}$$

where $V^* = (\rho, \rho v, \rho(v^2/2 + E))^{tr}, F(V^*) = (\rho v, p + \rho v^2, v(p + \rho(v^2/2 + E)))$.
RP is an initial value problem for the governing model (5.10) with the following initial boundary condition

$$V^*(x, 0) = V_0^*(x) = \begin{cases} V_-^*, & \text{if } x < 0 \\ V_+^*, & \text{if } x > 0 \end{cases}. \tag{5.14}$$

Here, V_-^* represents left constant state and V_+^* represents right constant state which is divided by jump discontinuity at $x = 0$. The explicit solution of RP (5.13) with the initial condition (5.14) consisting of three waves corresponding to distinct eigenvalues $\lambda_1 = v - c, \lambda_2 = v, \lambda_3 = v + c$ as represented in Figure 1. In hyperbolic system, the characteristics fields are genuinely non-linear as $\nabla\lambda_j k_j \neq 0$ and characteristics fields are linearly degenerate as $\nabla\lambda_j k_j = 0$. Hence, 1 and 3-characteristics fields are genuinely non-linear and 2-characteristics field is linearly degenerate. Genuinely non-linear characteristics field will be either rarefaction wave or shock wave and linearly degenerate characteristics field will be contact discontinuity.

Since,

$$p = p(\rho, S) \tag{5.15}$$

i.e. pressure is function of density and entropy.

Therefore, the Riemann invariants (Π_1^j, Π_2^j) for governing system (5.10) corresponding to the jth-characteristic field can be written as:

Across 1-characteristics field, we obtain

$$\frac{d\rho}{-\rho/c} = \frac{dv}{1} = \frac{dp}{-\rho c}. \tag{5.16}$$

Across 3-characteristics field, we obtain

$$\frac{d\rho}{\rho/c} = \frac{dv}{1} = \frac{dp}{\rho c}. \tag{5.17}$$

From Eq.(5.15) and (5.16), we have

$$j = 1, \Pi_1^1 = S, \Pi_2^1 = v + \frac{2c}{(\gamma - 1)}(1 - \bar{b}) - \frac{\bar{a}c}{(\gamma - 1)}(1 - \bar{b}), \tag{5.18}$$

where, $\bar{b} = b\rho$ and $\bar{a} = a\rho$.

From Eq.(5.15) and (5.17), we have

$$j = 3, \Pi_1^3 = S, \Pi_2^3 = v - \frac{2c}{(\gamma - 1)}(1 - \bar{b}) + \frac{\bar{a}c}{(\gamma - 1)}(1 - \bar{b}). \tag{5.19}$$

Now, the one parameter families of shock waves, contact discontinuities and simple waves will be calculated here.

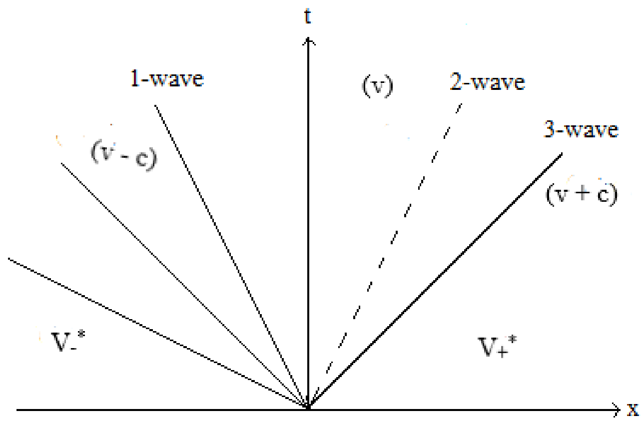

FIGURE 5.1: Solution of Riemann problem (RP) for 1-D Euler equations

5.4 Piecewise discontinuous solution (Shock wave)

Shock waves are bounded discontinuous solutions propagating with the speed of discontinuity σ corresponding to data which exist on both sides of jump discontinuity. For shock wave, Lax entropy conditions and Rankine-Hugoniot jump conditions [109] are satisfied. Also, the left constant state and right constant state are divided by either rarefaction wave or shock wave or contact discontinuity i.e.,

$$F(V_+^*) - F(V_-^*) = \sigma(V_+^* - V_-^*), \tag{5.20}$$

Hence, the R-H jump relations for system (5.10) is written as follows

$$-\sigma[\rho] + [\rho v] = 0, \tag{5.21}$$

$$-\sigma[\rho v] + [p + \rho v^2] = 0, \tag{5.22}$$

$$-\sigma[\rho(v^2/2 + E)] + [\rho v(E + v^2/2) + pv] = 0, \tag{5.23}$$

where, $[\rho] = \rho_+ - \rho_-$ denotes the jump in variable ρ.

For 1-shock wave, to solve (5.21)to (5.23) we obtain

$$\begin{cases} \sigma_1 = v - \sqrt{\frac{\rho_-[\rho]}{\rho[\rho]}}, \\ -[v] + \sqrt{\frac{\rho_-[p]}{\rho[\rho]}}[\rho] = 0. \end{cases} \tag{5.24}$$

For 3-shock wave,

$$\begin{cases} \sigma_3 = v + \sqrt{\frac{\rho_+[\rho]}{\rho[\rho]}}, \\ -[v] + \sqrt{\frac{\rho_+[p]}{\rho[\rho]}}[\rho] = 0. \end{cases} \tag{5.25}$$

The Lax entropy condition is written as

$$\lambda^{(j-1)}(V_-^*) < \sigma < \lambda^{(j)}(V_-^*), \lambda^{(j)}(V_+^*) < \lambda^{(j+1)}(V_+^*), j = 1, 3. \tag{5.26}$$

Using (5.24) to establish 1-shock wave curve,

$$S_1(\rho; V_-) : \begin{cases} \sigma_1 = v - \sqrt{\frac{\rho_-[\rho]}{\rho[\rho]}}, \\ -[v] + \sqrt{\frac{\rho_-[p]}{\rho[\rho]}}[\rho] = 0. \end{cases} \tag{5.27}$$

Using (5.25) to establish 3-shock wave curve

$$S_3(\rho; V_+) : \begin{cases} \sigma_3 = v + \sqrt{\frac{\rho_+[\rho]}{\rho[\rho]}}, \\ -[v] + \sqrt{\frac{\rho_+[p]}{\rho[\rho]}}[\rho] = 0, \end{cases} \tag{5.28}$$

In the above R-H conditions, by introducing new variable $u = v - \sigma$ and $m = \rho u$, we can write the above system of Equations (5.21) to (5.23) in more suitable form as

$$[m] = 0, \tag{5.29}$$

$$[p + mu] = 0, \tag{5.30}$$

$$m\left[u^2 + \frac{2(1-\overline{b})(c^2 + 2\overline{a})(\gamma - \overline{b})}{\gamma(\gamma - 1)} + 4\overline{a}\right]. \tag{5.31}$$

By using Lax entropy condition, for left shock curve i.e. 1-shock curve we obtain $\sigma < v_- - c_-$ which provides $c_- < u_-$ and $v_+ - c_+ < \sigma < v_+$ and therefore $0 < u_+ < c_+ < u_+ + \sigma$. Now, for left shock (1-shock wave), we have $u_- > c_-$ and $0 < u_+ < c_+$ which infers that $v_- > \sigma$ and $v_+ > \sigma$. Therefore, the shock speed is less than the velocity of the gas on the left side of the shock wave and right side of the shock wave. Therefore, we observe that in the case of 1-shock wave the particles move from left state to right state across the shock wave. In same manner for 3-shock curve, $v_- < \sigma < v_- + c_-$ and $v_+ + c_+ < \sigma$ provided that $-c_- < u_- < 0$ and $u_+ < -c_- < 0$. Hence, for 3-shock curve we have $\sigma > v_-$ and $\sigma > v_+$. Therefore, the velocity of gas on the left state of the shock wave and right state of the shock wave is less than shock speed. Thus, we observe that in the case of 3-shock wave the particles move from right state to left state across the shock wave. It is observed here that for left shock and right shock wave u_- and u_+ are non-zero, thus the quantity $m = \rho_- u_- = \rho_+ u_+ \neq 0$. Therefore in the case of shock families i.e. left shock wave and right shock wave, we have $u_-^2 > c_-^2$ and $u_+^2 < c_+^2$ respectively. Now, equation(5.31) written in the following form

$$u_-^2 + \frac{2(1-\bar{b}_-)(c_-^2+2\bar{a}_-)(\gamma-\bar{b}_-)}{\gamma(\gamma-1)} + 4\bar{a}_- = u_+^2 + \frac{2(1-\bar{b}_+)(c_+^2+2\bar{a}_+)(\gamma-\bar{b}_+)}{\gamma(\gamma-1)} + 4\bar{a}_+. \tag{5.32}$$

By using the fact $u_-^2 > c_-^2$ and $u_+^2 > c_+^2$, the Eq.(5.32) can be written as:

$$c_-^2 + \frac{2(1-\bar{b}_-)(c_-^2+2\bar{a}_-)(\gamma-\bar{b}_-)}{\gamma(\gamma-1)} + 4\bar{a}_- < c_+^2 + \frac{2(1-\bar{b}_+)(c_2^2+2\bar{a}_+)(\gamma-\bar{b}_+)}{\gamma(\gamma-1)} + 4\bar{a}_+. \tag{5.33}$$

Therefore, with the help of Eq.(5.33), we have $c_+^2 > c_-^2$ hence $u_-^2 > u_+^2$ which infers that $c_+ > c_-$ and $|u_-| > |u_+|$. Now, from Eq.(5.29) we obtain $\rho_+ > \rho_-$ and hence Eq.(5.30) implies that $p_+ > p_-$. Similarly, in the case of 3-shock wave $\rho_+ < \rho_-$ and $p_+ < p_-$. Hence both the shock families i.e. 1 and 3-shock waves are compressive waves in nature. Now, the one parameter family of shock waves are explicitly calculated here. For the parametrization of shock families, we define the constants which are given as

$$\begin{gathered} \alpha = \frac{p_+}{p_-}, \quad \beta = \frac{\rho_+}{\rho_-}, \quad A_- = \gamma - 2 + 2\bar{a}_-, \quad A_+ = \gamma - 2 + 2\bar{a}_+, \\ B_- = \gamma - 2 + 2\bar{b}_-, \quad B_+ = \gamma - 2 + 2\bar{b}_+. \end{gathered} \tag{5.34}$$

For 1-shock wave, with the help of above equation we have $\alpha > 1$ and $\beta > 1$. By utilizing the Eq(5.6), we have

$$\left(\frac{c_+}{c_-}\right)^2 = \frac{\alpha}{\beta}\frac{(1-\bar{b}_-)}{(1-\bar{b}_+)}\omega, \tag{5.35}$$

where, $w = \dfrac{(1-\bar{b}_+)(1-\bar{b}_-) + \dfrac{\bar{a}_+ B_+ (1-\bar{b}_-)}{d_+^2}}{(1-\bar{b}_+)(1-\bar{b}_-) + \dfrac{\bar{a}_- B_- (1-\bar{b}_+)}{d_-^2}}$.

From Eq.(5.29), we have

$$\frac{u_+}{u_-} = \frac{\rho_-}{\rho_+} = \frac{1}{\beta}. \tag{5.36}$$

We obtain the following equation by using Eq.(5.34) and Eq.(5.36) in Eq.(5.30)

$$\left(\frac{u_-}{c_-}\right)^2 = \left(\frac{\beta}{\beta-1}\right) D, \tag{5.37}$$

where,

$D = \dfrac{\tau\alpha}{\gamma}\left((1-\bar{b}_+) - \dfrac{B_+}{\gamma}\left(\dfrac{d_+^2}{\bar{a}_+} + \dfrac{B_+}{(1-\bar{b}_+)}\right)^{-1}\right) + \dfrac{1}{\gamma}\left(B_-\left(\dfrac{d_-^2}{\bar{a}_-} + \dfrac{B_-}{(1-\bar{b}_+)}\right)^{-1} - (1-\bar{b}_-)\right)$,

where, $\tau = \dfrac{(1-\bar{b}_-)}{(1-\bar{b}_+)}\left(\dfrac{(1-\bar{b}_+)(1-\bar{b}_-) + \dfrac{\bar{a}_+ B_+ (1-\bar{b}_-)}{d_+^2}}{(1-\bar{b}_+)(1-\bar{b}_-) + \dfrac{\bar{a}_- B_- (1-\bar{b}_+)}{d_-^2}}\right)$ and

$\left(\dfrac{d_+}{d_-}\right)^2 = \dfrac{\alpha}{\beta}\dfrac{(1-\bar{b}_-)}{(1-\bar{b}_+)}$. With the help of Equation.(5.31) we have

$$\left(\frac{u_-}{c_-}\right)^2 = \frac{\beta^2}{(\beta^2-1)}\left[A + 2\frac{\alpha}{\beta}\frac{(1-\bar{b}_-)^2}{(1-\bar{b}_+)}\frac{\omega(\gamma-\bar{b}_+)}{\gamma(\gamma-1)}\right], \tag{5.38}$$

where, $A = \dfrac{-4}{\gamma(\gamma-1)}\left[\dfrac{d_-^2}{\bar{a}_-(1-\bar{b}_-)(\gamma-\bar{b}_-)} + \dfrac{B_-}{(1-\bar{b}_-)^2(\gamma-\bar{b}_-)}\right]^{-1} - 4\left[\dfrac{d_-^2}{\bar{a}_-} + \dfrac{B_-}{(1-\bar{b}_-)}\right]^{-1} +$
$\dfrac{4}{\gamma(\gamma-1)}\left[\dfrac{d_-^2}{\bar{a}_+(1-\bar{b}_+)(\gamma-\bar{b}_+)} + \dfrac{\bar{a}_- B_-}{\bar{a}_+(1-\bar{b}_-)(1-\bar{b}_+)(\gamma-\bar{b}_+)}\right]^{-1} + 4\left[\dfrac{d_-^2}{\bar{a}_+} + \dfrac{\bar{a}_- B_-}{\bar{a}_+(1-\bar{b}_-)}\right]^{-1}$.

By comparing the Eq.(5.37) and Eq.(5.38), we have

$$\beta = \frac{1}{(A-D)}\left[D - 2\alpha\frac{(1-\bar{b}_-)^2}{(1-\bar{b}_+)}\frac{\omega(\gamma-\bar{b}_+)}{\gamma(\gamma-1)}\right]. \tag{5.39}$$

Now, with the help of Eq.(5.39) we obtain $\beta < \alpha$ and therefore $1 < \beta$. Hence we have $\rho_- < \rho_+$. By utilizing above equation and relation $u = v - \sigma$, we obtain

$$\frac{u_+ - u_-}{c_-} = \pm \left(\frac{\beta - 1}{\beta}\right) \sqrt{\frac{D\beta}{(\beta - 1)}}, \tag{5.40}$$

where,
$$D = \frac{\tau\alpha}{\gamma}\left((1-\bar{b}_+) - \frac{B_+}{\gamma}\left(\frac{d_+^2}{\bar{a}_+} + \frac{B_+}{(1-\bar{b}_+)}\right)^{-1}\right) + \frac{1}{\gamma}\left(B_-\left(\frac{d_-^2}{\bar{a}_-} + \frac{B_-}{(1-\bar{b}_+)}\right)^{-1} - (1-\bar{b}_-)\right).$$
Eq.(5.40) represents the variation in velocity across shock transition. Here, (+) sign denotes for 1-shock wave and (−) sign denotes for 3-shock wave. Hence, to obtain more explicit solution for shock curves, we utilize a new parameter η [21] which is defined as follows,

$$\eta = -\log\alpha, \tag{5.41}$$

From the Eq.(5.41), we have $e^{-\eta} = \alpha = \dfrac{p_+}{p_-} > 1$ therefore $\eta \leq 0$. Hence, by introducing this parametrization we obtain explicit formulations for shock curves which are written as

1-shock curve:

$$\frac{p_+}{p_-} = e^{-\eta}, \tag{5.42}$$

$$\frac{\rho_+}{\rho_-} = \frac{1}{(A - \overline{D})}\left[\overline{D} - 2e^{-\eta}\frac{(1-\bar{b}_-)^2}{(1-\bar{b}_+)}\frac{\omega(\gamma - \bar{b}_+)}{\gamma(\gamma - 1)}\right], \tag{5.43}$$

where,
$$\overline{D} = \frac{\tau e^{-\eta}}{\gamma}\left((1-\bar{b}_+) - \frac{B_+}{\gamma}\left(\frac{d_+^2}{\bar{a}_+} + \frac{B_+}{(1-\bar{b}_+)}\right)^{-1}\right) + \frac{1}{\gamma}\left(B_-\left(\frac{d_-^2}{\bar{a}_-} + \frac{B_-}{(1-\bar{b}_+)}\right)^{-1} - (1-\bar{b}_-)\right).$$

$$\frac{u_+ - u_-}{c_-} = \pm \left(\frac{\beta - 1}{\beta}\right) \sqrt{\frac{\overline{D}\beta}{(\beta - 1)}}. \tag{5.44}$$

3-shock curve:

$$\frac{p_-}{p_+} = e^{\eta}, \tag{5.45}$$

$$\frac{\rho_-}{\rho_+} = \frac{1}{(A-\overline{C})}\left[\overline{C} - 2e^{\eta}\frac{(1-\bar{b}_-)^2}{(1-\bar{b}_+)}\frac{\omega(\gamma-\bar{b}_+)}{\gamma(\gamma-1)}\right], \tag{5.46}$$

where,

$$\overline{C} = \frac{\tau e^{\eta}}{\gamma}\left((1-\bar{b}_+) - \frac{B_+}{\gamma}\left(\frac{d_+^2}{\bar{a}_+} + \frac{B_+}{(1-\bar{b}_+)}\right)^{-1}\right) + \frac{1}{\gamma}\left(B_-\left(\frac{d_-^2}{\bar{a}_-} + \frac{B_-}{(1-\bar{b}_+)}\right)^{-1} - (1-\bar{b}_-)\right).$$

$$\frac{u_- - u_+}{c_+} = \pm\left(\frac{\tilde{\beta}-1}{\tilde{\beta}}\right)\sqrt{\frac{\overline{C}\tilde{\beta}}{(\tilde{\beta}-1)}}, \tag{5.47}$$

where,

$$\tilde{\beta} = \frac{1}{(A-\overline{C})}\left[\overline{C} - 2e^{\eta}\frac{(1-\bar{b}_-)^2}{(1-\bar{b}_+)}\frac{\omega(\gamma-\bar{b}_+)}{\gamma(\gamma-1)}\right].$$

5.5 Smooth solutions (Simple wave)

For describing and building up solutions of flow problems, simple waves play a fundamental role. In particular, simple wave is always a flow in a region which is adjacent to a region of constant state. A simple wave zone is covered by arcs of characteristics which carry constant values of dependent variables and hence are straight lines. In 1-D hyperbolic system of PDEs, a centered rarefaction wave is defined as simple wave. In ith genuinely nonlinear characteristic field, the left constant state (V_-^*) and right constant state (V_+^*) are connected through a smooth transition and concurs with the following condition,

1. Riemann invariants are constant [108].

2. $\lambda_i(V_-^*) < \lambda_i(V_+^*), i = 1, 3$ i.e. The left characteristic of wave and right characteristic of wave diverge.

Now, we obtained the explicit formulation for simple wave curve. Across 1-simple wave, Π_i (Riemann Invariants), $i = 1, 2$ are constant i.e. $\Pi_2 = \Pi_1$

$$u_+ + \frac{2}{(\gamma-1)}c_+(1-\bar{b}_+) - \frac{\bar{a}_+ c_+}{(\gamma-1)}(1-\bar{b}_+) = u_- + \frac{2}{(\gamma-1)}c_-(1-\bar{b}_-) - \frac{\bar{a}_- c_-}{(\gamma-1)}(1-\bar{b}_-). \tag{5.48}$$

The equation of state (EOS) for van der Waals gas is given by

$$p = K\left(\frac{\rho}{1-b\rho}\right)^{\gamma} - a\rho^2. \tag{5.49}$$

From above equation, we have

$$\frac{p_+}{p_-} = \frac{c_+^2}{c_-^2}\frac{\rho_+}{\rho_-}\frac{1}{\tau} = \frac{\rho_+}{\rho_-}\frac{(\theta_+ - \bar{a}_+)}{(\theta_- - \bar{a}_-)}, \tag{5.50}$$

where,
$\theta_- = K\left(\frac{\rho_-}{1-b\rho_-}\right)^{\gamma}$ and $\theta_+ = K\left(\frac{\rho_+}{1-b\rho_+}\right)^{\gamma}$.

With the help of Eq. (5.48), we obtain

$$\frac{u_+ - u_-}{c_-} = \frac{(1-\bar{b}_-)}{(\gamma-1)}(\bar{a}_- - 2) - \frac{c_+}{c_-}\frac{(1-\bar{b}_+)}{(\gamma-1)}(\bar{a}_+ - 2). \tag{5.51}$$

The characteristic speed $\lambda_1 = u - c$ increases for 1-rarefaction wave. Hence, $\lambda_1^{(+)} \geq \lambda_1^{(-)}$ which provides $u_+ - u_- \geq c_+ - c_-$. From Eq.(5.51), we have

$$\frac{c_+ - c_-}{c_-} \leq \frac{1}{(\gamma-1)}\left[(1-\bar{b}_-)(\bar{a}_- - 2) - \frac{c_+}{c_-}(1-\bar{b}_+)(\bar{a}_+ - 2)\right]. \tag{5.52}$$

Now, with the help of Eq.(5.50) and Eq.(5.52) we obtain

$$0 < \frac{p_+}{p_-} \leq 1. \tag{5.53}$$

From Eq.(5.41), we have $e^{-\eta} = \alpha = \dfrac{p_+}{p_-} < 1$, therefore $\eta \geq 0$. Hence, from Eq.(5.50) and Eq.(5.51), by introducing this parametrization, we obtain explicit formulations for simple waves which are written as

1-simple wave:

$$\frac{p_+}{p_-} = e^{-\eta}, \tag{5.54}$$

$$\frac{\rho_+}{\rho_-} = e^{-\eta}\frac{(1-\bar{b}_+)}{(1-\bar{b}_-)}\tau\Psi, \tag{5.55}$$

where,
$$\Psi = \left(\frac{(1-\bar{b}_+)(1-\bar{b}_-)d_+^2 + \bar{a}_+B_+(1-\bar{b}_-)}{(1-\bar{b}_+)(1-\bar{b}_-)d_-^2 + \bar{a}_-B_-(1-\bar{b}_+)}\right),$$

$$\frac{u_+ - u_-}{c_-} = \frac{(1-\bar{b}_-)}{(\gamma-1)}(\bar{a}_- - 2) - \left(\frac{e^{-\eta}}{\beta}\frac{(1-\bar{b}_-)}{(1-\bar{b}_+)}\omega\right)^{1/2}\frac{(1-\bar{b}_+)}{(\gamma-1)}(\bar{a}_+ - 2). \tag{5.56}$$

3-simple wave:

$$\frac{p_+}{p_-} = e^{\eta}, \tag{5.57}$$

$$\frac{\rho_+}{\rho_-} = e^{\eta}\frac{(1-\bar{b}_+)}{(1-\bar{b}_-)}\tau\Psi, \tag{5.58}$$

where,
$$\Psi = \left(\frac{(1-\bar{b}_+)(1-\bar{b}_-)d_+^2 + \bar{a}_+B_+(1-\bar{b}_-)}{(1-\bar{b}_+)(1-\bar{b}_-)d_-^2 + \bar{a}_-B_-(1-\bar{b}_+)}\right),$$

$$\frac{u_+ - u_-}{c_-} = \frac{(1-\bar{b}_-)}{(\gamma-1)}(\bar{a}_- - 2) - \left(\frac{e^{\eta}}{\beta}\frac{(1-\bar{b}_-)}{(1-\bar{b}_+)}\omega\right)^{1/2}\frac{(1-\bar{b}_+)}{(\gamma-1)}(\bar{a}_+ - 2). \tag{5.59}$$

5.6 Contact discontinuities

The contact discontinuity originally separates two constant states with different values of density but with the same values of pressure and particle velocity. A contact discontinuity is a surface through which there is no mass flux, across which, however, density and temperature are discontinuous. In second linearly degenerate characteristic field, the left constant state (V_-^*) and right constant state (V_+^*) are connected through a jump discontinuity with shock speed σ_2 and concurs with the following conditions,

1. $F(V_+^*) - F(V_-^*) = \sigma_2(V_+^* - V_1^*)$ i.e.the R-H conditions.

2. $\lambda_2(V_+^*) = \lambda_2(V_-^*) = \sigma_2$, i.e. the parallel characteristic conditions.

$$\frac{p_+}{p_-} = 1, \tag{5.60}$$

$$\frac{\rho_+}{\rho_-} = e^{\eta}, -\infty < \eta < \infty, \tag{5.61}$$

$$u_+ - u_- = 0. \tag{5.62}$$

5.7 The properties of elementary wave curves(shock waves and rarefaction waves)

In this section, we analyze the properties of elementary waves.

Lemma 5.1. *Let $S_1(\rho; V_-)$ denote 1-shock and $S_3(\rho; V_+)$ denote 3-shock wave corresponding to characteristic fields λ_1 and λ_3 respectively. We consider that V_- and V_+ satisfy the R-H jump relations (5.21) to (5.23). Hence, the shock curves $S_1(\rho; V_-)$ and $S_3(\rho; V_+)$ satisfy*

$$v = v_- - \Phi(\rho_-, \rho), \tag{5.63}$$

where $\Phi(\rho_-, \rho) = \sqrt{(p - p_-)\left(\frac{\rho - \rho_-}{\rho\rho_-}\right)}$, such that for $1 < \gamma < 2$, we obtain for $\rho > \rho_-$, $v' < 0$ and $v'' > 0$ on $S_1(\rho; V_-)$, while for $\rho < \rho_-$, $v' > 0$ and $v'' < 0$ on $S_3(\rho; V_+)$.

Proof. *With the help of Eq.(5.21) to (5.23), we obtain Eq.(5.63). Let* $\mu(\rho) = \Phi(\rho_-, \rho)^2$*; hence from Eq.(5.63) we have* $v' = \dfrac{-\mu'(\rho)}{2\sqrt{\mu(\rho)}}$*. For* $\rho > \rho_-$*,* v' *is negative. It is noticed here that* $\mu(\rho_-) = \mu'(\rho) = 0$ *and* μ *and* μ' *are positive for* $\rho > \rho_l$*. Let us assume that* $\delta(\rho) = (\mu'(\rho))^2 - 2\mu(\rho)\mu''(\rho)$ *hence* $\delta'(\rho) = -2\mu(\rho)\mu'''(\rho)$ *and* $\delta(\rho_-) = 0$*. Therefore,* $\mu''(\rho) > 0$ *and* $\mu'''(\rho) < 0$ *for* $1 < \gamma < 2$ *and* $\delta'(\rho) > 0$*. Further, it follows that for* $\rho > \rho_-$*,* $\delta(\rho) > \delta(\rho_-) = 0$*. Therefore, we have* $v''(\rho) = \dfrac{(\mu'(\rho))^2 - 2\mu(\rho)\mu''(\rho)}{4\mu(\rho)^{3/2}} > 0$ *for* $\rho > \rho_-$ *on* $S_1(\rho; V_-)$*. In same manner, it is clear that* $v'(\rho) > 0$ *and* $v''(\rho) < 0$ *for* $1 < \gamma < 2$ *and* $\rho < \rho_-$ *on* $S_3(\rho; V_+)$*.*

Lemma 5.2. *If* p *satisfies* $p' > 0$ *and* $p'' \geq 0$*, then Lax conditions hold.*

Proof. *In case of 1-shock, we have to prove that* $\sigma < \lambda_1(V_-)$ *. On 1-shock, We have* $\rho_- < \rho$ *and given conditions* $p' > 0$ *and* $p'' \geq 0$*. Therefore, by Lagrange mean value theorem(LMVT)* $\exists$ *a* $\nu_- \in (\rho_-, \rho)$ *such that* $p'(\nu) = \dfrac{p - p_-}{\rho - \rho_-}$*. Therefore,* $p'' \geq 0$ *so we analyze* $p'(\nu) > p'(\rho_-)$*, which implies that* $p'(\rho_-) < \dfrac{(p - p_-)\rho}{\rho_-(\rho - \rho_-)}$*.*
Hence, $\dfrac{-\rho\sqrt{(p - p_-)(\rho - \rho_-)/\rho\rho_-}}{(\rho - \rho_-)} < -(p'(\rho_-))^{1/2}$*.*
With the help of Eq.(5.22) and above expression, we have the following inequality $\sigma < \lambda_1(V_-) = v_- - (p'(\rho_-))^{1/2}$*. For some* $\nu_+ \in (\rho_-, \rho)$*, we have*
$p'(\nu_+) = \dfrac{p - p_-}{\rho - \rho_-} < p'(\rho) \Rightarrow p'(\rho) > \dfrac{p - p_-}{\rho - \rho_-} \Rightarrow$

$$\frac{-\rho_-\sqrt{(p - p_-)(\rho - \rho_-)/\rho\rho_-}}{(\rho - \rho_-)} > -c. \tag{5.64}$$

Hence, with the help of Eqs.(5.21) and (5.63), we obtain $v - c < \dfrac{\rho v - \rho_- v_-}{\rho - \rho_-} = \sigma \Rightarrow \lambda_1(V) < \sigma$*. Now. Eq.(5.64) shows that* $-\sqrt{\dfrac{(p - p_-)\rho_-}{(\rho - \rho_-)\rho}} < p(\rho)$*.*
For 1-shock curve, by using Eq.(5.21) and above expression we have $\dfrac{(v - v_-)\rho}{\rho - \rho_-} < p'(\rho) \Rightarrow \sigma < \lambda_2(V)$*. Hence, 1-shock satisfies Lax condition. In the same manner 2-shock curve satisfies Lax condition.*

Theorem 5.3. *1-shock curve and 3-shock curve are star like with respect to* (ρ_-, v_-) *when* $p = K\left(\dfrac{\rho}{1-b\rho}\right)^\gamma - a\rho^2$, *for* $1 \le \gamma \le 2$.

Proof. *We have to show that any ray through the point* (ρ_-, v_-) *intersects 1-shock at most one point i.e. we shall show that for any two rays through* (ρ_-, v_-) *and different points* (ρ_-, v_-) *and* (ρ_+, v_+) *on 1-shock curve consisting of different slope i.e.* $\dfrac{v_1 - v_-}{\rho_1 - \rho_-}$ *and* $\dfrac{v_2 - v_-}{\rho_2 - \rho_-}$ *are different. On 1-shock curve Eq.(5.63) imply that* $\left(\dfrac{v - v_-}{\rho - \rho_-}\right)^2 = \dfrac{p - p_-}{\rho\rho_-(\rho - \rho_-)}$. *Let* $g(\rho) = \dfrac{p - p_-}{\rho\rho_-(\rho - \rho_-)} \implies g'(\rho) = \dfrac{\rho\rho_-(\rho-\rho_-)p'(\rho) - (p - p_-)(2\rho\rho_- - \rho_-^2)}{\rho_-^2\rho^2(\rho-\rho_-)^2}$.

Let $h(\rho) = \rho\rho_-(\rho - \rho_-)p'(\rho) - (p - p_-)(2\rho\rho_- - \rho_-^2)$ *then* $h(\rho_-) = 0$ *and* $h'(\rho) = \rho\rho_l(\rho - \rho_-)p''(\rho) - 2(p - p_-)\rho_-$, *therefore we have* $h'(\rho_-) = 0$. *Hence,* $h''(\rho) = \rho\rho_-(\rho - \rho_-)p'''(\rho) + (2\rho\rho_- - \rho_-^2)p''(\rho) - 2p'(\rho)\rho_-$.

For $1 \le \gamma \le 2$, *we analyze that* $h''(\rho) < 0$. *Therefore, for* $\rho > \rho_-$ *we have* $h'(\rho) < h'(\rho_-)$ *i.e.* $h'(\rho)$ *is monotonically decreasing function of* ρ. *But* $h'(\rho_-) = 0 \Rightarrow$ *for* $\rho > \rho_-$ *we have* $h(\rho) < h(\rho_-)$ *i.e.* $h(\rho)$ *is decreasing function of* ρ. *Further,* $h(\rho_-) = 0 \Rightarrow g(\rho)$ *is monotonically decreasing function of* ρ. *Hence,* $\dfrac{v - v_-}{\rho - \rho_-}$ *is decreasing function of* ρ. *Therefore, from above discussion we obtain 1-shock curve is star like with respect to* (ρ_-, v_-). *In the same manner, we can prove that 3-shock curve is star like with respect to* (ρ_+, v_+).

Lemma 5.4. *Across 1 and 3-rarefaction waves, characteristic speed increases from left hand state to right hand state if and only if across 1-rarefaction waves* $\rho \le \rho_-$ *and* $v_- \le v$, *Similarly for 3-rarefaction waves* $\rho \ge \rho_-$ *and* $v_- \le v$.

Proof. *Let us suppose that for 1-rarefaction wave, we have* $\rho \le \rho_-$ *and* $v_- \le v$. *Since,* $c = \sqrt{\dfrac{\partial p}{\partial \rho}}$, *hence* $\dfrac{dc}{d\rho} = \dfrac{p''(\rho)}{2c} \implies \dfrac{dc}{d\rho} > 0$ *so* $c(\rho) \le c(\rho_-)$ *i.e.* c *is increasing function. Further, from above assumption we obtain that* $v_- - c_- \le v - c \implies \lambda_1(V_-) \le \lambda_1(V)$. *In the same manner, for 3-rarefaction wave we can obtain* $\lambda_3(V_-) \le \lambda_3(V)$. *Conversely, let us consider that for 1-rarefaction wave,* λ_1 *increases* *from left hand state to right hand state i.e.* $\lambda_1(V_-) \leq \lambda_1(V) \Longrightarrow c - c_- \leq v - v_-$. *Across 1-rarefaction wave Riemann invariants are constant, so we have* $v - v_- = \int_0^{\rho_-} \frac{c(\rho)}{\rho} d\rho - \int_0^{\rho} \frac{c(\rho)}{\rho} d\rho$. *Hence,* $c - c_- \leq \int_0^{\rho_-} \frac{c(\rho)}{\rho} d\rho - \int_0^{\rho} \frac{c(\rho)}{\rho} d\rho \Longrightarrow \rho \leq \rho_-$ *and* $v - v_- \geq 0$. *Therefore,* $\rho \leq \rho_-$ *and* $v_- \leq v$. *In same manner, for 2-rarefaction wave we can show that* $\rho \geq \rho_-$ *and* $v_- \leq v$.

Theorem 5.5. *The 1-rarefaction curve is convex and monotonic decreasing and 2-rarefaction curve is concave and monotonic decreasing.*

Proof. *For 1-rarefaction wave,*

$$v = v_- + \int_{\rho}^{\rho_-} \frac{c(\rho)}{\rho} d\rho, \qquad if\, \rho \leq \rho_-, \tag{5.65}$$

$$\frac{dv}{d\rho} = -\frac{c}{\rho} < 0 \Rightarrow \frac{d^2 v}{d\rho^2} = -\frac{c'}{\rho} + \frac{c}{\rho^2}. \tag{5.66}$$

Since, $p = K\left(\frac{\rho}{1 - b\rho}\right)^{\gamma} - a\rho^2$. *Hence Eq.(**??**)* $\Rightarrow \frac{d^2 v}{d\rho^2} = \frac{2(p'(\rho)) - \rho p''(\rho)}{2(p'(\rho))^{1/2}\rho^2} > 0$. *Therefore, v is convex for 1-rarefaction waves. In same manner, we can show that v is concave for 2-rarefaction waves with respect to* ρ.

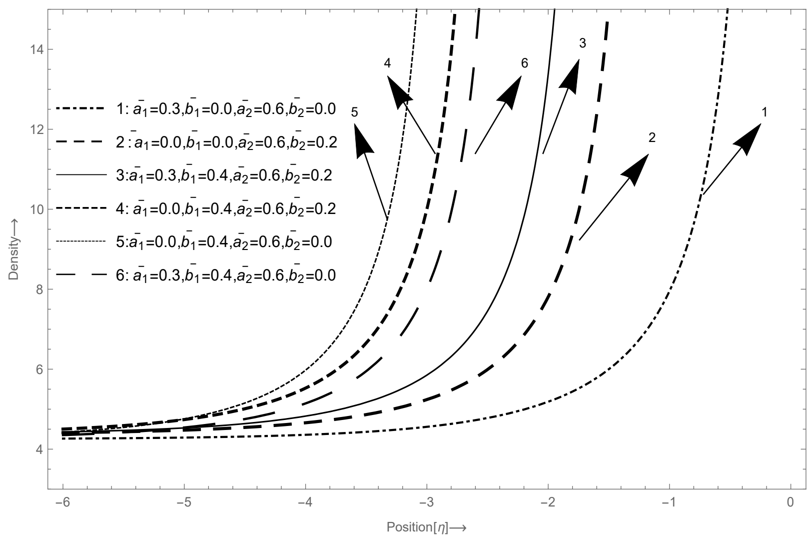

FIGURE 5.2: Density diagram for compressive waves (1-shock).

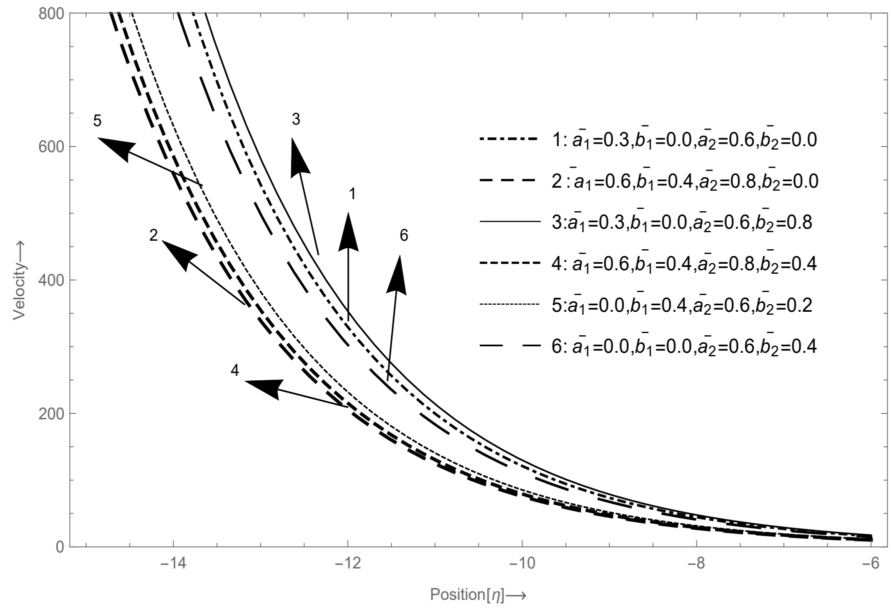

FIGURE 5.3: Velocity diagram for compressive waves (1-shock).

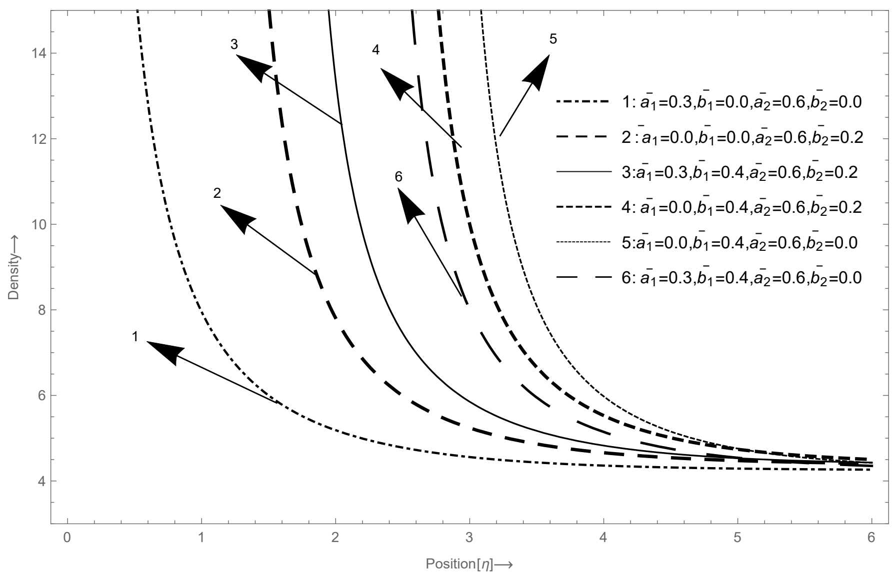

FIGURE 5.4: Density diagram for compressive waves (3-shock).

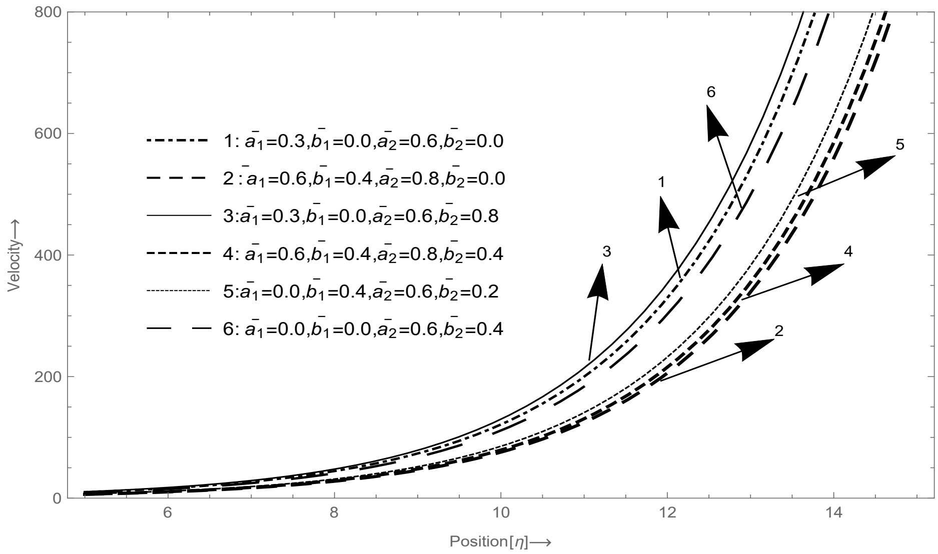

FIGURE 5.5: Velocity diagram for compressive waves (3-shock).

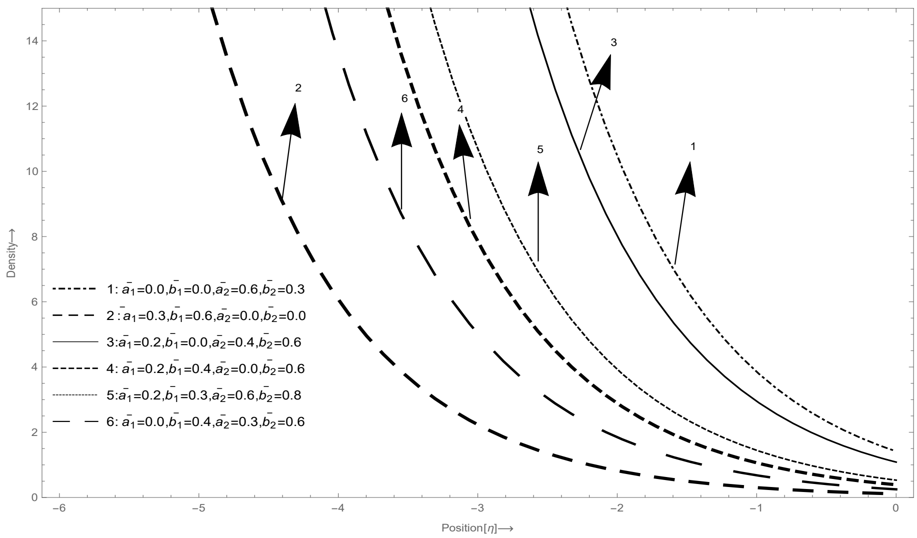

FIGURE 5.6: Density diagram for rarefaction waves (1-shock).

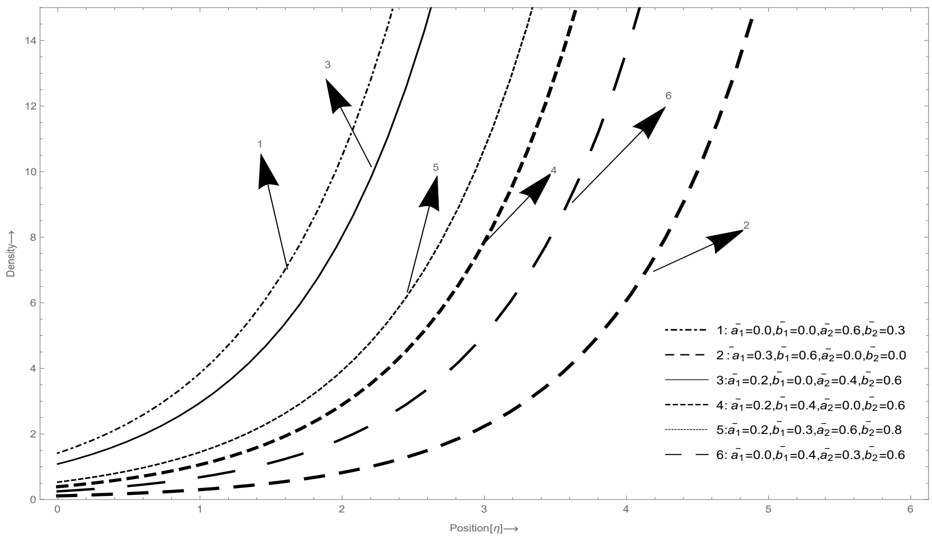

FIGURE 5.7: Density diagram for rarefaction waves (1-shock).

5.8 Results and discussion

The classical solution of the RP for the governing model for the van der Waals gasdynamic flow is derived here. When the parameters $\overline{a}_- = 0.0, \overline{a}_+ = 0.0$ which corresponds to non-ideal gas dynamic case the results are in close agreement with Ambika et al.[95] as well as in the absence of magnetic field strength results are same as reported in Pooja et al.[152]. Here, the effect of attractive force between the gas molecules and covolume of the gas appears into the solution through the parameters $\overline{a}_-, \overline{a}_+$ and $\overline{b}_-, \overline{b}_+$ respectively. The density(ρ) and velocity(u) versus position(η) profiles for compressive waves and rarefaction waves corresponding to different values of parameters $\overline{a}$ and $\overline{b}$ are drawn in Fig.2 to Fig.7. From Fig 2 and Fig.4, it is noticed that the density(ρ) versus position(η) curves for 1 and 3-shock wave for compressive waves is convex downward. From Fig.2, we observe that the presence of covolume of the gas i.e. $\overline{b}_-, \overline{b}_+$ causes to decrease the ρ (density). Also, we notice that the increasing values of covolume of the gas i.e. $\overline{b}_-, \overline{b}_+$ in the presence of attractive forces between the gas molecules $(\overline{a}_-, \overline{a}_+)$ is to further decrease the density. Further, it is noticed here that the effect of non-idealness parameters $\overline{b}_{-}$ and $\overline{b}_{+}$ have reverse effect on the curves corresponding to the density for 1-shock for compressive waves. Hence, the combined effect of covolume of the gas and attractive force between the gas molecules on the density of shock wave is to further hastened. The solution profiles presented in Fig.2 representing the variation of density for 1-shock show opposite behaviour as compared to the solution profiles of Fig.4 corresponding to 3-shock. From Fig.3 and Fig.5, we obtain that the velocity(u) versus position(η) curve is convex downward for compressive waves. From Fig.3 and Fig.5, we observe from the results shown in the solution profiles have the same trend as compared to the results presented in Fig.2. The density curves for 1 and 3-shock waves for rarefaction wave is convex downward which is presented in the Fig.6 and Fig.7. The density of 1-shock wave decreases with position(η) for rarefaction wave. From Fig.6, we observe that the presence of intermolecular forces of attraction between the particles $(\overline{a}_{-}, \overline{a}_{+})$ causes to increase the density. Also, the increasing values of attractive forces between the gas molecules $(\overline{a}_{-}, \overline{a}_{+})$ in the presence of covolume of the gas i.e. $\overline{b}_{-}, \overline{b}_{+}$ is to further increase the density. Further, it is noticed here that the effect of non-idealness parameters, $\overline{b}_{-}$ and $\overline{b}_{+}$ have completely different effect on the density profile for 1-shock of rarefaction waves. Similarly, the effect of intermolecular forces of attraction between the particles, $\overline{a}_{-}$ and $\overline{a}_{+}$ have the same effect on the density profiles for 1-shock of rarefaction waves. From Fig.7, we obtain that the density of 3-shock wave increases with position(η) for rarefaction wave. Further, it is noticed here that the effect of non-idealness parameters $\overline{b}_{-}$ and $\overline{b}_{+}$ have opposite effect on the density profiles for 3-shock of rarefaction waves. Similarly, the effect of intermolecular forces of attraction between the particles $\overline{a}_{-}$ and $\overline{a}_{+}$ have the same effect on the density profiles for 3-shock of rarefaction waves. Also, the combined effect of covolume of the gas and attractive forces between the gas molecules causes to decrease the density (ρ) and velocity (u) of the shock wave to further hastened.

5.9 Conclusion

In the present study, the elementary wave solution of RP for van der Waals gasdynamics is analyzed. The velocity(u) and density(ρ) profiles for 1 and 3-shock wave for compressive wave and rarefaction wave is plotted here. Velocity(u) and density(ρ) profiles for 1 and 3-shock wave for compressive wave and rarefaction wave is convex downward. Also, it is observed here that the effect of intermolecular forces of attraction between the particles $\overline{a}_-$ and $\overline{a}_+$ have the same effect on the density(ρ) and velocity(u) profiles for rarefaction and compressive wave. Similarly, the effect of non-idealness parameters $\overline{b}_-$ and $\overline{b}_+$ have opposite effect on the density(ρ) and velocity(u) profiles for rarefaction wave and compressive wave. Also, the presence of intermolecular forces of attraction between the particles is to further enhance the effect of covolume of the gas. Also, the properties of classical wave curves is discussed.

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