Chapter 4

Interaction of waves in one-dimensional dusty gas flow

"Mathematicians do not study objects, but the relations between objects."

-Henri Poincare

4.1 Introduction

The study of elementary wave interactions consist of either interaction between two waves colliding, or one wave overtaking another, or one wave meeting a discontinuity. Such a phenomenon frequently happens while studying the wave propagation in the field of space science, space re-entry, astrophysical phenomenon and aerodynamics etc. Also, the analysis of how shock waves interact with each other, as well as with the exhaust plume of an aircraft, has been an area of great interest among scientists and engineers.

The propagation of waves in a medium is governed by quasilinear hyperbolic system of PDE's. One often encounters certain kinds of discontinuities known as acceleration waves, shock waves and weak waves. The study of these waves has been of great significance in engineering science and nonlinear science due to its application in various field such as nuclear physics, plasma physics, geophysics, astrophysical sciences and interstellar gas masses. In the present chapter, we use the method of asymptotic analysis to study the evolutionary behavior of shock wave which is widely used by many researchers e.g. Choquet-Bruhat [58], Hunter et al. [59], Majda et al. [60] over decades. Also, the qualitative analysis of interaction of non-linear waves can be obtained by the interaction coefficients which occur in transport equation and these coefficients are measure the coupling strength between different type of wave modes. The method of asymptotic analysis has been widely used to study the propagation of weak shock waves governed by the non-linear hyperbolic system of partial differential equations. The study of resonantly interaction of shock waves by using "Asymptotic analysis method" for one-dimensional ideal gas flow in presence of the solid dust particles have not been analyzed by any author till now. To analyze the evolutionary behavior of shock wave in ideal gas with dust particles is more complex in comparison to ideal gas flow. A different kind of physical phenomenon which occurs in various processes such as space re-entry, chemical explosion, nuclear explosions, supersonic flow and collision of two or more galaxies are described by mathematical model of quasilinear hyperbolic system of partial differential equations.

In last few decades many attempts have been made to analyze the asymptotic properties of shock waves in various gasdynamic regimes where the governing equation is a system of quasilinear hyperbolic partial differential equations. The "Weakly non-linear geometrical acoustics theory" provides a methodical technique for dealing with the interaction of non-linear high frequency small amplitude waves. The wave propagation phenomenon with an added effect of nonlinearity have been analyzed in past but the closed form exact analytic solution of the equations governing the motion of waves have never been obtained. In most of the literature, only approximate analytical or numerical solutions are discussed. In this context it is worth to mention the contributions made by many authors like Hunter and Ali [132], Gunderson [92], He and Moodie [133], Whitham [19], Moodie et al. [134], Arora and Sharma [135], Arora [136], D.Fusco [39].

From the physical and mathematical view point, the discussion of shock waves in an ideal gas consisting of solid particles is a topic of great interest because of its numerous applications such as underground explosions, interstellar masses, lunar ash flow and explosive volcanic eruptions etc. Dusty gas is composed of small solid particles and gas in which solid particles do not attain more than five percent of its entire volume. In mixture of gas and solid particles, the study of shock wave has more significance due to its wide applications in several areas such as supersonic-vehicle in sand storms, supersonic flights in polluted air, nuclear reaction, aerospace engineering science etc. Vishwakarma et al. [137, 138] have discussed the propagation of shock wave in dusty gas with varying density. Chaturvedi et al. [115] have discussed the evolution of weak shock wave in two-dimensional, steady supersonic flow in dusty gas. Sharma et al. [139] have used the scheme of multiple time scales to study the wave interaction in a non-equilibrium gas flow. Pooja et al. [140] and Nath et al. [43] have used an asymptotic technique to analyze the evolution of weak shock waves in non-ideal magnetogasdynamics and non-ideal radiating gas flow. Singh et al. [118] have theoretically investigated the propagation of shock wave in radiative magnetogasdynamics. Propagation of shock wave in a mixture of gas and dust particles has been widely investigated by several authors such as Nath et al. [141], Nath [141],[142] and Nandkeolyar et al. [143]. Singh et al. [116, 116, 128, 144] have studied the evolutionary behavior of shock wave in various gasdynamic regimes. Bhattacharyya et al. [127] have discussed about the simulation of Cattaneo-Christov heat flux on the flow of single and multi-walled carbon nanotubes between two stretchable coaxial rotating disks. Seth et al. [145] have studied the partial slip mechanism on free convection flow of viscoelastic fluid past a nonlinearly stretching surface. Jena et al. [146] and Radha et al. [147] have applied the methods of relatively undistorted waves and weakly nonlinear geometrical optics to study the situations when the disturbance amplitude is finite, arbitrarily small, and not so small in non-ideal gas flow and relaxing gas.

The main motive of the present chapter is to apply the method of resonantly interacting multiple time scales to study the small amplitude high frequency waves for one dimensional, unsteady planar flow, cylindrically symmetric flow and spherically symmetric flow in a dusty gas. The transport equations for the amplitude of resonantly interacting high-frequency waves in a dusty gas is derived. Also the existence of weak shock waves in a dusty gas is discussed here. Further, the evolutionary behavior of weak shock waves propagating in ideal gas flow with dust particles is examined here.

This chapter is organized as follow: In section 2, we describe the basic equations for the dusty gas flow. Also, we reformulate the governing equations into quasilinear system and derive the characteristic for the system. In section 3, we use the multiple time scale method to obtain high frequency small amplitude asymptotic solution to the system written in section 2. The transport equations for the propagation of shock is derived in section 4. In section 5, we investigate the conditions which explain the evolutionary behavior of shock wave for the planar and non-planar cases. In the last section 6, we discuss the results and conclusion of the present work.

4.2 Problem formulation and characteristics

The basic equations governing the one dimensional compressible, inviscid, unsteady planar and non-planar flows in a dusty gas mixture following the equation of state of Mie Grüneisen type

$$p = \frac{(1-k_p)\rho RT}{(1-Z)}, \tag{4.1}$$

are written as [6, 5, 148, 43, 149]

$$\frac{\partial \rho}{\partial t} + \nu\frac{\partial \rho}{\partial x} + \rho\frac{\partial \nu}{\partial x} + \frac{m\rho\nu}{x} = 0 \tag{4.2}$$

$$\rho\left(\frac{\partial \nu}{\partial t} + \nu\frac{\partial \nu}{\partial x}\right) + \frac{\partial p}{\partial x} = 0, \tag{4.3}$$

$$\frac{\partial p}{\partial t} + \nu\frac{\partial p}{\partial x} + \rho c^2\left(\frac{\partial \nu}{\partial x} + \frac{m\nu}{x}\right) = 0, \tag{4.4}$$

where ν is the velocity of the particle along the spatial coordinate. The symbols ρ, p and t represent the density, pressure and time respectively. T denotes the temperature and R is the gas constant. Here, $m = 0$ exhibits the planar flow, $m = 1$ exhibits the cylindrically symmetric flow and $m = 2$ exhibits the spherically symmetric flow. The entity Z is the volume fraction and k_p is the mass fraction of solid particles in the mixture which are defined as $Z = \frac{V_{sp}}{V_g}$, $k_p = \frac{m_{sp}}{m_g}$, where m_{sp} is the total mass of the solid particles, V_{sp} is volumetric extension of the solid particles, V_g is the total volume of the mixture and m_g is the total mass of the mixture.
The quantity $c = (\Gamma p/((1-\theta\rho)\rho)^{1/2}$ is the equilibrium speed of sound with

$$\Gamma = \gamma(1+\lambda\beta)/(1+\lambda\beta\gamma), \tag{4.5}$$

where

$$\gamma = \frac{c_p}{c_v}, \lambda = \frac{k_p}{(1-k_p)}, \beta = \frac{c_{sp}}{c_p}. \tag{4.6}$$

Here c_{sp} is the specific heat of the solid particles, c_p and c_v are the specific heats of the gas at constant pressure and at constant volume respectively. The relation between the entities Z and k_p is defined as $Z = \phi\rho$, where $\phi = \dfrac{k_p}{\rho_{sp}}$ with ρ_{sp} as the specific density of solid particles and e is the internal energy per unit mass of the mixture which is given by

$$e = \frac{(1-Z)p}{(\Gamma-1)\rho}. \tag{4.7}$$

Now we write equations (4.2) to (4.4) in the following matrix form

$$\frac{\partial U}{\partial t} + P\frac{\partial U}{\partial x} + Q = 0, \tag{4.8}$$

where $U = (\rho, \nu, p)^{tr}$, $Q = (m\rho\nu/x, 0, \rho c^2 m\nu/x)^{tr}$ and P is the coefficient matrix of order 3×3 having the components P^{ij} ,

$$\begin{aligned} P^{11} &= P^{22} = P^{33} = \nu, \\ P^{13} &= P^{21} = P^{31} = 0, \\ P^{12} &= \rho,\ P^{23} = \tfrac{1}{\rho}, P^{32} = \rho c^2 = \Gamma p/(1-\phi\rho). \end{aligned} \tag{4.9}$$

Here the superscript 'tr' represents transposition.
The eigenvalues of the matrix P are given as $\lambda_1 = \nu + c$, $\lambda_2 = \nu$, $\lambda_3 = \nu - c$. Therefore, the system (4.8), which has distinct eigenvalues, is strictly hyperbolic and has three families of characteristics corresponding to three distinct eigenvalues, out of these three characteristics two represent waves moving in $\pm x$ directions with speed $\nu \pm c$. The remaining one characteristics exhibits the particle path propagating with velocity ν. Now we suppose that the shock waves are propagating into an initial back ground state $U_0 = (\rho_0, 0, p_0)^{tr}$. At constant speed $\nu_0 = 0$, the characteristics speeds are provided by $\lambda_1 = c_0$, $\lambda_2 = 0$, $\lambda_3 = -c_0$, where the subscript 0 indicates

the evaluation at $U = U_0$ which is identical with an equilibrium state.

4.3 Weakly non-linear resonant waves

In this segment the multiple time scale method will be applied to obtain high frequency small amplitude asymptotic solution to the system of equations (4.8) when the attenuation time scale (τ_{at}) is large in comparison to the characteristic time scale (τ_{ch}) , it means $\xi = \tau_{ch}/\tau_{at} \ll 1$. Let $l^{(i)}$ and $r^{(i)}\,(i = 1, 2, 3)$ respectively exhibit the left and right eigenvectors of the matrix corresponding to the eigenvalues $\lambda_1 = c_0, \lambda_2 = 0, \lambda_3 = -c_0$ .The eigenvectors $l^{(i)}$ and $r^{(i)}\,(i = 1, 2, 3)$ satisfy the normalization conditions $l^{(i)}r^{(j)} = \delta_{ij}\,(1 \le i \le 3, 1 \le j \le 3)$, where δ_{ij} is Krönecker delta. Now under the above assumptions the left and right eigenvectors are given by

$$\begin{aligned} l^{(1)} &= \left(0, \tfrac{\rho_0}{2c_0}, \tfrac{1}{2c_0^2}\right), \quad r^{(1)} = \left(1, \tfrac{c_0}{\rho_0}, c_0^2\right), \\ l^{(2)} &= \left(-c_0^2, 0, 1\right), \quad r^{(2)} = \left(\tfrac{-1}{c_0^2}, 0, 0\right), \quad . \\ l^{(3)} &= \left(0, \tfrac{-\rho_0}{2c_0}, \tfrac{1}{2c_0^2}\right), \quad r^{(3)} = \left(1, \tfrac{-c_0}{\rho_0}, c_0^2\right) \end{aligned} \tag{4.10}$$

Now we explore the asymptotic solution of equation (4.8) as $\xi \to 0$ in the following form

$$U(x,t) = U_0 + \xi U_1\left(x, t, \vec{\vartheta}\right) + \xi^2 U_2\left(x, t, \vec{\vartheta}\right) + O\left(\xi^3\right), \tag{4.11}$$

where U_1 is smooth bounded function of its arguments and U_2 is bounded function in (x, t) coordinate in a definite bounded portion containing atmost sublinear growth $\vec{\vartheta}$ as $\vec{\vartheta} \to \pm\infty$ Sharma and Srinivasan [139]. Here $\vec{\vartheta} = (\vartheta_1, \vartheta_2, \vartheta_3)$ is fast variable and is symbolized as $\vec{\vartheta} = \vec{\theta}\big/\xi$, with $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$, $\theta_i\,(1 \le i \le 3)$ being the phase function of i^{th} wave communicated with the characteristics speed λ_i . Now using the Taylor's series expansion of matrices P and Q, about the constant state U_0

in powers of ξ and using equation (4.11) in (4.8) and then substituting the partial derivatives $\partial/\partial Y$ (Y will either be x or t) by $\partial/\partial Y + \xi^{-1}\sum_{i=1}^{3}(\partial\theta_i/\partial Y)\partial/\partial\vartheta_i$ and equating to zero the coefficients of $\xi^0$and ξ^1 in resulting expression, we get

$$\sum_{i=1}^{3}\left(I\frac{\partial\theta_i}{\partial t}+P_0\frac{\partial\theta_i}{\partial x}\right)\frac{\partial U_1}{\partial\vartheta_i} = 0, \tag{4.12}$$

$$\sum_{i=1}^{3}\left(I\frac{\partial\theta_i}{\partial t}+P_0\frac{\partial\theta_i}{\partial x}\right)\frac{\partial U_2}{\partial\vartheta_i} = -\frac{\partial U_1}{\partial t}-P_0\frac{\partial U_1}{\partial x}-(U_1.\nabla Q)_0-\sum_{i=1}^{3}\frac{\partial\theta_i}{\partial x}(U_1.\nabla P)_0\frac{\partial U_1}{\partial\xi_i}, \tag{4.13}$$

where ∇ is the gradient operator with regard to the dependent variable U , I is the identity matrix of order 3×3. Now all the phase functions θ_i $(1\le i\le 3)$ propitiate the Eikonal equation:

$$\det\left(I\frac{\partial\theta_i}{\partial t}+P_0\frac{\partial\theta_i}{\partial x}\right) = 0, \tag{4.14}$$

where $\langle\det\rangle$ represents the determinant. Now we consider a simplest phase function given as

$$\theta_i\,(x,t) \;=\; x-\lambda_i t,\;\; 1\le i\le 3\;. \tag{4.15}$$

We infer from equation (4.12) that for all phase functions θ_i, the derivative terms $\frac{\partial U_1}{\partial\vartheta_i}$ are parallel to the right eigenvectors $r^{(i)}$ of the matrix P_0, thus we have

$$U_1 \;=\; \sum_{i=1}^{3}\sigma_i\,(x,t,\vartheta_i)\,r^{(i)}, \tag{4.16}$$

where the scalar function $\sigma_i = (l^i\cdot U_1)$, is recognized as wave amplitude that depends on the i^{th} fast variable ϑ_i . The wave which forms here, whether it is an oscillatory wave or a pulse, is based on dependency of σ_i on ϑ_i. Let us suppose that

$\sigma_i\,(x,t,\vartheta_i)$ has zero mean value with regard to the fast variable ϑ_i, it means

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sigma_i\,(x,t,\vartheta_i)d\vartheta_i\;=\;0. \tag{4.17}$$

We utilize the equation (4.16) in equation (4.13) which gives the following relationship for U_2

$$U_2\;=\;\sum_{j=1}^{3}m_j r^{(j)}. \tag{4.18}$$

Further, utilizing equation (4.18) in equation (4.13) and pre-multiplying the resulting expression by $l^{(i)}$ yields the system of decoupled inhomogeneous first order PDEs which are given as

$$\sum_{j=1}^{3}(\lambda_i\;-\;\lambda_j)\frac{\partial m_i}{\partial\vartheta_j}\;=\;-\frac{\partial\sigma_i}{\partial t}-\lambda_i\frac{\partial\sigma_i}{\partial x}$$
$$-\;l^{(i)}(U_1\cdot\nabla Q)_0-\sum_{j=1}^{3}l^{(i)}(U_1\cdot\nabla P)_0\frac{\partial U_1}{\partial\vartheta_j},\;1\le i\le 3. \tag{4.19}$$

Now the term, $\dfrac{\partial}{\partial t}\;+\;\lambda_i\dfrac{\partial}{\partial x}$, which exhibits the ray derivative, is denoted by $\dfrac{d\sigma_i}{ds}$, hence the equation (4.19) may be recast as

$$\sum_{j=1}^{3}(\lambda_i\;-\;\lambda_j)\frac{\partial m_i}{\partial\vartheta_j}\;=\;-\frac{d\sigma_i}{ds}-\beta_i\sigma_i-\Omega^i_{ii}\sigma_i\frac{\partial\sigma_i}{\partial\vartheta_i}$$
$$-\sum_{j,k}\Omega^i_{jk}\sigma_j\frac{\partial\sigma_k}{\partial\vartheta_k}=H_i\,(x,t,\vartheta_1,\vartheta_2,\vartheta_3)\,,1\le i\le 3. \tag{4.20}$$

The i^{th} characteristics in equation (4.20) is provided by
$\overset{\bullet}{\vartheta}=\;(\lambda_i\;-\;\lambda_j)$ for $i\neq j,\;\overset{\bullet}{\vartheta}\;=\;0,\overset{\bullet}{m}_i\;=\;H_i.$
Therefore, we determine the asymptotic average of equation (4.20) along the characteristics and then supplicate to the sub linearity of U_2 in ϑ, which ensures that there is no secular term in equation (4.13). In view of equation (4.20) we have, along the

characteristics, ϑ_i are stable and the asymptotic mean value of $\overset{\bullet}{m}$ vanishes which implies that the wave amplitude $\sigma_i\,(1 \le i \le 3)$ propitiating the succeeding system of coupled integro-differential equations:

$$\begin{aligned}\frac{\partial \sigma_i}{\partial t}&+\lambda_i \frac{\partial \sigma_i}{\partial x} + \beta_i \sigma_i + \Omega^i_{ii}\sigma_i \frac{\partial \sigma_i}{\partial \vartheta_i} \\ &+ \sum_{i\neq j\neq k} \Omega^i_{jk} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma_j\left[\vartheta_i + (\lambda_i-\lambda_j)\, s\right] \overset{\wedge}{\sigma}_k \left[\vartheta_i + (\lambda_i - \lambda_j)\, s\right] ds = 0,\end{aligned} \tag{4.21}$$

where $\overset{\wedge}{\sigma}_k = \frac{\partial \sigma_k}{\partial \vartheta_k}$ and the coefficient β_i and Ω^i_{jk} are written as

$$\beta_i = l^{(i)}\big(r^i \cdot \nabla Q\big)_0\,,\ \Omega^i_{jk} = l^{(i)}\big(r^{(j)} \cdot \nabla P\big)_0 r^{(k)}. \tag{4.22}$$

Now we have to determine the coefficients β_i which vanish for plane waves ($m = 0$), therefore in this case the governing system does not contain any source term. Further, in the absence of dust particles, equation (4.21) changes to the same equation as discussed in Ref.[60]. Furthermore the interaction coefficients Ω^i_{jk} are asymmetric in j and k which quantify the coupling strength between j^{th} and k^{th} wave modes where $j \neq k$ that can produce a i^{th} wave ($i \neq j \neq k$). The interaction coefficients Ω^i_{jk} for $i = j = k$ assign to the non-linear self-interaction. Here for genuinely non-linear waves the interaction coefficients are non-zero and zero for linearly degenerate waves. Also it is noticed that if each of the coupling coefficients Ω^i_{jk} ($i \neq j \neq k$) are zero or the integral in equation (4.21) becomes zero, then we conclude that the wave do not resonate and equation (4.21) detracts to the uncoupled system of Burger's equations. Now from the equation (4.20), in the governing system, the coefficients β_i and Ω^i_{jk} provide the qualitative picture of the nonlinear interaction procedure and it can be demonstrated by the formulae as given in equation (4.22).

Therefore these coefficients are written as

$$\beta_1 = \frac{mc_0}{2x}\,,\ \beta_2 = 0\,,\ \beta_3 = \frac{-mc_0}{2x}\,,$$

$$\Omega^1_{23} = -\Omega^3_{21} = \frac{1}{2c_0\rho_0(1-\phi\rho_0)} = \alpha_1\,(say)\,,$$

$$\Omega^2_{13} = -\,\Omega^2_{31} = \frac{-2c_0^3\phi\rho_0 + c_0^3(1-\Gamma)}{\rho_0(1-\phi\rho_0)} = \alpha_2\,(say)\,, \tag{4.23}$$

$$\Omega^1_{32} = \Omega^3_{12} = \Omega^2_{22} = 0\,,$$

$$\Omega^1_{11} = -\,\Omega^3_{33} = \frac{c_0(1+\Gamma)}{\rho_0(1-\phi\rho_0)} = \alpha_3\,(say)\,.$$

Now after some simplification the resonant equation (4.21) can be written as

$$\frac{\partial\sigma_1}{\partial t} + c_0\frac{\partial\sigma_1}{\partial x} + \frac{mc_0}{2x}\sigma_1 + \alpha_3\sigma_1\frac{\partial\sigma_1}{\partial\vartheta_1} + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} k\left(x,t,\frac{\vartheta_1+\theta}{2}\right)\sigma_3\,(x,t,\theta)\,d\theta = 0, \tag{4.24}$$

$$\frac{\partial\sigma_2}{\partial t} = 0, \tag{4.25}$$

$$\frac{\partial\sigma_3}{\partial x} - c_0\frac{\partial\sigma_3}{\partial t} - \frac{mc_0}{2x}\sigma_3 - \alpha_3\sigma_3\frac{\partial\sigma_3}{\partial\vartheta_3} - \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} k\left(x,t,\frac{\vartheta_3+\theta}{2}\right)\sigma_1\,(x,t,\theta)\,d\theta = 0. \tag{4.26}$$

Here k is kernel which is written as

$$k\left(x,t,\frac{\vartheta+\theta}{2}\right) = \frac{\alpha_1}{2}\frac{\partial\sigma_2}{\partial\vartheta_2}\left(x,t,\frac{\vartheta+\theta}{2}\right). \tag{4.27}$$

In equation (4.24), the integral average term shows the contribution to the wave amplitude σ_1 as a result of the nonlinear interactions of the wave field σ_2 with the wave field σ_3. Equivalently from equation (4.26) the integral average term shows the contribution to the wave amplitude σ_3 as a result of the nonlinear interaction of the wave field σ_2 with the wave field σ_1. The nonlinear term proportional to $\sigma_1\sigma_1{}'$ and $\sigma_3\sigma_3{}'$ in (4.24) and (4.26) account for self-interaction which generate higher harmonics leading to the distortions of the wave profile and consequent shock formation. The result is that the two acoustic wave fields σ_1 and σ_3 exhibit a strong effect from the nonlinearity present in the system under study. More detailed analysis may be presented on the similar lines as given in Sharma and Srinivasan [139].

Let us consider that the initial value of σ_i at time $t = 0$ given by $\sigma_i^0(x, \vartheta_i)$. Further from equation (4.25) we obtain $\sigma_2(x, t, \vartheta_1) = \sigma_2^0(x, \vartheta_1)$. Therefore the system of equations (4.24) to (4.26) reconstruct to the pair of equations for the wave field σ_2 and wave field σ_3 which connect through the linear integral operator containing the kernel which is given by

$$k(x, t, \vartheta) = \frac{\alpha_1}{2}\frac{\partial \sigma_2}{\partial \vartheta_2}(x, \vartheta). \tag{4.28}$$

Now if the initial data $\sigma_i^0(x, \vartheta)$ are periodic function of period 2π of the phase variable ϑ , the pair of resonant asymptotic equations are written as

$$\frac{\partial \sigma_1}{\partial t} + c_0\frac{\partial \sigma_1}{\partial x} + \frac{mc_0}{2x}\sigma_1 + \alpha_3\sigma_1\frac{\partial \sigma_1}{\partial \vartheta_1} + \frac{1}{2\pi}\int_{-\pi}^{\pi} k\left(x, t, \frac{\vartheta_1 + \theta}{2}\right)\sigma_3(x, t, \theta)\, d\theta = 0, \tag{4.29}$$

$$\frac{\partial \sigma_3}{\partial t} - c_0\frac{\partial \sigma_3}{\partial x} - \frac{mc_0}{2x}\sigma_3 - \alpha_3\sigma_3\frac{\partial \sigma_3}{\partial \vartheta_3} - \frac{1}{2\pi}\int_{-\pi}^{\pi} k\left(x, t, \frac{\vartheta_3 + \theta}{2}\right)\sigma_1(x, t, \theta)\, d\theta = 0. \tag{4.30}$$

The kernel k appearing in equations (4.29) and (4.30) is given by the equation (4.28).

4.4 Non-linear geometrical acoustics solution

The approximate asymptotic solution of the form (4.11) of the system of equation (4.2) to (4.4) or (4.8) satisfy the small amplitude oscillating initial data given by

$$U\left(x,0\right) = U_0 + \xi U_1^0\left(x, x/\xi\right) + O\left(\xi^2\right), \tag{4.31}$$

which is non-resonant when the functions $U_1^0\left(x, x/\xi\right)$ are smooth with a compact support [60]. In fact the expression (4.11), with U_1 as provided by equation (4.16), is uniformly valid to the leading order if the shock waves are present in the solution. Further, the characteristics equations are written as

$$\frac{d\vartheta_i}{dx} = \frac{\alpha_3 \sigma_i}{c_0}, \quad \frac{dt}{dx} = \frac{e_i}{c_0}, \tag{4.32}$$

where e_i is either +1 or -1 for $i = 1$ and $i = 3$ respectively.
Thus in view of equation (4.31) the decoupled equations (4.29) and (4.30) may be recast as

$$\frac{d\vartheta_i}{dx} = -\frac{m\sigma_i}{2x}. \tag{4.33}$$

Now equation (4.33) produces on integration along the rays, $s_i = x - e_i\, c_0\, t =$ constant

$$\sigma_i = \sigma_i^0\left(s_i, \tau_i\right)\left(\frac{x}{s_i}\right)^{-m/2}, \tag{4.34}$$

where the function σ_i^0 is derived from the initial data (4.31) and the fast variable τ_i parameterizes the set of characteristics curves which are given in equation (4.32). Therefore, one can derive from equation (4.32),

$$\tau_i = \vartheta_i - e_i \alpha_3 \sigma_i^0\left(s_i, \tau_i\right) B_i^{(m)}\left(t\right), \tag{4.35}$$

where $B_i^{(m)}(t) = \int\limits_0^t \left(1 + \frac{e_i c_0 t}{s_i}\right)^{-m/2} dt$. Here we infer that the wave amplitude decays as which is similar to corresponding classical gasdynamic case Hunter and Keller [59]. In view of and the initial data given by equation (4.31), one can find the solution of the system of equations (4.8) as

$$\begin{aligned}\rho(x,t) =&\rho_0 + \xi\left(\sigma_1^0(s_1,\tau_1)(x/(x-c_0t))^{-m/2} + \sigma_3^0(s_3,\tau_3)(x/(x+c_0t))^{-m/2}\right)\\ &-\frac{\xi}{c_0^2}\sigma_2^0(x,x/\xi) + O\left(\xi^2\right),\end{aligned} \tag{4.36}$$

$$\begin{aligned}\nu(x,t) =&\xi\left(\frac{c_0}{\rho_0}\sigma_1^0(s_1,\tau_1)(x/(x-c_0t))^{-m/2} - \frac{c_0}{\rho_0}\sigma_3^0(s_3,\tau_3)(x/(x+c_0t))^{-m/2}\right)\\ &+O\left(\xi^2\right),\end{aligned} \tag{4.37}$$

$$\begin{aligned}p(x,t) =&p_0 + \xi\left(c_0^2\sigma_1^0(s_1,\tau_1)(x/(x-c_0t))^{-m/2} + c_0^2\sigma_3^0(s_3,\tau_3)(x/(x+c_0t))^{-m/2}\right)\\ &+O\left(\xi^2\right).\end{aligned} \tag{4.38}$$

Now let us suppose that the fast variable τ_i given by equation (4.35), may be written at $t = 0$, $\tau_i = x/\xi$. Therefore, with the help of solution (4.36) to (4.38) at $t = 0$ i.e. the initial values of $\sigma_i\,(1 \le i \le 3)$ are written as

$$\sigma_1^0(x,\tau_1) = \frac{\rho_0}{2c_0}\nu_1^0(x,\tau_1) + \frac{1}{2c_0^2}p_1^0(x,\tau_1), \tag{4.39}$$

$$\sigma_2^0(x,\tau_1) = -c_0^2\rho_1^0(x,x/\xi) + p_1^0(x,x/\xi), \tag{4.40}$$

$$\sigma_3^0(x,\tau_1) = -\frac{\rho_0}{2c_0}\nu_1^0(x,\tau_3) + \frac{1}{2c_0^2}p_1^0(x,\tau_3). \tag{4.41}$$

Therefore we have obtained the complete solution of the system of equations (4.8) in view of equation (4.31) which is given by equations (4.36) to (4.41). Further if there is any multi valued overlap in the given solution then it has to be dealt by introducing shock waves into the solution. Now by using the Rankine-Hugoniot jump conditions, shock wave is introduced into the solution to prohibit the multi-valuedness.

4.5 Shock waves

The shock location ϑ_i^s may be obtained from the following relation Hunter and Keller [59],

$$\frac{d\vartheta_i^s}{dt} = \frac{1}{2}\Omega_{ii}^i\left(\sigma_i^{(-)} + \sigma_i^{(+)}\right),\ (i = 1,\ i = 3), \tag{4.42}$$

which is the shock speed in the $t - \vartheta_i$ plane, where $\sigma_i^{(-)}$ and $\sigma_i^{(+)}$ exhibit the value of the σ_i just ahead and behind the shock wave serially. Furthermore, we have $\sigma_i^{(-)} = 0$ in the undisturbed region for the shock front. Now by using the equation (4.42) and (4.34) and leaving the superscripts on ϑ_i^s and $\sigma_i^{(+)}$ in equation (4.42) we obtain

$$\frac{d\vartheta_i}{dt} = \frac{\alpha_3}{2}e_i\sigma_i^0\left(s_i, \tau_i\right)\left(\frac{x}{s_i}\right)^{-m/2}. \tag{4.43}$$

Further, using the equation (4.35) and equation (4.43) we obtain the following relation between e_i and t on the shock,

$$B_i^{(m)}\left(t\right) = -\left(\frac{2e_i}{\alpha_3(\sigma_i^0)^2}\right)\int_0^{\tau_i}\sigma_i^0\left(t'\right)dt'. \tag{4.44}$$

Hence because of equation (4.35), equation (4.44) provide the following equation which negotiate the shock path parametrically.

$$\vartheta_i = \tau_i - \frac{2}{\sigma_i^0}\int_0^{\tau_i}\sigma_i^0\left(t\right)dt. \tag{4.45}$$

Now we conclude that if $\sigma_i^0(0) \neq 0$ then shock forms immediately right from the origin [60, 136]. The approximation between τ_i and t is provided as

$\tau_i \sim -\left(\frac{\alpha_3}{2}\right) e_i \sigma_i^* B_i^{(m)}(t)$; $\sigma_i^* = \sigma_i^0|_{\tau_i=0}$, hence $\vartheta_i \sim \left(\frac{\alpha_3}{2}\right) e_i \sigma_i^* B_i^{(m)}(t)$. Now it is indicating that the shock moves with velocity $\frac{\alpha_3}{2} e_i \sigma_i^* \left(\frac{x}{s_i}\right)^{-m/2}$, which is totally same as equation (4.43). We also consider the case when only outgoing waves are generated from the source, that is compression, is followed by rarefaction implying $\sigma_i^0(\tau_i) \to 0$ as $\tau_i \to \tau_i^*$ which shows that in the surroundings of the wavelet $\tau_i = \tau_i^*$, the wavelets will never approach the shock and the shock is asymptotic to $\vartheta_i \sim \tau_i^* + De_i\sqrt{2\alpha_3 B_i^{(m)}}$, where $D = \left(-e_i \int\limits_0^{\tau_i^*} \sigma_i^0(t')\, dt'\right)^{1/2}$, and from equation (4.34), the amplitude is approximated as

$$\sigma_i \sim \sqrt{\frac{2}{\alpha_3}} D \left(1 + \frac{e_i c_0 t}{s_i}\right)^{-m/2} \left(\int\limits_0^t \left(1 + \frac{e_i c_0 t'}{s_i}\right)^{-m/2} dt'\right)^{-1/2}. \tag{4.46}$$

The effect of dust particles enters into the expression for amplitude through the parameter α_3 and c_0. From the relation (4.46) we infer that the amplitude is inversely proportional to α_3 . Also, the amplitude σ_i varies according to $(\alpha_3)^{-1/2}$. Also an increase in α_3 causes to decrease the wave amplitudes. An increase in the dust particle density causes to decrease the wave amplitude which causes to enhance the wave decay process. Further, it is observed here that the decreasing (increasing) values of the mass fraction of dust particles causes to enhance (slow down) the amplitude of the shock waves, as a result the shock formation distance increases (decreases) i.e. the shock formation is delayed (early). Also, the amplitude has been computed to see the effect of dust particles which is presented in the Table 1. It is clear from the Table 1 that the presence of dust particles have same influence on the wave amplitude as reported qualitatively. Further, in the absence of dust particles our results are in close agreement to the results as reported in [19]. It is clear from the equation (4.46) that in the absence of dust particles, shock waves decay like ≈

$$\begin{cases} t^{-1/2}, & \text{if } m = 0 \\ t^{-3/4}, & \text{if } m = 1 \\ t^{-1}(\log)^{-1/2}, & \text{if } m = 2. \end{cases} \cdot \tag{4.47}$$

We obtained that shock waves decay like $t^{-1/2}$ for planar flow. In same manner shock waves decay like $t^{-3/4}$ and $t^{-1}(\log t)^{-1/2}$ for cylindrically symmetric flow and spherically symmetric flow respectively. Hence in case of non-planar flows, the shock formation distance increases in comparison to the corresponding planar flows. Also, the results obtained here are similar to the results as reported in [149].

4.6 Results and Conclusion

The present study uses the multiple time scales method to derive the small amplitude high frequency asymptotic solution for the system of nonlinear partial differential equations characterizing one-dimensional compressible unsteady, planar and nonplanar flows in a dusty gas. The theory of weakly non-linear geometrical acoustics is utilized to examine the resonant interaction of waves and to analyze the evolution of shock wave in a dusty gas flow. The transport equations for the wave amplitude along the rays for the dusty gas flow, comprising of a system of inviscid Berger's equations with known kernel, has been derived. The qualitative analysis of non-linear wave interaction process and self-interaction of non-linear waves which exist in the system under study can be made by using the coefficients occurring in the transport equations. In our discussion the Euler equations reduces to a pair of asymptotically resonant equations for the fields of acoustic wave. The nonlinear interaction of the wave fields and self-interactions which generate higher harmonics leading to the distortions of the wave profile and consequent shock formation has been discussed. A non-resonant multiwave mode matter has been discussed by Hunter and Keller [59]. Here it is obtained that the wave fields do not get across with each other which is connected with the particle path way, however they interact with an acoustic wave field to yield resonant contribution regarding the other acoustic wave fields. The acoustic wave fields may or may not get across to each other but in either case their entire contribution towards the entropy field must be zero. We require a suitable value of ϕ , to assess the influence of dust particles, which enters into the solution through the parameter c_0. Now any change in value of ϕ, affects the velocity, pressure and density of the high frequency small amplitude type waves. It is clear from equations (4.36) to (4.38) that the increasing values of ϕ, causes the density, velocity and pressure of the high frequency small amplitude waves to increase. Also, we have discussed here about the presence of shock and their position in the dusty gas. It is noticed from equation (4.35) that in a contracting piston motion having cylindrical symmetry, if the initial wave amplitude exceeds a critical value then the shock forms before the focus, but in case of spherical symmetry, a shock is always formed before the formation of focus, does not matter how small be the initial wave amplitude. The existence of shock wave in a dusty gas is also discussed here. Therefore, we conclude that for nonplanar flows the formation of shock wave delays in comparison to the similar cases of the planar flows. Also, the decreasing (increasing) values of the mass fraction of dust particles causes to increase (decrease) the amplitude of the shock waves, as a result the shock formation distance increases (decreases) i.e. the shock formation is delayed (early). Also, the results obtained here are similar to the results as reported in [149].

Table 1: Wave amplitudes for non-planar flow for different value of k_p and β

k_p	β	Γ	Computed (σ)	
			$m=1$	$m=2$
0.1	0.05	1.39587	10.7903488	14.7759488
0.1	0.1	1.39223	10.77965228	14.7253477
0.1	0.5	1.36556	10.70230137	14.4457068
0.2	0.05	1.39587	10.77689045	14.7261661
0.2	0.1	1.39223	10.75707175	14.5132022
0.2	0.5	1.36556	10.58033799	13.9716292
0.4	0.05	1.39587	10.74662136	14.5476528
0.4	0.1	1.39223	10.69203578	14.3121624
0.4	0.5	1.36556	10.37485453	12.86823074
0.6	0.05	1.39587	10.68405753	14.2726328
0.6	0.1	1.39223	10.57655768	13.76453187
0.6	0.5	1.36556	10.11834066	11.841303255

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