Chapter 2

Riemann problem for non-ideal polytropic magnetogasdynamic flow

“Mathematics is the key and door to the sciences.”

-Galileo Galilei

2.1 Introduction

The Riemann problem consists of an initial value problem composed of the Euler equations together with piecewise constant initial data having single contact discontinuity. For the numerical and theoretical point of view, the study of solution to the Riemann problem is extensively utilized for the system of conservation laws in real gas flow, shallow water flow, gasdynamics etc. In the solution of the Riemann problem, all the features such as rarefactions waves, shock waves occur in the form of characteristics hence it is very convenient for the readers to understand the Euler equations in the form of conservation laws. The solution of the Riemann problem consists of three waves with middle one is always contact discontinuity and other remaining waves are either rarefaction waves or shock waves. Lax [74] determined the solution of the Riemann problem for the condition when the initial data of the problem consists of two constant states U_*^1 and U_*^2, where U_*^1 and U_*^2 are respectively the vector of conserved variables to the left and right of $x = 0$ such that $\|U_*^1 - U_*^2\|$ is appropriately small and left and right constant states are divided by jump discontinuity at $x = 0$. In the consideration of Euler equations, the Riemann problem consists of the shock tube problem and for detailed discussion of shock tube problem and other physical problems in form of conservation laws of gasdynamics, the readers are recommended to study the book by Courant and Friedrichs [23]. Hu and Sheng [98] solved the Riemann problem of conservation laws in magnetogasdynamics. The explicit solution to the Riemann problem is of great significance in relativistic gasdynamics and magnetogasdynamics Godunov [20], Smoller [21] and Chorin [22]. The Riemann problem in one-dimensional planar flow of ideal gas subjected to transverse magnetic field have been studied by Shekhar and Sharma [99, 12], Singh and Singh [100]. Gupta et al. [101] have used a direct approach to solve Riemann problem for dusty gas flow. Ambika and Radha [95] studied the uniqueness and existence of elementary wave solution to the Riemann problem for van der Waals gases. Kuila and Sekhar [102, 103] presented the solution of the Riemann problem for ideal and non-ideal isentropic magnetogasdynamic flows. Mentrelli and Ruggeri [104] investigated shock and rarefaction waves in hyperbolic model of incompressible fluids. Yang and Sun [105] solved the Riemann problem for a class of coupled hyperbolic systems of conservation laws with delta initial data. Kuila et al. [103] have solved the Riemann problem for two pressure model of non-ideal isentropic compressible two phase flows.

Bressan [90] provided a self-contained introduction to the mathematical theory of hyperbolic system of conservation laws, with particular emphasis on the study of discontinuous solutions, characterized by the appearance of shock waves. Lihui Guo [106] solved the Riemann problem with a source term for Chaplygin gas equations with delta initial data. Pang and Hu [107] studied the Reimann solutions to compressible fluid described by the generalized chaplygin gas. Toro [108] proposed the Riemann solvers and numerical methods for explicit solution of the Riemann problem. A detail deliberation on the explicit solution of the Riemann problem can be obtained in Toro [108], Smoller [21]. Among the several authors such as Godunov [20], Toro [108] and Quartapelle [109] modified the iterative scheme for the Riemann problem of classical gasdynamics to demonstrate the flow field.

The important contribution of the present paper is to derive the analytical solution of the Riemann problem for one-dimensional motion of non-ideal gas subjected to transverse magnetic field with infinite electrical conductivity. The elementary wave solution i.e. shock wave, simple wave and contact discontinuities is obtained and discussed about their properties. Also the effect of parameter of non-idealness in the presence of magnetic field on the density and velocity profile is analyzed.

2.2 Basic Equations

The equations governing the motion of one-dimensional unsteady planar flow of non-ideal polytropic gas subject to a transverse magnetic field with infinite electrical conductivity is written as (Shekhar and Sharma [12]),

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\rho^{-1}\left(\frac{\partial p}{\partial x}+\frac{B}{\nu}\frac{\partial B}{\partial x}\right)=0, \tag{2.2}$$

$$\frac{\partial E}{\partial t}+u\frac{\partial E}{\partial x}-\frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right)=0, \tag{2.3}$$

$$\frac{\partial B}{\partial t}+u\frac{\partial B}{\partial x}+B\frac{\partial u}{\partial x}=0, \tag{2.4}$$

where ρ, u and p denote the fluid density , particle velocity and pressure respectively. x and t represent the spatial coordinate and time respectively. Here, $B=\nu H$, with ν, B and H as the magnetic permeability, magnetic induction and magnetic field vector respectively. In Eq. (2.3), E represents the internal energy per unit mass of the mixture expressed as

$$E=\frac{p(1-b\rho)}{\rho(\gamma-1)}, \tag{2.5}$$

where b is the van der Waals gas constant and $\gamma=\frac{c_p}{c_v}$ is the ratio of specific heats of the gas with c_p and c_v being respectively the specific heats of the gas at constant pressure and volume. Also the equation of state for polytropic non-ideal gas is

$$p=ke^{\frac{S}{c_v}}\left(\frac{\rho}{(1-b\rho)}\right)^{\gamma}, \tag{2.6}$$

where S is the specific entropy and k is constant. By utilizing Eq.(2.5) in Eq. (2.3) we get

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+d^2\rho\frac{\partial u}{\partial x}=0, \tag{2.7}$$

where d is the sound velocity which is defined as $d=\left(\frac{\gamma p}{\rho(1-b\rho)}\right)^{1/2}$.

Now system of equations (2.1) to (2.4) are modified as

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0, \tag{2.8}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\left(\frac{\partial p}{\partial x}+\frac{B}{\nu}\frac{\partial B}{\partial x}\right)=0, \tag{2.9}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + d^2\rho\frac{\partial u}{\partial x} = 0, \tag{2.10}$$

$$\frac{\partial B}{\partial t} + u\frac{\partial B}{\partial x} + B\frac{\partial u}{\partial x} = 0. \tag{2.11}$$

The system of equation (2.8) to (2.11) may be represented in the following matrix form

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0, \tag{2.12}$$

where

$$U = \begin{bmatrix} \rho \\ u \\ p \\ B \end{bmatrix}, \quad A = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & \frac{1}{\rho} & \frac{B}{\nu\rho} \\ 0 & \rho d^2 & u & 0 \\ 0 & B & 0 & u \end{bmatrix}.$$

The eigenvalues of the matrix A can be written as

$$\lambda_1 = u - c, \lambda_2 = u, \lambda_3 = u + c, \lambda_4 = u. \tag{2.13}$$

Here, c denotes the magneto-acoustic speed which is defined as
$c = \sqrt{d^2 + e^2}$, where $e = \sqrt{\frac{B^2}{\nu\rho}}$ is the Alfvén speed.
The eigenvectors corresponding to eigenvalues $\lambda_i, i = 1, 2, 3, 4$ of matrix A are

$$r_1 = \begin{bmatrix} \frac{\rho}{c} \\ 1 \\ \rho c \\ 0 \end{bmatrix}, r_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, r_3 = \begin{bmatrix} \frac{-\rho}{c} \\ 1 \\ -\rho c \\ 0 \end{bmatrix}, r_4 = \begin{bmatrix} 1 \\ 0 \\ -e^2 \\ 1 \end{bmatrix}. \tag{2.14}$$

Since all the eigenvalues of matrix A are real and corresponding eigenvectors are linearly independent, therefore we infer that the system (2.12) is hyperbolic. Under the assumption $B = k_1\rho$ (Shekhar and Sharma [12], Sheng [98]), which is an interpretation of the frozen-in law where k_1 is positive constant, the Eq. (2.8) is equivalent to Eq. (2.11). In view of the above assumption the system (2.12) may be rewritten in the following form

$$\frac{\partial U^*}{\partial t} + A\frac{\partial U^*}{\partial x} = 0, \tag{2.15}$$

where

$$U^* = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}, A^* = \begin{bmatrix} u & \rho & 0 \\ \frac{e^2}{\rho} & u & \frac{1}{\rho} \\ 0 & \rho d^2 & u \end{bmatrix}.$$

Since $c > 0$, the eigenvalues of matrix A^* are also distinct and real, eigenvectors corresponding to distinct eigenvalues are linearly independent which are written as

$$r^{(1)} = \begin{bmatrix} \frac{-\rho}{c} \\ 1 \\ \frac{-\rho d^2}{c} \end{bmatrix}, r^{(2)} \begin{bmatrix} 1 \\ 0 \\ -e^2 \end{bmatrix}, r^{(3)} = \begin{bmatrix} \frac{\rho}{c} \\ 1 \\ \frac{\rho d^2}{c} \end{bmatrix}. \tag{2.16}$$

Therefore the reduced system (2.15) is also strictly hyperbolic.

2.3 Riemann problem and Riemann invariants

The conservative form of the system (2.15) may be presented as

$$\frac{\partial U^*}{\partial t} + \frac{\partial F(U^*)}{\partial x} = 0, \tag{2.17}$$

where $U^* = (\rho, \rho u, \rho(u^2/2 + E) + B^2/\nu\rho)^{tr}, F(U^*) = (\rho u, p + \rho u^2 + (B^2/2\nu), u(p + \rho(u^2/2 + E) + B^2/2\nu))$.

The initial data of the Riemann problem for the system of equations (2.17) is given by

$$U^*(x,0) = U_0^*(x) = \begin{cases} U_1^*, & \text{if } x < 0 \\ U_2^*, & \text{if } x > 0 \end{cases}. \tag{2.18}$$

Here, U_1^* and U_2^* denote the left and right constant state respectively which is separated by the jump discontinuity at $x = 0$. The explicit solution of the Riemann problem (2.17) and (2.18) contains three waves, which is related with distinct eigenvalues as shown in Fig. 1. In system (2.15), all the characteristic fields are either genuinely non-linear or linearly degenerate depending on the quantity $\Delta\lambda_j r^{(j)}$, which is non-zero and zero respectively. Hence, the 1-characteristic field and 3-characteristic field are genuinely non-linear. In similar manner 2-characteristic field is linearly degenerate. Therefore, 1-characteristic field and 3-characteristic field will always be either shock wave or rarefaction wave and 2-characteristic field will be always contact discontinuity.

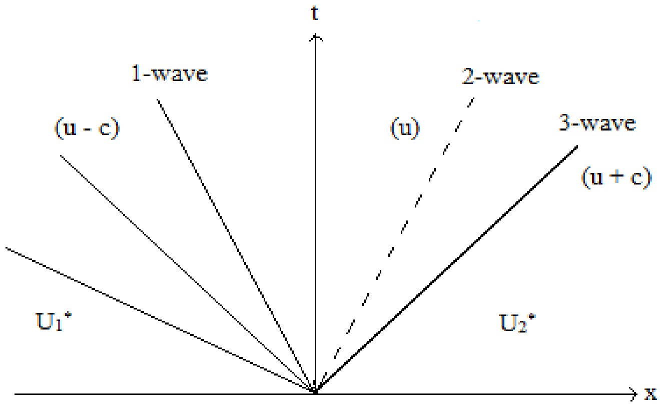

FIGURE 2.1: Solution of the Riemann problem for the one dimensional Euler equations of gasdynamics in the x-t plane.

We consider the following matrix of order 3×3 consisting of the the right eigenvectors of matrix A^*

$$\begin{bmatrix} \dfrac{-\rho}{c} & 1 & \dfrac{\rho}{c} \\ 1 & 0 & 1 \\ \dfrac{-\rho d^2}{c} & -e^2 & \dfrac{\rho d^2}{c} \end{bmatrix}.$$

The inverse of the above matrix is written as

$$\begin{bmatrix} \dfrac{1}{\psi^2} & 0 & \dfrac{-1}{d^2\psi^2} \\ \dfrac{-d(\psi^2-1)}{2\rho\psi} & \dfrac{1}{2} & \dfrac{-1}{2d\rho\psi} \\ \dfrac{d(\psi^2-1)}{2\rho\psi} & \dfrac{1}{2} & \dfrac{1}{2d\rho\psi} \end{bmatrix},$$

where $\psi = \left(1+\frac{e^2}{d^2}\right)^{1/2}$ denotes Alfvén number.
Integration of the above expression yields,

$$\frac{1}{\psi^2}d\rho - \frac{1}{d^2\psi^2}dp, \left(\frac{1}{\psi}-\psi\right)\frac{d}{\rho}d\rho + du - \frac{1}{d\rho\psi}dp, \left(\psi - \frac{1}{\psi}\right)\frac{d}{\rho}d\rho + du + \frac{1}{d\rho\psi}dp. \quad (2.19)$$

The first term of above expression can be written as,
$\frac{1}{\psi^2}d\rho - \frac{1}{d^2\psi^2}dp = \frac{(1-b\rho)}{\psi^2}d\left(\frac{(\rho/(1-b\rho))^\gamma}{p}\right)$,
which represents that the entropy is constant along the particle path through the smooth solutions. On introducing the entropy condition, the remaining two terms of Eq. (2.19) give the following relation,

$$R^{\pm} = u \pm \frac{2}{(\gamma-1)}\psi d(1-b\rho)^{1/2}. \quad (2.20)$$

The Riemann invariants (Π_1^j, Π_2^j) with respect to the jth-characteristic field are written as,

$$j = 1, \Pi_1^1 = S, \Pi_2^1 = u + \frac{2}{(\gamma-1)}\psi d(1-b\rho)^{1/2}, \quad (2.21)$$

$$j = 2, \Pi_1^2 = u, \Pi_2^2 = p + \frac{B^2}{2\nu}, \quad (2.22)$$

$$j = 3, \Pi_1^1 = S, \Pi_2^1 = u - \frac{2}{(\gamma-1)}\psi d(1-b\rho)^{1/2}. \quad (2.23)$$

2.4 Shock wave

Shock waves are piecewise discontinuous solutions, which satisfy the Lax entropy condition. Let us assume that shock wave propagates with velocity σ, dependent on the data existing on the two sides of jump discontinuity. Also the conserved variables satisfy the R-H relations [109]. For a shock wave two constant states, U_1^*

and U_2^* such that U_1^* represents the left constant state and U_2^* represents the right constant state separated by either a shock or rarefaction wave or contact discontinuity. Therefore the conserved variables satisfy the following conditions:

1. The R-H jump relation,i.e.

$$F(U_2^*) - F(U_1^*) = \sigma(U_2^* - U_1^*). \tag{2.24}$$

Hence, the jump relations for the system (2.17) is given by

$$\sigma[\rho] = [\rho u], \tag{2.25}$$

$$\sigma[\rho u] = [p + \rho u^2 + (B^2/2\nu)], \tag{2.26}$$

$$\sigma[E + (B^2/2\nu)] = [u(E + p + (B^2/2\nu))]. \tag{2.27}$$

2. The Lax entropy condition [74], which is written as

$$\lambda^{(j-1)}(U_1^*) < \sigma < \lambda^{(j)}(U_1^*), \lambda^{(j)}(U_2^*) < \lambda^{(j+1)}(U_2^*), j = 1, 3. \tag{2.28}$$

By introducing new variables $v = u - \sigma$, $n = \rho v$ and using in the above jump relation, we obtain

$$[n] = 0, \tag{2.29}$$

$$[p + nv + (B^2/2\nu)] = 0, \tag{2.30}$$

$$n\left[v^2 + e^2 + \frac{2}{\gamma(\gamma-1)}d^2(1 - b\rho)(\gamma - b\rho)\right]. \tag{2.31}$$

Using Eq. (2.28) in case of 1- shock wave, we obtain $\sigma < u_1 - c_1$ which gives $c_1 < v_1$ and $u_2 - c_2 < \sigma < u_2$ which implies $0 < v_2 < c_2 < v_2 + \sigma$. Hence in the case of 1-shock wave, we obtain $v_1 > c_1$ and $0 < v_2 < c_2$, which implies that $u_1 > \sigma$ and $u_2 > \sigma$. Hence the shock velocity is less than the velocity of the gas on both sides of the shock wave. Therefore, for 1-shock wave the particles move across the shock wave from left side to right side. In similar manner for 3-shock wave, $u_1 < \sigma < u_1 + c_1$ and $u_2 + c_2 < \sigma$ implying that $-c_1 < v_1 < 0$ and $v_2 < -c_1 < 0$. Hence for 3-shock wave, we obtain $\sigma > u_1$ and $\sigma > u_2$. Thus the velocity of gas on the both sides of the shock wave is less than the velocity of shock wave. Therefore, the particles move across a 3-shock wave from right side to left side. It is noticed that for 1-shock wave and 3-shock wave v_1 and v_2 are non-zero, therefore the quantity $n = \rho_1 v_1 = \rho_2 v_2 \neq 0$. Thus for 1-shock wave and 3-shock wave we obtain $v_1^2 > c_1^2$ and $v_2^2 < c_2^2$ respectively. With the help of Eq. (2.31) we have the following relation,

$$v_1^2 + e_1^2 + \frac{2}{\gamma(\gamma-1)} d_1^2 (1-\bar{b}_1)(\gamma - \bar{b}_1) = v_2^2 + e_2^2 + \frac{2}{\gamma(\gamma-1)} d_2^2 (1-\bar{b}_2)(\gamma - \bar{b}_2), \quad (2.32)$$

where $\bar{b}_1 = b\rho_1$ and $\bar{b}_2 = b\rho_2$.

Now utilizing the fact $v_1^2 > c_1^2$ and $v_2^2 > c_2^2$ in the above equation we obtain,

$$c_1^2 + e_1^2 + \frac{2}{\gamma(\gamma-1)} d_1^2 (1-\bar{b}_1)(\gamma - \bar{b}_1) < c_2^2 + e_2^2 + \frac{2}{\gamma(\gamma-1)} d_2^2 (1-\bar{b}_2)(\gamma - \bar{b}_2). \quad (2.33)$$

Putting $c = \sqrt{d^2 + e^2}$ in Eq. (2.33), we get $e_1^2 < e_2^2$ and $d_1^2 < d_2^2$ which implies that $c_1^2 < c_2^2$ thus $v_1^2 > v_2^2$. This gives $c_2 > c_1$ and $|v_1| > |v_2|$. Therefore from Eq.(2.29) we obtain for 1-shock wave $\rho_1 < \rho_2$ then $p_1 < p_2$ and $B_1 < B_2$. In similar manner, we can verify $\rho_1 > \rho_2$, $p_1 > p_2$ and $B_2 < B_1$ for 3-shock. Hence 1-shock and 3-shock families are compressive waves.

Now the one parameter family of shock waves are computed explicitly. Hence for the 1-shock wave, we assume the following constants,

$$\mu = \frac{p_2}{p_1}, \ \omega = \frac{\rho_2}{\rho_1}, \ \alpha = \frac{\gamma+1}{\gamma-1}. \quad (2.34)$$

The above expressions show that $\mu > 1$ and $\omega > 1$.

Using the relation $d^2 = \dfrac{\gamma p}{\rho(1-b\rho)}$, we get

$$\left[\frac{d_2}{d_1}\right]^2 = \frac{\mu}{\omega}\frac{(1-\bar{b}_1)}{(1-\bar{b}_2)}. \tag{2.35}$$

With the help of Eq. (2.29), we obtain

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{1}{\omega}. \tag{2.36}$$

By utilizing Eq. (2.34) and Eq. (2.36) in Eq. (2.30) we obtain

$$\left(\frac{v_1}{d_1}\right)^2 = \frac{\omega}{2(1-\omega)}\left[\frac{2(1-\mu)(1-\bar{b}_1)}{\gamma} + (\psi_1^2 - 1) - \frac{\mu(\psi_2^2-1)(1-\bar{b}_1)}{(1-\bar{b}_2)}\right]. \tag{2.37}$$

Here

$$\psi_1^2 = 1 + \frac{e_1^2}{d_1^2} \textit{ and } \psi_2^2 = 1 + \frac{e_2^2}{d_2^2}. \tag{2.38}$$

Also from Eq. (2.31) we have

$$\left(\frac{v_1}{d_1}\right)^2 = \frac{\omega}{(1-\omega^2)}\left[\frac{2(1-\bar{b}_1)}{\gamma(\gamma-1)}(\omega(\gamma-\bar{b}_1)\right] - \frac{\omega}{(1-\omega^2)}\left[\mu(\gamma-\bar{b}_2))\right] + \frac{\omega^2(\psi_1^2-1)}{(1-\omega^2)} - \frac{\omega}{(1-\omega^2)}\left[\frac{\mu(\psi_2^2-1)(1-\bar{b}_1)}{(1-\bar{b}_2)}\right]. \tag{2.39}$$

Now equating Eq. (2.37) and Eq. (2.39), we obtain

$$\omega = \frac{\Pi + \left(\dfrac{1-\bar{b}_1}{\gamma}\right)\left(\mu\alpha + 1 - \dfrac{2\mu\bar{b}_2}{\gamma-1}\right)}{\Pi + \left(\dfrac{1-\bar{b}_1}{\gamma}\right)\left(\mu + \alpha - \dfrac{2\bar{b}_1}{\gamma-1}\right)}, \tag{2.40}$$

where $\Pi = \frac{1}{2}(\psi_1^2-1) + \dfrac{\mu}{2}\dfrac{(1-\bar{b}_1)}{(1-\bar{b}_2)}(\psi_2^2-1)$.

From the above equation, we have $\omega < \alpha$ and therefore $1 < \omega$, hence we obtain the bounds for ρ_2 in terms of ρ_1 i.e., $\rho_1 < \rho_2 < \alpha\rho_1$.

Let $\zeta = \dfrac{2\bar{b}_1}{(\gamma-1)}$ and $\delta = \dfrac{2\bar{b}_2}{(\gamma-1)}$, using in Eq. (2.40) yields the following relation

$$\omega = \frac{\mathit{\Pi} + \left(\dfrac{1-\bar{b}_1}{\gamma}\right)(\mu(\alpha-\delta)+1)}{\mathit{\Pi} + \left(\dfrac{1-\bar{b}_1}{\gamma}\right)(\mu+\alpha-\zeta)}. \tag{2.41}$$

Now using Eq. (2.41) and the relation $v = u - \sigma$ in Eq. (2.40), we have

$$\frac{u_2-u_1}{d_1} = \pm\left(\frac{\omega-1}{\omega}\right)\sqrt{\frac{D\omega}{(1-\omega^2)}}, \tag{2.42}$$

where

$$D = \left(\frac{1-\bar{b}_1}{\gamma}\right)(\delta\mu-\zeta\omega) + (\omega-\mu)\left(\frac{2}{(\gamma-1)}-\zeta\right) - \frac{\mu(\psi_2^2-1)(1-\bar{b}_1)}{(1-\bar{b}_2)} + \omega(\psi_1^2-1).$$

The above equation shows the velocity variation across a shock transition. Here positive and negative sign denote for 1-shock and 3-shock wave respectively.

Now for more explicit formulation of shock curves, we construct a new parameter ξ which is determined as (Smoller [21]) where

$$\xi = -\log\mu. \tag{2.43}$$

From the equation Eq. (2.43) it is noticed that $e^{-\xi} = \mu = p_2/p_1 > 1$, therefore $\xi \leq 0$. By using this parameterization, we have more explicit formulation for shock curves which is given as

For 1-shock curve:

$$\frac{p_2}{p_1} = e^{-\xi}, \tag{2.44}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\psi_1^2-1)/2 + e^{-\xi}(1-\overline{b}_1)(\psi_2^2-1)/(2(1-\overline{b}_2)) + (1-\overline{b}_1)(e^{-\xi}(\alpha-\delta)+1)/\gamma}{(\psi_1^2-1)/2 + e^{-\xi}(1-\overline{b}_1)(\psi_2^2-1)/(2(1-\overline{b}_2)) + (1-\overline{b}_1)(e^{-\xi}+\alpha-\zeta)/\gamma}, \tag{2.45}$$

$$\frac{u_2-u_1}{d_1} = \left(\frac{\omega^*-1}{\omega^*}\right)\sqrt{\frac{D^*\omega^*}{(1-{\omega^*}^2)}}, \tag{2.46}$$

where

$$\omega^* = \left[\frac{(\psi_1^2-1)/2 + e^{-\xi}(1-\overline{b}_1)(\psi_2^2-1)/2(1-\overline{b}_2) + (1-\overline{b}_1)(e^{-\xi}(\alpha-\delta)+1)/\gamma}{(\psi_1^2-1)/2 + e^{-\xi}(1-\overline{b}_1)(\psi_2^2-1)/2(1-\overline{b}_2) + (1-\overline{b}_1)(e^{-\xi}+\alpha-\zeta)/\gamma}\right],$$

and

$D^* = [(1-\overline{b}_1)(\delta e^{-\xi} - \zeta\omega^*)/\gamma + 2(\omega^* - e^{-\xi})/(\gamma-1) - \zeta(\omega^* - e^{-\xi}) - e^{-\xi}(\psi_2^2-1)(1-\overline{b}_1)/(1-\overline{b}_2) + \omega^*(\psi_1^2-1)]$.

Similarly, for 3-shock curve:

$$\frac{p_1}{p_2} = e^{\xi}, \tag{2.47}$$

$$\frac{\rho_1}{\rho_2} = \frac{(\psi_1^2-1)/2 + e^{\xi}(1-\overline{b}_1)(\psi_2^2-1)/2(1-\overline{b}_2) + (1-\overline{b}_1)(e^{\xi}(\alpha-\delta)+1)/\gamma}{(\psi_1^2-1)/2 + e^{\xi}(1-\overline{b}_1)(\psi_2^2-1)/2(1-\overline{b}_2) + (1-\overline{b}_1)(e^{\xi}+\alpha-\zeta)/\gamma}, \tag{2.48}$$

$$\frac{u_1-u_2}{d_2} = \left(\frac{\tilde{\omega}-1}{\tilde{\omega}}\right)\sqrt{\frac{\tilde{D}\tilde{\omega}}{(1-\tilde{\omega}^2)}}, \tag{2.49}$$

where

$$\tilde{\omega} = \left[\frac{(\psi_1^2-1)/2 + e^{\xi}(1-\overline{b}_1)(\psi_2^2-1)/2(1-\overline{b}_2) + (1-\overline{b}_1)(e^{\xi}(\alpha-\delta)+1)/\gamma}{(\psi_1^2-1)/2 + e^{\xi}(1-\overline{b}_1)(\psi_2^2-1)/2(1-\overline{b}_2) + (1-\overline{b}_1)(e^{\xi}+\alpha-\zeta)/\gamma}\right],$$

and

$\tilde{D} = [(1-\overline{b}_1)(\delta e^{\xi} - \zeta\omega^*)/\gamma + 2(\omega^* - e^{\xi})/(\gamma-1) - \zeta(\omega^* - e^{\xi}) - e^{\xi}(\psi_2^2-1)(1-\overline{b}_1)/(1-\overline{b}_2) + \omega^*(\psi_1^2-1)]$.

2.5 Simple wave

In one-dimensional hyperbolic system of partial differential equations a simple wave is a centered rarefaction wave. In case of simple wave, all the dependent variables are constant along the characteristics and these characteristics are straight lines. For rarefaction wave, the constant states U_1^* and U_2^* are joined through a smooth transition in k^{th} genuinely nonlinear characteristic field and agrees with the conditions given as:

1. Across the wave, the Riemann invariants are constant [108].
2. The left and right characteristics of wave diverge i.e. $\lambda_k\left(U_1^*\right) < \lambda_k\left(U_2^*\right)$, $k = 1, k = 3$.

Now we determine the simple wave curve. Here we determine only 1-simple wave and for 3-simple wave, the details are similar. Therefore in case of 1-simple wave, Riemann invariants are constant. Hence we obtain

$$S_2 = S_1, \tag{2.50}$$

and

$$u_2 + \frac{2}{(\gamma-1)}\psi_2 d_2(1-\bar{b}_2)^{1/2} = u_1 + \frac{2}{(\gamma-1)}\psi_1 d_2(1-\bar{b}_1)^{1/2}. \tag{2.51}$$

From Eq. (2.6) we have

$$\frac{p_2}{p_1} = \left(\frac{d_2(1-\bar{b}_2)}{d_1(1-\bar{b}_1)}\right)^{\dfrac{2\gamma}{\gamma-1}} = \left(\frac{\rho_2(1-\bar{b}_2)}{\rho_1(1-\bar{b}_1)}\right)^{\gamma}. \tag{2.52}$$

Hence from Eq. (2.51), we obtain

$$\frac{u_2 - u_1}{d_1} = \frac{2\psi_1}{(\gamma - 1)} \left[(1 - \bar{b}_1)^{1/2} - \frac{\psi_2}{\psi_1} \frac{d_2}{d_1} (1 - \bar{b}_2)^{1/2} \right]. \tag{2.53}$$

In case of 1-rarefaction wave, the characteristic speed $\lambda_1 = u - c$ increases therefore $\lambda_1^{(2)} \geq \lambda_1^{(1)}$ which yields $u_2 - u_1 \geq \psi_2 d_2 - \psi_1 d_1$.

Hence with the help of Eq. (2.51) we obtain

$$\frac{\psi_2 d_2 - \psi_1 d_1}{d_1} \leq \frac{2\psi_1}{(\gamma - 1)} [(1 - \bar{b}_1)^{1/2} - \frac{\psi_2}{\psi_1} \frac{d_2}{d_1} (1 - \bar{b}_2)^{1/2}].$$

By utilizing the above expression in Eq.(2.52), we obtain

$$0 < \frac{p_2}{p_1} \leq 1. \tag{2.54}$$

Hence from Eq. (2.43), we get

$$\xi = -\log \mu. \tag{2.55}$$

From above equation it is noticed that $e^{-\xi} = \mu = p_2/p_1 < 1$, therefore $\xi \geq 0$. Hence we can derive more explicit formulation for simple waves. Therefore, Eq. (2.52) and Eq. (2.53) is rewritten as

For 1-simple wave:

$$\frac{p_2}{p_1} = e^{-\xi}, \tag{2.56}$$

$$\frac{\rho_2}{\rho_1} = \left[\frac{1 - \bar{b}_2}{1 - \bar{b}_1} \right] e^{\frac{-\xi}{\gamma}}, \tag{2.57}$$

$$\frac{u_2 - u_1}{d_1} = \frac{2\psi_1}{(\gamma - 1)} (1 - \bar{b}_2)^{1/2} \left[1 - \frac{\psi_2}{\psi_1} e^{-\psi\beta} \frac{(1 - \bar{b}_1)^{1/2}}{(1 - \bar{b}_2)^{1/2}} \right]. \tag{2.58}$$

Similarly for 3-simple waves:

$$\frac{p_1}{p_2} = e^{\xi}, \tag{2.59}$$

$$\frac{\rho_1}{\rho_2} = \left[\frac{1-\bar{b}_2}{1-\bar{b}_1}\right] e^{\frac{\xi}{\gamma}}, \tag{2.60}$$

$$\frac{u_1 - u_2}{d_2} = \frac{2\psi_1}{(\gamma-1)}(1-\bar{b}_2)^{1/2}\left[1 - \frac{\psi_2}{\psi_1}e^{\psi\beta}\frac{(1-\bar{b}_1)^{1/2}}{(1-\bar{b}_2)^{1/2}}\right]. \tag{2.61}$$

2.6 Contact discontinuities

Contact discontinuities are the contact surface which move with the gas and divide two zones of different densities(and temperatures); but the pressure and flow velocity are the same on both sides. A contact discontinuity may separate not only parts of the same gas but also two different gases. For contact discontinuities, in the case of second characteristic field which is linearly degenerate, the constant states U_1^* and U_2^* are combined through a single jump discontinuity with speed σ_2 and satisfying the following conditions:

1. The R-H jump conditions i.e. $F(U_2^*) - F(U_1^*) = \sigma_2(U_2^* - U_1^*)$.

2. The parallel characteristic conditions $\lambda_2(U_2^*) = \lambda_2(U_1^*) = \sigma_2$.

Therefore, in the case of 1-parameter family of contact discontinuities, we have the following formulation

$$\frac{p_2}{p_1} = 1, \tag{2.62}$$

$$\frac{\rho_2}{\rho_1} = e^{\xi}, -\infty < \xi < \infty, \tag{2.63}$$

$$u_2 - u_1 = 0. \tag{2.64}$$

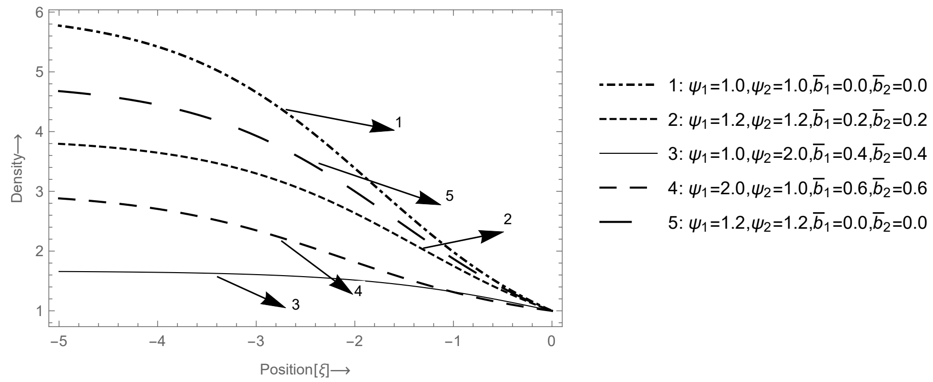

FIGURE 2.2: Density profiles for compressive waves: 1-shock.

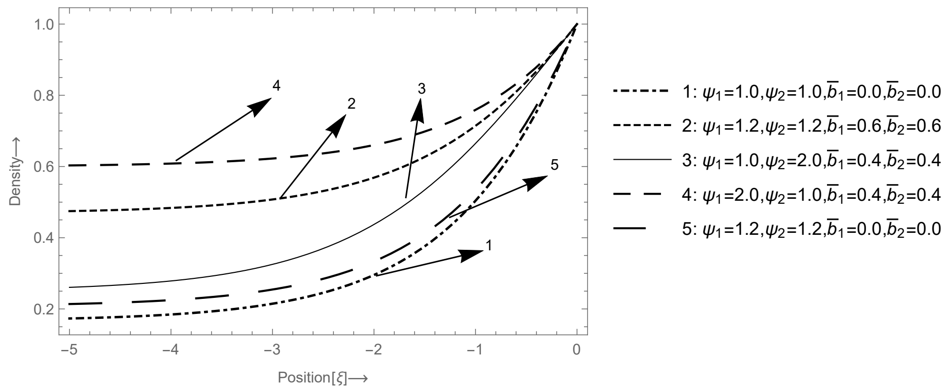

FIGURE 2.3: Density profiles for compressive waves: 3-shock.

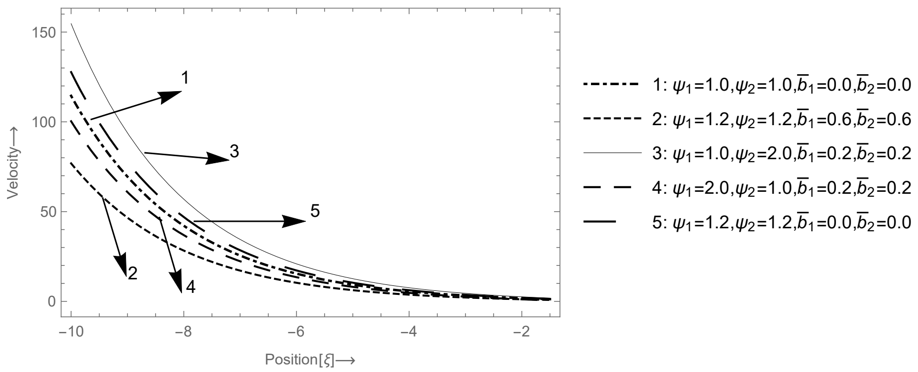

FIGURE 2.4: Velocity profiles for compressive waves: 1-shock.

2.7 Result and discussion

The analytical solution of the Riemann problem for non-ideal polytropic gas with added effect of transverse magnetic field is obtained here. The case when $\bar{b}_1 = 0$, $\bar{b}_2 = 0$ corresponds to ideal polytropic gas with magnetic field and results are in close

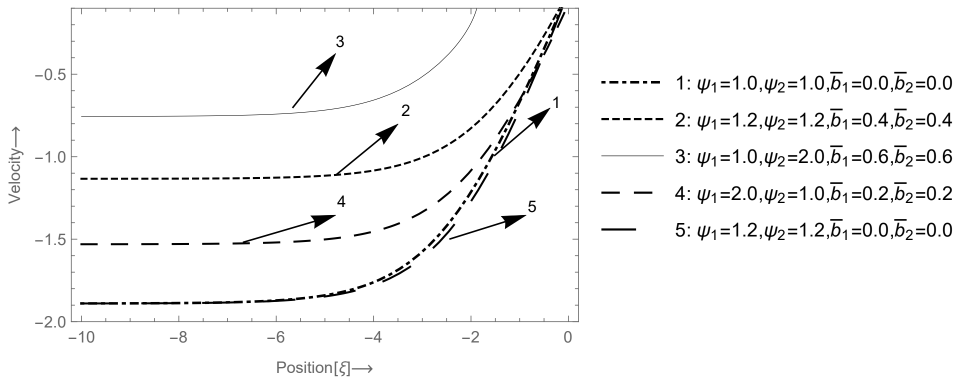

FIGURE 2.5: Velocity profiles for compressive waves: 3-shock.

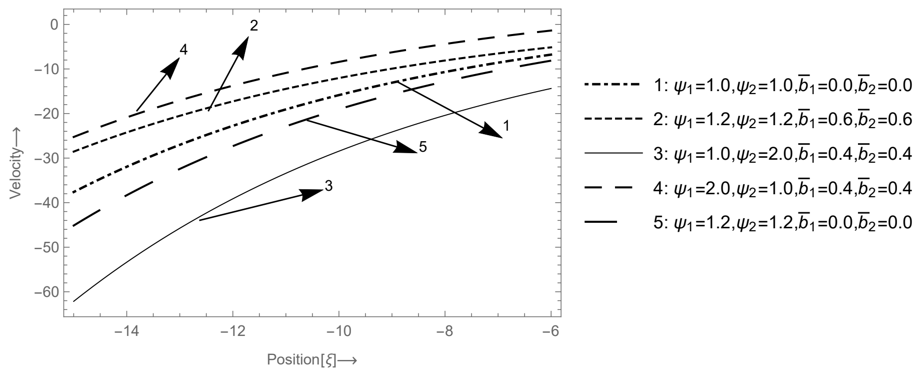

FIGURE 2.6: Velocity profiles for rarefaction waves: 1-shock.

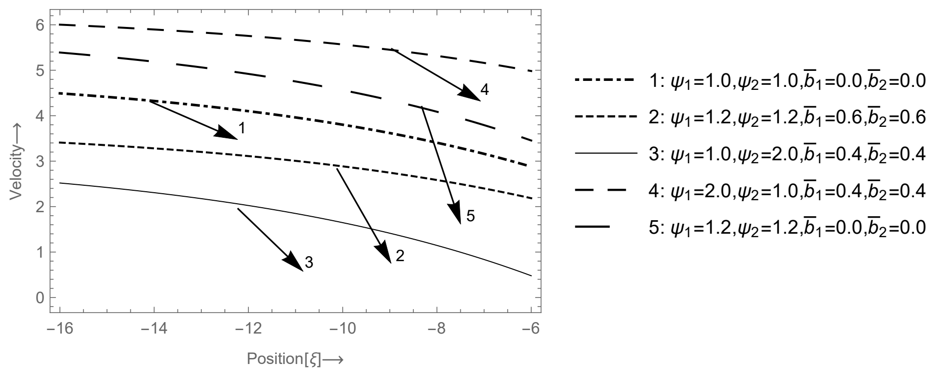

FIGURE 2.7: Velocity profiles for rarefaction waves: 3-shock.

agreement with Singh et al. [100]. Further, the case $\psi_1 = 1.0$, $\psi_2 = 1.0$, $\overline{b}_1 = 0$, $\overline{b}_2 = 0$ corresponds to ideal gas dynamics case and the results are in close agreement as reported in Smoller [21]. Computed values are presented in Fig.[2-7] for different values of parameter of non-idealness and magnetic field strength. MATHEMATICA 11.1 is used for all computations in the chapter. Here the effect of magnetic field strength enters into the solution through the parameters ψ_1 and ψ_2. The solution curves 1 and 5 in all figures correspond to the ideal gas flow and magnetogasdynamic flow respectively. The density versus position(ξ) profiles for 1-shock and 3-shock for compressive wave is presented in Fig.2.2 and Fig.2.3. It is important to note that the density profile for 1-shock of compressive wave is concave downward and for 3-shock of compressive wave is concave upward respectively. From Fig.2.2, we infer that the effect of increasing values of magnetic field strength is to decrease the density. Also the effect of increasing value of non-ideal gas parameter in the presence of magnetic field strength, both ψ_1 and ψ_2, is to further decrease the density for 1-shock wave. From Fig.2.3, we find that the solution curves have opposite trend in comparison to the results obtained in Fig.2.2. The velocity versus position(ξ) profile for 1-shock and 3-shock for compressive waves is shown in Fig.2.4 and Fig.2.5 respectively. It is noted here that the velocity profile for 1-shock and 3-shock for compressive wave is concave upward. From Fig.2.4, It is observed that the effect of an increase in the magnetic field strength parameters ψ_1 and ψ_2 have reverse effect on the velocity profiles for 1-shock wave. Also the effect of non-ideal gas parameter is to further enhance the effect of magnetic field strength on the velocity profiles. Also from Fig.2.5, we observe that all the results have opposite trend as compared to the results obtained in Fig.2.4. The velocity versus position(ξ) profiles for 1-shock and 3-shock for rarefaction wave is presented by Fig.2.6 and Fig.2.7 respectively. Here, we note that the velocity profile for both 1-shock and 3-shock of rarefaction wave is concave downward. From Fig.2.6 and Fig.2.7, we infer that the effect of magnetic field strength parameters ψ_1 and ψ_2 on the velocity profile have reverse effect. Also the effect of non-ideal gas parameter is to further enhance the effect of magnetic field strength.

2.8 Conclusion

In the present chapter, the analytical solution of the Riemann problem for non-ideal polytropic gas with an added effect of transverse magnetic field is presented. The density and velocity profiles for compressive waves, 1-shock and 3-shock and velocity profiles for rarefaction waves, 1-shock and 3-shock is presented. It is observed that the effect of magnetic field strength is to decrease the density for 1-shock and have increasing effect for 3-shock . Also the effect of magnetic field strength parameters, ψ_1 and ψ_2 have opposite effect on the velocity profiles for both compressive wave and rarefaction waves, 1-shock and 3-shock, which is in agreement with the earlier results reported by [100]. Also the effect of presence of the parameters of non-idealness is to further enhance the effect of magnetic field strength.

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