Chapter 1

Introduction

"Mathematicians stand on each other's shoulders".

-Carl Friedrich Gauss

1.1 Background

Any mathematical model of continuum is given by a system of partial differential equations (PDEs). In continuum mechanics, the conservation laws of mass, momentum and energy form a common starting point, and each medium is then characterized by its constitutive laws. The conservation laws and constitutive equations for the field variables, under quite natural assumptions, reduce to field equations, i.e., partial differential equations, which, in general, are nonlinear and nonhomogeneous. For nonlinear problems, neither the methods of their solutions nor the main characteristics of the motion are as well understood as in the linear theory.

1.1.1 Nonlinear Waves and Hyperbolic Equations

A wave is any recognizable signal that is transferred from one part of the medium to another with a recognizable velocity of propagation. Waves occur in most scientific and engineering disciplines, for example: fluid mechanics, optics, electromagnetism, solid mechanics, structural mechanics, quantum mechanics, etc. The waves for all these applications are described by solutions to PDEs.

The most important classification criterion is to distinguish PDEs as linear or nonlinear. Roughly, a homogeneous PDE is linear if the superposition principle for the solutions of PDEs holds, otherwise it is nonlinear. The division of PDEs into these two categories is significant. The mathematical methods devised to deal with these two classes of equations are often entirely different, and the qualitative behavior of solutions differ substantially.

One underlying cause is the fact that the solution space to a linear, homogeneous PDE is a vector space, and the linear structure of that space can be used with advantage in constructing solutions with desired properties that can meet diverse boundary and initial conditions. Such is not the case for nonlinear equations. It is easy to find examples where nonlinear PDEs exhibit behavior with no linear counterpart.

One is the breakdown of solutions and the formation of singularities; such as shock waves. A shock wave is a surface of discontinuity propagating in a gas at which density and velocity experience abrupt change. One can imagine two types of shock waves: (positive) compression shocks which propagate into the direction where the density of the gas is minimum, and (negative) rarefaction waves which propagate into the direction of maximum density.

A second is the existence of solitions, which are solutions to nonlinear dispersion equations. These solitary wave solutions maintain their shapes through collisions, in much the same was as linear equations do, even though the interactions are not linear.

In hyperbolic systems, the nonlinearity brings about progressively more and more deformation in the wave profile. As a result the profile breaks down after a finite span of time, and the smooth solution ceases to be valid beyond this point due to the blow up of its derivatives. Further, the solution, admissible for all times, belongs to a class of discontinuous functions, which brings the notion of weak solutions, but with a difficulty that the uniqueness of the solution is lost. It is then required to select a unique and physically meaningful solution by using some admissibility criterion.

Most of the physical problems, arising in gasdynamics, lead to the formulation of a quasilinear system of first order partial differential equations. These equations are linear in the first derivative of dependent variables, but the coeffcients may be functions of dependent variables. Let us consider first order partial differential equation of the form

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{m} a_{ij}(x, t, u_1, u_2, ..., u_m)\frac{\partial u_j}{\partial x} + b_i(x, t, u_1, u_2, ..., u_m) = 0 \tag{1.1}$$

for $i = 1, ..., m$. This is a system of m equations in m unknowns u_i that depend on space x and a time–like variable t. Here u_i are the dependent variables and x, t are the independent variables. We also make use of subscripts to denote partial derivatives. System (1.1) can also be written in matrix form as

$$U_t + AU_x + B = 0, \tag{1.2}$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_m \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}, A = \begin{bmatrix} a_{11} & . & . & a_{1m} \\ a_{21} & . & . & a_{2m} \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & . & . & a_{mm} \end{bmatrix}. \tag{1.3}$$

If the entries a_{ij} of the matrix A are all constant and the components b_i of the vector B are also constant then system (1.2) is linear with constant coefficients. If $a_{ij} = a_{ij}(x,t)$ and $b_i = b_i(x,t)$, the system is linear with variable coefficients. The system is still linear if B depends linearly on U and is called quasi-linear if the coefficient matrix A is a function of the vector U, that is A=$A(U)$. Note that quasi-linear systems are in general system of non-linear equations. System (1.2) is called homogeneous if $B = 0$. A system (1.2) is said to be hyperbolic at a point (x,t) if A has m real eigenvalues λ_1,λ_2,.....,λ_m and a corresponding set of m linearly independent eigenvectors K_1,K_2,.....,K_m. The system is said to be strictly hyperbolic if the eigenvalues λ_i are all distinct [1].

1.1.2 Ideal and Non-Ideal Gas

An ideal gas is a theoretical gas composed of many randomly moving point particles that are not subject to interparticle interactions. Under various conditions of temperature and pressure, many real gases behave qualitatively like an ideal gas where the gas molecules (or atoms for monatomic gas) play the role of the ideal particles. The ideal gas model tends to fail at lower temperatures or higher pressures, when intermolecular forces and molecular size becomes important. It also fails for most heavy gases, such as many refrigerants and for gases with strong intermolecular forces, notably water vapour. At high pressures, the volume of a real gas is often considerably larger than that of an ideal gas. At low temperatures, the pressure of a real gas is often considerably less than that of an ideal gas. At some point of low temperature and high pressure, real gases undergo a phase transition, such as to a liquid or a solid. The model of an ideal gas, however, does not describe or allow phase transitions. These must be modeled by more complex equations of state. However, if the temperature of the gas is very high and density is too low then the hypothesis that the gas is ideal is no longer valid. Then there is no choice but to relax the assumptions of ideal gas.

The equation of state of an ideal gas is written as:

$$PV = nRT,$$

where n is the number of molecules of the gas, R is the gas constant, T is the absolute temperature, P is the pressure and V is the volume of the gas.

The approximation to the properties of real matter is represented by an equation given by Dutch Physicist van der Waals. In spite of its simplicity, it comprehends both the gaseous and the liquid state and brings out, in a most remarkable way, all the phenomena pertaining to the continuity of these two states.

This equation has the form

$$P = \frac{RT}{V-b} - \frac{a}{V^2},$$

where a and b are two numerically small constants and V is the molal volume. If we consider a gas with molecules of finite size, mutually exclusive as to their extension but not interacting in any other way, the difference is that the centers of the molecules cannot spread out in the whole volume of the gas but only in that part of it which is not occupied by other molecules and not immediately adjacent to them. In the first approximation, for the molal volume V there must be substituted the covolume, $V - b$, where the constant b is proportional to the sum of the volumes of all molecules in one mol of the gas.

Therefore, the equation of state becomes,

$$P = \frac{RT}{V - b} \ ,$$

i.e. covolume equation of state. However, if the molecules of the gas do interact at a distance, say, attract one another, then the internal pressure due to this attraction must be taken into account. This means that, when the density of the gas in a given vessel is changed by adding more gas or subtracting it, all the internal forces change in the ratio $\frac{1}{V^2}$. Since the pressure is defined as the force per unit area, this applies also to the internal pressure and we obtain for it the expression $\frac{a}{V^2}$ which is added to P in Van der Waals' equation.

Roberts and Wu [2] obtained the similarity solutions and determined the stability conditions of a spherical shock wave for both ideal and Van der Waals gases. Roberts and Somogyi [3] analyzed the stability of an imploding spherical shock wave in Van der Waals gas. Sharma and Pandey [4] studied the evolutionary behaviour of an unsteady three-dimensional motion of a shock wave of arbitrary strength propagating through a non-ideal gas.

1.1.3 Dusty Gas

The study of a two-phase flow of gas and dust particles has been of great interest because of many applications to different engineering problems. Gas flows, which carry an appreciable amount of solid particles, may exhibit significant relaxation effects as a result of particles being unable to follow rapid changes of the velocity and temperature of the gas. When the mass concentration of the particles is comparable with that of the gas, the flow properties become significantly different from that of a pure gas. Here, we consider a mixture of a perfect gas and a large number of small dust particles of uniform spherical shape.

Dusty gas is considered to be mixture of gas and small solid dust particles where these dust particles attain less than five percent of total volume [5]. At very high speed of fluid, these small solid particles behave as a pseudo fluid [6]. We consider the mixture as the mixture of two fluids: one is gas and the other is the pseudo fluid of solid particles. The solid particles are spheres of identical mass m_{sp}, radius r_{sp} and specific heat c_{sp}. We consider an element of mixture of gas and solid particles (dusty gas) with total mass $M = M_g + M_{sp}$ and with total volume $V = V_g + V_{sp}$, where subscript g refers to the value for the gas and subscript sp refers to that of the solid particles. The volume of solid particles in mixture is obtained as:

$$V_{sp} = n_{sp}.V.\tau_{sp},$$

where τ_{sp} and n_{sp} is the volume of a solid particle and the number of solid dust particles per unit volume of dusty gas respectively.

The mass of solid particles in the volume V of the mixture is written as:

$$M_{sp} = n_{sp}.V.m_{sp}.$$

The species density of the solid particles is defined as:

$$\rho_{sp} = \frac{M_{sp}}{V_{sp}} = \frac{m_{sp}}{\tau_{sp}}.$$

Also, The partial density of the pseudo-fluid of solid particles is defined as:

$$\overline{\rho}_{sp} = \frac{M_{sp}}{V_{sp}} = n_{sp}.m_{sp} = Z\rho_{sp} = n_{sp}.\rho_{sp}.\tau_{sp},$$

where Z represents the volume fraction of solid particles in the mixture. Further, volume fraction of solid particles is given as:

$$Z = \frac{V_{sp}}{V} = n_{sp}.\tau_{sp}.$$

The species density of the gas is defined as:

$$\rho_g = \frac{M_g}{V_g}.$$

Similarly, the partial density of a gas is defined as:

$$\overline{\rho}_g = \frac{M_g}{V} = (1 - Z)\rho_g.$$

Let us consider the thermodynamic equilibrium condition such as:

$$T_{sp} = T_g = T.$$

The density of the mixture is obtained as

$$\rho = Z\rho_{sp} + (1 - Z)\rho_g = \overline{\rho}_{sp} + \overline{\rho}_g.$$

The mass concentration of the pseudo fluid of the solid particles is obtained as:

$$k_p = \frac{\overline{\rho}_{sp}}{\rho} = \frac{Z\rho_{sp}}{\rho}.$$

The pressure of the mixture is written as:

$$p = p_{sp} + p_g.$$

The total pressure of the mixture is p which is obtained from the perfect gas law as:

$$p = R\rho_g T_g.$$

With the help of above analysis, the pressure of the mixture as a whole is obtained as:

$$p_m = p = R\rho_g T_g = R\left(\frac{\rho_m - Z\rho_{sp}}{1-Z}\right)T_g = R\rho_m\left(\frac{1-k_{sp}}{1-Z}\right)T.$$

Therefore,

$$p_m = \frac{\rho_m R_m T}{1-Z},$$

where $R_m = (1-k_p)R$. Here, R may be considered as an effective gas constant of the mixture and subscript m refers to the value of the gas constant in the mixture as a whole.

1.1.4 Radiating Gas

In the present space age, there are many technological developments of interest, for example hypersonic flight, gas-cooled nuclear reactors, power plants for space exploration needs, fission and fusion reactions in which the temperature is very high and the density is rather low. As a result, thermal radiation becomes an important mode of heat transfer. A complete analysis of such a high-temperature flow field should be based upon a study of the gasdynamic field and the thermal radiation field simultaneously. We use the term "Radiative gasdynamics" for such a new branch of fluid dynamics.

In radiative gasdynamics, the basic equations governing the flow form a system of coupled integro-differential equations of considerable complexity. The consequence of these complexities has been to stimulate a search for approximate formulation of the equation of radiative transfer which leads merely to a system of nonlinear differential equations. In radiative gasdynamics, the basic equations governing the flow form a system of coupled integro-differential equations of considerable complexity. The consequence of these complexities has been to stimulate a search for approximate formulation of the equation of radiative transfer which leads merely to a system of nonlinear differential equations.

Thermal radiation may be considered as either a stream of photons or as electromagnetic waves Pai [7]. Here we shall consider the thermal radiation as electromagnetic waves.

The heat rays may be specified by a specific intensity I_ν which is defined as follows:

$$I_\nu = \lim_{d\sigma_0, d\omega, dt, d\nu \to 0} \left(\frac{dE_\nu}{d\sigma_0\, cos\theta\, d\omega\, dt\, d\nu} \right), \tag{1.4}$$

where I_ν is a function of time t, spatial coordinates, direction θ the angle between the direction of heat rays and the normal of area $d\sigma_0$, and the frequency of wave ν. The amount of radiative energy flowing through the area $d\sigma_0$, in the frequency range ν and $\nu + d\nu$ in the direction of heat ray L, which makes an angle θ with the normal of $d\sigma_0$ within a solid angle $d\omega$ in the time interval dt is dE_ν. One can calculate the effects of thermal radiation on the flow field, which are specified as follows:

1. The flux q_R^i of heat energy by thermal radiation defined as

$$q_R^i = \int_{4\pi} \int_0^\infty I_\nu n^i \, d\nu \, d\omega, \tag{1.5}$$

   where n^i is the directional cosine of the radiation ray with respect to the $i - th$ axis. One should add the divergence of this radiation heat flux in the energy equation of gasdynamics, i.e.

$$Q_R = \frac{q_R^i}{x_i}, \tag{1.6}$$

where the summation convention is used. In general, Q_R is a differential-integral expression and the energy equation in radiation gasdynamics is an integro-differential equation.

2. The radiation energy density E_R: The radiation energy density within the frequency range ν to $\nu + d\nu$ is

$$U_\nu = \frac{1}{c} \int_{4\pi} I_\nu \, d\omega, \tag{1.7}$$

where c is the velocity of light. The radiation energy density for the whole spectrum is then

$$E_R = \int_0^\infty U_\nu \, d\nu = \frac{1}{c} \int_{4\pi} \int_0^\infty I_\nu \, d\nu \, d\omega. \tag{1.8}$$

This radiation energy should be added to the total energy of a gas.

3. Radiation stress tensor τ_R^{ij}: The ij -th component of the radiation stress tensor is

$$\tau_R^{ij} = \frac{-1}{c} \int_{4\pi} \int_0^\infty I_\nu n^i n^j \, d\nu \, d\omega. \tag{1.9}$$

The radiation pressure p_R may be defined as

$$\begin{aligned} p_R =& (-1/3)(\tau_R^{11} + \tau_R^{22} + \tau_R^{33}) \\ =& (1/3c) \int_{4\pi} \int_0^\infty I_\nu \, d\nu \, d\omega = (1/3) E_R. \end{aligned} \tag{1.10}$$

It may be noted here that the radiation pressure p_R and the radiation energy density E_R are of the same order of magnitude.

The equation of radiative transfer can be written as [7]:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + n^i \frac{\partial I_\nu}{\partial x_1} = \rho k_\nu (J_\nu - I_\nu), \tag{1.11}$$

where k_ν is the absorption coefficient of radiation, and

$$J_\nu = \frac{j_\nu}{k_\nu}, \tag{1.12}$$

is the source function of radiation with j_ν as the emission coefficient of radiation. When the gas is in local thermodynamic equilibrium, i.e., the emission determined by the local temperature; equation (1.11) becomes

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = \alpha_\nu (B_\nu - I_\nu), \tag{1.13}$$

where $\alpha_\nu = \rho k'_\nu$ is the volumetric absorption coefficient with

$$k'_\nu = k_\nu [1 - \exp(-h\nu/kT)]. \tag{1.14}$$

Here h and k are respectively the Plank constant and Boltzmann constant, B_ν is the Plank radiation function defined as

$$B_\nu = (2h\nu^3/c^2)[\exp(h\nu/kT) - 1], \tag{1.15}$$

and s is the distance along a radiation ray which has the direction n^i with respect to i-th axes.

As the velocity of light c is a very large quantity, the unsteady term $\frac{1}{c}\frac{\partial I_\nu}{\partial t}$ in equation (1.13) is very small and may be neglected. Then the radiation term Q_R appearing in energy equation given by equation (1.6) may be written as

$$
\begin{aligned}
q_R = \frac{\partial q_R^i}{\partial x_i} &= \int_0^\infty \int_0^{4\pi} n^i \frac{\partial I_\nu}{\partial x_i} \, d\Omega \, d\nu \\
&= \int_0^\infty \int_0^{4\pi} \frac{\partial I_\nu}{\partial x_i} \, d\Omega \, d\nu.
\end{aligned} \tag{1.16}
$$

In view of the radiative transfer equation, this can be written as

$$
\frac{\partial q_R^i}{\partial x_i} = - \int_0^\infty \alpha_\nu \int_0^{4\pi} (I_\nu d\Omega - 4\pi B_\nu) \, d\nu. \tag{1.17}
$$

The two terms on the right hand side of equation (1.17) account for the energy addition to the gas.

Now, the following important approximation are considered which occur in most of the physical problems:

1. When the mean free path of radiation is large: In this case the gas radiation interaction is described as being emission dominated and the inequality $I_\nu \ll B_\nu$ holds.

   This corresponds to an optically thin gas. In this approximation equation (1.17) reduces to [7, 8]:

$$
\frac{\partial q_R^i}{\partial x_i} = 4\alpha_P \sigma T^4, \tag{1.18}
$$

   where σ is the Stefan-Boltzmann constant, and

$$
\alpha_P = \frac{\pi}{\sigma T^4} \int_0^\infty \alpha_\nu B_\nu \, d\nu, \tag{1.19}
$$

   is the mean absorption coefficient.

2. When mean free path of radiation is small: In this case I_ν differs by only a small amount from B_ν. For this approximation gas is said to be optically thick,

and the radiation flux vector q_R^i may be written as

$$q_R^i = -(16\sigma T^3/3\alpha_R)\left(\frac{\partial T}{\partial x_i}\right), \tag{1.20}$$

where α_R is the Rosseland mean absorption coefficient defined as

$$\frac{1}{\alpha_R} = \frac{\pi}{4\sigma T^3}\int_0^\infty \frac{1}{\alpha_\nu}\frac{dB_\nu}{dt}\, d\nu. \tag{1.21}$$

The above approximation is valid only when (i) $E_R = 3p_R = \sigma T^4$, (ii) all the shearing stresses of radiation vanish , and (iii) radiation heat flux is given by equation (1.20) [9]. The investigation of high-temperature flow fields, such as those in gas-cooled nuclear reactors, hypersonic flights, fission and fusion reactions, and power plants for space researches, will be based on the analysis of radiation and gas-dynamical fields simultaneously.

1.1.5 Magnetogasdynamics

Magnetogasdynamics is also an important example of the hyperbolic system's theory. The governing system of magnetogasdynamics is highly non-linear and complicated, it is necessary to study various simplified models, in which the magnetic field and velocity field are orthogonal everywhere. Magnetic fields permeating the universe plays a crucial role in a number of astrophysical situations and probably affect all astrophysical plasmas. Many interesting astrophysical and aerodynamical problems involve magnetic fields, and shock waves in those fields have various industrial applications. Cylindrical shock waves could be originated in the processes where a large amount of energy is liberated instantly.

Magnetic fields play an important role in energy and momentum transport and can rapidly release energy in flares. Many interesting problems involve magnetic fields. The shock waves in the presence of a magnetic field in conducting perfect gas can be important for description of shocks in supernova explosion and explosion in the ionosphere. Complex filamentary structures in molecular clouds, shapes and the shaping of planetary nebulae, synchrotron radiation from supernova remnants, magnetized stellar winds, galactic winds, gamma-ray bursts, dynamo effects in stars, galaxies, and galaxy clusters as well as other interesting problems all involve magnetic fields. The industrial applications are drag reduction in duct flows, design of efficient coolant blankets in tokamak fusion reactors, control of turbulence of immersed jets in the steel casting process and advanced propulsion and flow control schemes for hypersonic vehicles, involving applied external magnetic fields (see Hartmann [10], Balick and Frank [11]).

The equations governing the motion of unsteady flow of perfectly conducting fluid in the absence of thermal conduction and viscosity may be written as [12],

$$\frac{\partial \rho}{\partial t} + div(\rho v) = 0, \tag{1.22}$$

$$\frac{\partial u}{\partial t} + (v.\bigtriangledown)v = \frac{-1}{\rho} \bigtriangledown p + (\bigtriangledown \times H) \times B, \tag{1.23}$$

$$\frac{\partial p}{\partial t} + v. \bigtriangledown p + \rho C^2. \bigtriangledown v = 0, \tag{1.24}$$

$$\frac{\partial B}{\partial t} = curl(v \times B), \tag{1.25}$$

$$div(B) = 0, \tag{1.26}$$

where ρ is the fluid density, p the pressure, c the speed of sound, $u = (u_1, u_2, u_3)$ the velocity vector and B the magnetic induction satisfying the relation $B = \mu H$ with μ being the magnetic permeability, assumed to be constant, and $H = (H_1, H_2, H_3)$ being the magnetic field vector. For a perfect fluid, $c = \sqrt{\frac{\gamma p}{\rho}}$ with a constant ratio of the specific heat capacities $\gamma = c_p/c_v$.

Here, we consider the one-dimensional motion with plane waves, which is encountered very frequently in problems of magnetohydrodynamics. In a planar flow, the trajectories of the particles form a family of straight lines perpendicular to some fixed plane. If we choose the x-axis perpendicular to the plane, then the velocity vector will have only one non-zero component, that is, $u = (u(x,t), 0, 0)$, while $p = p(x,t)$ and $\rho = \rho(x,t)$. We now envisage a one-dimensional planar motion of plasma, which is assumed to be an ideal gas with infinite electrical conductivity and to be permeated by a magnetic field $H = (0, H(x,t), 0)$ orthogonal to the trajectories of the gas particles.

It is noticed here that Equation (1.25) is identical to the equation for vortex velocity in the hydrodynamics of a non-viscous fluid that implies that the magnetic field varies as if the magnetic force lines were rigidly coupled with the matter; on using Equations (1.22) and (1.26), Equation (1.25) can be written as

$$\frac{d}{dt}\left(\frac{B}{\rho}\right) = \left(\frac{B}{\rho}.\bigtriangledown\right)u, \tag{1.27}$$

where the total derivative signifies the rate of change in a given fluid particle as it moves about. If δL is an element of a 'fluid line' which moves with the fluid, then during a time interval dt, the rate of change of δL and (B/ρ) is given by identical formulae. Hence, it follows that for a planar flow with a transverse magnetic field, (B/ρ) is a constant that can be determined from the initial conditions; in other words, every line of force moves with the fluid particles which lie on it.

1.1.6 Riemann Problem

The Riemann problem consists of an initial value problem composed of the Euler equations together with piecewise constant initial data having single contact discontinuity. In the solution of the Riemann problem, all the features such as rarefractions waves, shock waves occur in the form of characteristics hence it is very convenient for the readers to understand the Euler equations in the form of conservation laws. The solution of the Riemann problem consists of three waves with middle one is always contact discontinuity and other remaining waves are either rarefraction waves or shock waves.

The Riemann problem for the one-dimensional time-dependent Euler equations is the Initial Value Problem for the conservation laws

$$\frac{\partial V}{\partial t} + \frac{\partial F(V)}{\partial x} = 0, \tag{1.28}$$

where, $V = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{bmatrix}.$

The initial conditions of the Riemann problem for the system of equations (1.28) is given by

$$V(x,0) = V_0(x) = \begin{cases} V_l = (\rho_l, v_l, p_l), & \text{if } x < 0 \\ V_r = (\rho_r, v_r, p_r), & \text{if } x > 0 \end{cases}. \tag{1.29}$$

Here, V_l and V_r denote the left and right constant state respectively which is separated by the jump discontinuity at $x = 0$.

1.1.7 Shock wave and Rankine-Hugoniot condition

Gasdynamic shocks form each time a high velocity supersonic flow is stopped by an obstacle. One of the most important distinction between space shocks and ordinary shocks in gases is that the medium space shocks form in is a plasma ionized gas. Another very significant property of these shocks is that the plasma almost always is embedded in an ambient magnetic field, which is compressed by the shock. Therefore, these shocks form whenever the plasma flow velocity exceeds the corresponding signal velocity in the magnetized plasma.
Shocks are ubiquitous phenomena in space. Many different kinds of shocks have been observed in the solar system. Planetary bow shocks are formed when the solar wind encounters a planetary magnetosphere, whether natural (as on Earth, Jupiter, and Saturn, which have their own magnetic field) or induced (as on Mars and Venus). Cometary shocks are produced by the interaction of the solar wind with charged particles of cometary origin. Interplanetary shocks appear whenever fast solar wind overtakes slow wind. Another feature of these shocks that distinguishes them from gasdynamic shocks is that the mean free path for Coulomb collisions in the system is much larger than the system size itself.
We consider the integral conservation law

$$\frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t), \tag{1.30}$$

where u is the density and ϕ is the flux. Equation (1.30) states that the time rate of change of the total amount of u inside the interval [a b] must equal the rate that u flows into [a b] minus the rate that u flows out of [a b]. Under suitable smoothness assumptions (e.g., both u and ϕ continuously differentiable).
Equation (1.30) implies

$$u_t + \phi_x = 0, \tag{1.31}$$

which is the differential form of the conservation law. Recall that ϕ may depend on x and t through dependence on u (i.e. $\phi = \phi(u)$), and Equation (1.31) can be written as

$$u_t + c(u)u_x = 0, \quad c(u) = \phi'(u). \tag{1.32}$$

But if u and ϕ have simple jump discontinuities, we still insist on the validity of the integral form Equation (1.30).

Now assume that $x = s(t)$ is a smooth curve in space time along which u suffers a simple discontinuity ; i.e., assume that u is continuously differentiable for $x > s(t)$ and $x < s(t)$, and that u and its derivatives have finite one-sided limits as $x \to s(t)^-$ and $x \to s(t)^+$. Then choosing $a < s(t)$ and $b > s(t)$, Equation (1.30) may be written as

$$\frac{d}{dt}\int_a^{s(t)} u(x,t)dx + \frac{d}{dt}\int_{s(t)}^{b} u(x,t)dx = \phi(a,t) - \phi(b,t), \tag{1.33}$$

Leibniz' rule for differentiating an integral whose integrand and limits depend on a parameter (here the parameter is time t) can be applied on the left side of Equation (1.32), because the integrands are smooth. We therefore obtain

$$\int_a^{s(t)} u_t(x,t)dx + \int_{s(t)}^{b} u_t(x,t)dx + u(s^-,t)s' - u(s^+,t)s' = \phi(a,t) - \phi(b,t), \tag{1.34}$$

where $u(s^-,t)$ and $u(s^+,t)$ are the limits of $u(x,t)$ as $x \to s(t)^-$ and $x \to s(t)^+$ respectively and $s' = ds/dt$ is the speed of discontinuity $x = s(t)$. In Equation (1.34) we now take the limit as $a \to s(t)^-$ and $b \to s(t)^+$. The first two terms to go zero because the integrand is bounded and the interval of integration shrinks to zero. Therefore, we obtain

$$-s'[u] + [\phi(u)] = 0, \tag{1.35}$$

where the brackets [ ] denote the jump of the quantity inside across the discontinuity (the value on the left minus the value on the right). Equation (1.35) is called the jump condition. In fluid mechanical problems, conditions across a discontinuity are known as Rankine-Hugoniot conditions. It relates conditions both ahead of and behind the discontinuity to the speed of the discontinuity itself. In this context, the discontinuity in u that propagates along the curve $x = s(t)$ is called a shock wave, and the curve $x = s(t)$ is called the shock path,or just the shock: s' is called the shock speed, and the magnitude of the jump in u is called the shock strength. In the case of gasdynamics, a discontinuous solution for a system of equations written in conservation form satisfying the generalized Rankine-Hugoniot conditions is called a shock [13, 14].

1.2 Motivation

The study of waves can be traced back to antiquity where philosophers, such as Pythagoras, studied the relation of pitch and length of string in musical instruments. The first analytical solution for a vibrating string was given by Brook Taylor (1685-1731). After this, advances were made by Daniel Bernoulli (1700-82), Leonard Euler (1707-83) and Jean d'Alembert (1717-83) who found the first solution to the linear wave equation. The superposition of solution, reflection and refraction are very common phenomena for a linear wave. If the governing partial differential equations are non-linear, the familiar laws of superposition, reflection, refraction and transmission of signals etc. ceases to be valid, but more interesting flow features appear such as shock wave. The knowledge of nonlinear waves is vital in aerodynamics for spacecraft propulsion, in medical sciences to disintegrate Kidney stone, in black hole theory etcetera. Over a hundred years ago, the study of nonlinear waves got attention with the pioneering work of Stokes [15], Earnshaw [16], Riemann [17] and Hadamard [18].

The system of non-linear PDEs is classified into elliptic, parabolic and hyperbolic. Out of these, the hyperbolic systems of conservation laws are one of the most important classes of non-linear PDEs. Euler's equations are the most common examples of hyperbolic PDEs. The Euler equations arise from the compressible Navier-Stokes equations by neglecting the viscosity and heat conduction. The theoretical foundation of gasdynamics is formed by the application of the basic conservation laws of mechanics and the second law of thermodynamics to a moving volume of a compressible gas.

The most outstanding new phenomenon of the nonlinear theory is the appearance of shock waves, which are abrupt jumps in pressure, density, and velocity: the blast waves of explosions and the sonic booms of high speed aircraft. But the whole intricate machinery of nonlinear hyperbolic equations had to be developed for their prediction, and a full understanding required analysis of the viscous effects and some aspects of kinetic theory. Whitham [19] brought up two main class of waves; one is nonlinear waves, such waves can be obtained from the quasilinear hyperbolic partial differential equations (PDE's), and secondly dispersive waves, these waves can be described by dispersive relation connecting the frequency and the wave number. The solution of such PDEs breaks down in a finite length of time which shows the appearance of discontinuity and smooth solution does not valid beyond this point due to the blow up of its derivatives. The occurrence of shock generally ceases the existence and uniqueness of admissible classical solutions for all the time, which leads the idea of weak solutions.

The analytical study of non-linear hyperbolic conservation law is interesting but leads to cumbersome task in Mathematics. In various areas of natural science and physical science, the analytical solution of the quasilinear hyperbolic system of partial differential equations plays a prominent role for the qualitative characterization of many physical processes and phenomena. The comprehension of the nonlinearity is very essential to understand our surroundings. Non-linear models arising in the real world often face serious mathematical difficulties related to the occurrence of discontinuities, singularities, the resonance between wave speeds, etc. Since the Law of Superposition does not hold for the non-linear PDEs, and this may probably be the reason that even today we do not have a single methodology that can solve all kinds of non-linear PDEs or the systems of PDEs. Thus, to find an analytical solution to these non-linear PDEs, we require special techniques.

Fluid/Gas dynamics has been the topic of great interest for the researchers in view of the systems of hyperbolic conservation laws since the pioneering work of Riemann (1860). It provides the motivation for many of the basic ideas in the analysis of the quasi-linear hyperbolic systems of PDEs. The Riemann problem models the one-dimensional interaction between a pair of uniform states of compressible fluids that are initially separated by a plane of discontinuity and plays a key role to understand the phenomena of wave structure for the hyperbolic systems of conservation laws. More generally, for a system of conservation laws, a Riemann problem is an initial value problem such that the initial data are scale-invariant (i.e., constant on rays). The Riemann problem is of great significance as its solution constitutes the basic building blocks for the construction of a solution to the general initial value problems. In the solution of the Riemann problem, all the features such as rarefractions waves, shock waves occur in the form of characteristics hence it is very convenient for the readers to understand the Euler equations in the form of conservation laws. The explicit solution to the Riemann problem is of great significance in relativistic gasdynamics and magnetogasdynamics Godunov [20], Smoller [21] and Chorin [22]. In the consideration of Euler equations, the Riemann problem consists of the shock tube problem and for detailed discussion of shock tube problem and other physical problems in form of conservation laws of gasdynamics, the readers are recommended to study the book by Courant and Friedrichs [23].

Continuum physics is rich source of hyperbolic problems and can be obtained from the conservation laws of mass, momentum and energy. Each medium is then characterized by its constitutive laws such as the shallow water equations, Euler's equations of gasdynamics, chaplygin gas and nonideal radiating gas, which in general, are nonlinear. The evolutionary behavior of nonlinear waves, such as shock waves and acceleration waves, in diverse branches of continuum mechanics has long been a subject of great interest from both the physical and mathematical points of view [19, 13, 24]. The practical importance of these waves has increased in last decades due to their particular applications in blast wave phenomena, supersonic flows, sonic booms and more recently their application to the motion of satellites and other bodies moving through the interplanetary and interstellar media.

1.3 Review

In this section, the literature review is divided into two parts. The review of literature on the shock waves in gasdynamics and Riemann problem is given in subsections (1.3.1) and (1.3.2) respectively.

1.3.1 Literature Review of shock waves in Gasdynamics

Shock waves are mechanical waves of finite amplitudes and arise when matter is subjected to a rapid compression. Compared to acoustic waves, which are waves of very small, almost infinitesimal amplitudes, shock waves can be characterized by four unusual properties:

- a pressure-dependent, supersonic velocity of propagation;
- the formation of a steep wave front with abrupt change of all thermodynamic quantities;
- for nonplanar shock waves, a strong decrease of the propagation velocity with increasing distance from the center of origin;
- nonlinear superposition (reflection and interaction) properties.

The phenomenon of shock waves is mainly associated with aerospace engineering and in particular with the supersonic flight. The development of this particular branch of physics began, in 1746 when a mathematician Robins determined velocity of the bullet by ballistic pendulum and noticed a growth in aerodynamic drag as velocity tends to the sound speed. However, in the 19th century, the phenomenon of the shock wave was still a mystery to many researchers. In 1759, without mentioning the word shock waves, Euler talked about the "size of disturbance" of a sound wave meaning its amplitude. Nevertheless, his assumption that velocity would diminish with increasing amplitude was incorrect. In 1808, Poisson was the first researcher to solve the Euler equation for the one-dimensional unsteady fluid-flow and got the exact solutions . In 1823, Poisson [25] created a milestone in non-linear wave theory by constructing isentropic gas law for the sound wave with infinitesimal amplitude. In 1848, Stokes[15] used the term "surface of discontinuity". He further extended the theory of acoustic wave, having a finite amplitude, by considering the problem of wave steepening. Stokes was not sure about the possibility of discontinuous motion, in this confusion he used isentropic relation which decides the role of dissipation and energy conservation in shock formation, instead of the correct energy equation. He derived the conservation laws for mass and momentum which are used very frequently in modern days. In 1889, Hugoniot [26] independently derived the correct jump conditions for the shock waves. The formulated theory of Rankine and Hugoniot is, even at present also, the basic model for the propagation of shock waves.

The modern concise definition of the shock wave was first given by a young Hungarian physicist Zempln from the University of Budapest in the year 1905. He proved that only a compression wave can be a shock wave, and rarefaction waves, i.e., negative shock waves do not exist, this is the so-called "Zempln's theory." He reported his results in Gttingen at a Felix Klein's seminar and in France in friendly conversations with Pierre Duhem and Jacques Hadamard, two experts in the field of shock waves.

When a body is in a relative motion with respect to the fluid (inside the fluid), the disturbance (if sufficiently small) produced by the body moves through the fluid with the speed of sound. These disturbances can be rarefaction waves or compression waves. The compressions of finite amplitude usually give rise to a discontinuous growth of pressure leading to a shock wave in the flow field. There is a likewise increase in temperature, density, entropy and other fluid properties. If initially, the fluid is at rest and the shock wave is moving then, after the passage of the shock, the fluid will move in the direction of the shock.

Gas compressions, which have finite amplitude, travel faster than the speed of sound, as in the case of strong explosions. In recent years the formation of shock waves has received considerable attention in the literature with the shock formation time or distance being used as important parameters characterising the relative importance of convective nonlinear steepening and dissipative flattening and setting a limit for the use of certain approximate theoretical approaches.

In formulating the general theory of the propagation of weak discontinuities in solutions to quasilinear hyperbolic systems and establishing the time of shock wave formation several methods are in order, such as the method of wavefront analysis [27], parameter expansion technique [28], wavefront expansion method [19], asymptotic method [29], reductive perturbation method [30] and the singular surface method [31, 32]. These different techniques prove effective according to the number of dependent and independent variables involved, the number and forms of the equations coupling them and the form of the evolution law that is required. The instant of formation of shock waves was largely investigated by many authors.

Basics of gasdynamics can be found in Zierep [33] (1978). The general theory of propagation of shock waves was presented by Boillat [29]. Shifrin [34] studied the formation of a shock wave for planar flow of a perfect gas. Ardavan-Rhad [35] studied the propagation of plane shock wave into a non-viscous, non-isentropic and non-heat conducting medium. Saldatov [36] determined the instant of formation of a shock wave in a symmetric two-way traffic flow, by using the Riemann method. Fusco [37], Germain [38], Fusco and Engelbrecht [39] , Sharma et al. [40], Singh et al. [41] and Nath et al. [42, 43] have utilized the asymptotic technique to study the non-linear wave propagation in various gaseous media. Flack and Wittig [44] presented the general solution for the normal shock wave moving in a medium where all flow properties vary arbitrarily. Macpherson [45] used the molecular-dynamic approach to study the formation of a shock wave in dense Argon. Chen [46] studied the propagation of shock waves in elastic non-conductors. The effect of thermodynamic properties on the propagation of shock waves has been studied by Chen [47]. Chen and Gurtin [48] and Cole-mann and Gurtin [49] studied the growth and decay of shock waves with internal state variables. Bowen and Chen [50] studied the same problem in the ideal mixture with several temperature layers.

Shock waves are characterized by an abrupt, nearly discontinuous change in the characteristics of the medium Anderson . Analytical solutions to the wave equations for steady vertical compression waves in a isothermal hydrostatics atmosphere with uniform horizontal magnetic field have been presented by Musielak et al. [51]. Cheng-Yue et al.[52] presented one dimensional relativistic shock model for the light curve of gamma-ray bursts.

The study of shock structure also received prominence in the recent decades. A lot of work on the shock structure was carried out by Kuznetsov [53], Goldman and Sirovich [54]. Wave fronts which are concave in the direction of propagation exhibit different kinds of behaviour depending on the strength of the wave-front. In 1999, Ruggeri [55] discussed the shock wave structure solutions of a general dissipative quasi-linear hyperbolic system of balance laws. Generally, wave front propagates normal to itself and therefore has a tendency to converge. The shocks of weak strength are called weak shocks. Focusing of weak shock is an important problem. This problem of focusing of weak shocks was studied by Wanner et al.[56]. Observers of atomic explosion are also known to have seen shock waves of strong strength, called blast wave. In case of blast waves, the shock becomes very strong and the pressure ahead is generally neglected in comparison to the pressure behind the shock wave. This leads to similarity formulation of the problem. The first work on the converging shock waves was done by Guderley [57] in 1942. Guderley emphasized that some physical assumptions lead to the formulation of the selfsimilar problem and the solution of which depends upon determining the similarity exponent that defines the property of shock wave, like, the space-time path in proximity to the collapse location.

The study of elementary wave interactions consist of either interaction between two waves colliding, or one wave overtaking another, or one wave meeting a discontinuity. Choquet-Bruhat [58] proposed a method to discuss shockless solutions of hyperbolic systems which depend upon a single phase function. Germain [38] has given the general discussion of single phase progressive waves.Hunter and Keller [59] established a general nonresonant multi-wave mode theory which has led to several interesting generalizations by Majda and Rosales [60] and Hunter et al.[61] to include resonantly interacting multi-wave mode features. Radha et al. [62] have shown that the general theory of wave interaction problem which originated from the work of Jeffrey [63] leads to the results obtained by Brun [64] and Boillat and Ruggeri [65]. This theory of wave interaction has been used to study the interaction of a bore with the weak discontinuity wave in shallow water [66] and the interaction of discontinuous waves in a gas with dust particles [67].

In 2009, Arora et al. [68] obtained small amplitude high frequency asymptotic solution to the basic equations in Eulerian coordinates governing one dimensional unsteady planar, spherically and cylindrically symmetric flow in a reactive hydrodynamic medium.

In 2015, Chadha and Jena [69] discussed the propagation of weak disturbances in a non-ideal gas with dust particles by using relatively undistorted method.

In 2016, Singh et al. [70] investigated the problem of the propagation of weak shock waves in an inviscid, electrically conducting fluid under the influence of a magnetic field. A system of two coupled nonlinear transport equations, governing the strength of a shock wave and the first-order discontinuity induced behind it, are derived which admit a solution that agrees with the classical decay laws for a weak shock.

In 2017, Shukla et al. [71] studied the evolution of planar and cylindrically symmetric magneto-acoustic waves in a van der Waals fluid. Also, an asymptotic method is used to derive an evolution equation that governs the wave amplitude in the far field.

In 2019, the propagation of a spherical shock wave in a non-ideal gas with or without gravitational effects is investigated under the action of monochromatic radiation. It

is manifested that the gravitational parameter and the radiation parameter have in general opposite behaviour on the flow variables and the shock strength [72] .
In 2020, G. Nath [73] investigated the non-similarity solution for unsteady isothermal flow behind the cylindrical shock wave in a rotational axisymmetric perfect gas in the presence of azimuthal magnetic field. Solutions are obtained for MHD shock in a rotating medium with the vorticity vector and its components in one-dimensional flow case.

1.3.2 Literature review of Riemann problem

The study of Riemann problem started with the work " theory of waves of finite amplitude" by great mathematician G. F. B. Riemann (1859), which was not limited to a single progressive wave and suited to calculate the propagation of planar waves of finite amplitude proceeding in both directions. In 1860, Riemann [17] introduced the Riemann problem for a system of conservation laws in gas dynamics. Lax [74] determined the solution of the Riemann problem for the condition when the initial data of the problem consists of two constant states U_*^1 and U_*^2, where U_*^1 and U_*^2 are respectively the vector of conserved variables to the left and right of $x = 0$ such that $\|U_*^1 - U_*^2\|$ is appropriately small and left and right constant states are divided by jump discontinuity at $x = 0$. In the consideration of Euler equations, the Riemann problem consists of the shock tube problem and for detailed discussion of shock tube problem and other physical problems in form of conservation laws of gasdynamics, the readers are recommended to study the book by Courant and Friedrichs [75]
Later, Courant and Friedrichs [23] extended the result of Riemann [17] to adiabatic flows and presented a new kind of elementary wave, i.e., contact discontinuity. Godunov [76] is credited with the first exact Riemann solver for the Euler equations.

By today's standards Godunov's first Riemann solver is cumbersome and computationally inefficient. Later, Godunov [20] proposed a second exact Riemann solver. Smoller [21] determined a solution to the Riemann problem for an extended class of hyperbolic systems with arbitrary constant states. A detailed deliberation on the explicit solution of the Riemann problem can be obtained in Toro [77], Schleicher and Pike [78].

The exact solution to the Riemann problem is of great significance. For instance, it constitutes the basic building block for the construction of solutions to general initial value problems using the well known random choice method proposed by Glimm [79]. Exact solutions of the Riemann problem are proposed by Godunov [20] and Chorin [22]; however Smoller [80] proposed a rather different approach. Smoller and Temple [81] demonstrated the existence of solutions with shocks for equations describing a perfect fluid in special relativity; this work was generalized by Chen [82] for the general isentropic relativistic gases. Toro [83] presented an efficient solver for computing the exact solution of the Riemann problem for ideal and covolume gases; for detailed methodologies, the reader is referred to the book by Toro [84]. Gottlieb and Groth [85] presented another Riemann solver for ideal gases. Chorin [22] proposed the new approach to obtain the exact solution to the Riemann problem. Another improvement to the Godunov's first Riemann solver was presented by Leer [86].

For an illuminating treatment on Riemann problem, we also refer to an article by Liu [87] and the books of Li-Tsien [88], Dafermos [89], Bressan [90], LeFloch [91] and LeVeque [91]. The special solution of Euler equations in which one of the Riemann invariants remain constant throughout the flow field is called a simple wave. In simple wave solutions, waves break and the solution has to be complemented by the introduction of shock waves. When the shock strength is small and even moderate, jumps in entropy and the Riemann invariants are surprisingly small, see Whitham

[19]; this, indeed, formed the basis for an approximate theory for shocks of weak or moderate strength developed by Friedrichs [23], where the actual shock conditions are replaced by the transition through a corresponding simple compression wave. Thus, for shocks of weak or moderate strength, it is a reasonable approximation to neglect changes in the entropy and Riemann invariant, as they are of third order in strength, and the simple wave approximation can be retained and used. As the theory governing the motion of a compressible fluid, whose electrical conductivity may be assumed to be infinite, and the theory of conventional gasdynamics are quite simple, it is possible to use the simple wave approximation for magnetogasdynamic flows involving shocks, which are not too strong, in a form that includes the solution for conventional gasdynamics as a special case, and yields the principle characteristics of magnetogasdynamic flows (see, for instance, Gundersen [92, 93]).

Liu and Sun [94] solved the Riemann problem for a class of coupled hyperbolic systems of conservation laws with delta initial data. Ambika and Radha [95] studied the uniqueness and existence of elementary wave solution to the Riemann problem for van der Waals gases. Bressan [90] provided a self-contained introduction to the mathematical theory of hyperbolic system of conservation laws, with particular emphasis on the study of discontinuous solutions, characterized by the appearance of shock waves. Mentrelli and Ruggeri [96] investigated shock and rarefaction waves in hyperbolic model of incompressible fluids. Mentrelli et al.[97] studied thoroughly the problem of the interaction of waves originated from Riemann problems in an Euler fluid.

1.4 Problem statement and Thesis Objectives

The main objective of this thesis is achieved by determining the analytical and/or numerical solutions to some selected non-linear realistic problems governed by the partial differential equations. The numerical work is carried out using the symbolic software packages MATLAB and MATHEMATICA. Our objective of investigation would be to focus on a detailed analytical and numerical study of the following specific problems, formulated mathematically as IVPs/BVPs, associated with quasilinear hyperbolic system of partial differential equations.
This Ph.D. thesis fulfils the following objectives:

- To derive the analytical solution of the Riemann problem for magnetogasdynamic equations governing an inviscid unsteady one-dimensional flow of non-ideal polytropic gas subjected to the transverse magnetic field with infinite electrical conductivity.
- Analytical and numerical studies of long time evolution of solution from an initial data to investigate for never breaking solutions.
- Study of problem of wave interaction in a dusty gas, where many wave modes coexist and interact resonantly, using asymptotic and numerical methods.

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