Chapter 5

A characterization of the category FCS

5.1 Introduction

Manes [24] (see also [25], p. 155) gave a characterization (up to an isomorphism) of the category **TOP** of topological spaces and continuous maps, in a suitable class of categories satisfying certain conditions, with the help of the Sierpinski space which is a Sierpinski object in the category **TOP**. Srivastava [39] defined fuzzy Sierpinski space and proved that it is a Sierpinski object in the category **FTOP** of fuzzy topological spaces (in the sense of Chang [10]) and fuzzy continuous maps and using it, obtained an analogous characterization of the category **FTOP**. Subsequently, Srivastava and Srivastava [38] defined fuzzy Sierpinski space for the class of Lowen's stratified fuzzy topological spaces [22] and obtained a similar characterization for the category **SFTOP** of stratified fuzzy topological spaces and fuzzy continuous maps. Singh and Srivastava [30] obtained a characterization of the category Q-**TOP** of Q-topological spaces and Q-continuous maps, using Q-Sierpinski space introduced in [36].

The contents of this chapter, in the form of a research paper, has been published in 'Soft Comput. **23** (2019) 13001-13005'.

Srivastava et al. [40] had defined a fuzzy closure space and studied the category **FCS** of fuzzy closure spaces and fuzzy closure preserving maps.

In this chapter, we have introduced the Sierpinski fuzzy closure space and shown that it is a Sierpinski object in the category **FCS** of fuzzy closure spaces. Further, we have given a characterization (up to an isomorphism) of the category **FCS**, using the Sierpinski fuzzy closure space, which is similar to the characterization of the category **TOP** given by Manes [24].

5.2 The category FCS

The following definition is from Manes [24] (see also [30]).

Definition 5.2.1. A **category C of sets with structures** is a concrete category $(\mathbf{C}, F)$ over the category **Set**, such that

1. a class $\mathbf{C}(X)$ of '**C**-structures' is assigned to every set X, which is in one-to-one correspondence with the 'fibre' $\{A \in Ob(\mathbf{C}) \mid F(A) = X\}$ of X; thus each **C**-object A can be identified with a pair (X, s) (and called a 'set with a structure'), where $F(A) = X$ and $s \in \mathbf{C}(X)$,

2. given a bijection $f : F(A) \to F(B)$ and $p \in \mathbf{C}(Y)$, there exists a unique $s \in \mathbf{C}(X)$ such that $f : (X, s) \to (Y, p)$ is a **C**-isomorphism.

Definition 5.2.2. [3] An **isomorphism** of the categories **C**, **D** of sets with structure is given by bijections $\Psi_X : \mathbf{C}(X) \to \mathbf{D}(X)$ with the property that for all s in $\mathbf{C}(X)$, p in $\mathbf{C}(Y)$ and maps $f : X \to Y$, we have that $f : (X, s) \to (Y, p)$ is a **C**-morphism if and only if $f : (X, \Psi_X(s)) \to (Y, \Psi_Y(p))$ is a **D**-morphism.

Definition 5.2.3. ([1], [24]) Let $\mathbf{C} = (\mathbf{C}, F)$ be a category of sets with structure.

1. A source $\{f_j : (X, s) \to (X_j, s_j) \mid j \in J\}$ in **C** is **initial** if whenever we have a **C**-structured set (Y, p) and $f : Y \to X$ is a function such that $f_j \circ f : (Y, p) \to (X_j, s_j)$ is a **C**-morphism for each j, then $f : (Y, p) \to (X, s)$ is a **C**-morphism.

2. Given a F-structured source $\{f_j : X \to F(X_j, s_j) \mid j \in J\}$, if there exists s in $\mathbf{C}(X)$ such that the source $\{f_j : (X, s) \to (X_j, s_j) \mid j \in J\}$ in **C** is

initial, then $\{f_j : (X, s) \to (X_j, s_j) \mid j \in J\}$ is called an **initial lift** of the F-structured source $\{f_j : X \to F(X_j, s_j) \mid j \in J\}$.

We mention here that the unit closed interval $[0, 1]$ will be denoted by I and for any fuzzy set u in X, u' will denote the complement of u.

The following definitions and propositions in this section are from Srivastava et al. [40].

Definition 5.2.4. Let X be a set. A map $c : I^X \to I^X$ is called a **fuzzy closure operation** on X if for all $u,\ v \in I^X,\ a \in I$:

1. $u \leq c(u)$,
2. $u \leq v \Rightarrow c(u) \leq c(v)$,
3. $c(u) = c(c(u))$,
4. $c(f_a) = f_a$, where $f_a : X \to I$ is such that $f_a(x) = a$, for every $x \in X$.

The pair (X, c) is called a **fuzzy closure space** and fuzzy sets $v \in I^X$ such that $c(v) = v$, are called c-**closed fuzzy sets** in X. Let (X_1, c_1) and (X_2, c_2) be two fuzzy closure spaces and $\gamma : X_1 \to X_2$ be a map. Then $\gamma : (X_1, c_1) \to (X_2, c_2)$ is called **fuzzy closure preserving** if $\gamma(c_1(v)) \leq c_2(\gamma(v))$ for every $v \in I^{X_1}$.

A family $\mathcal{F} \subseteq I^X$ is called a **Moore family** if it is closed under infima and contains all constant fuzzy sets.

Proposition 5.2.5. Let (X, c) be a fuzzy closure space. Then the family of all c-closed fuzzy sets of (X, c) is a Moore family.

Proposition 5.2.6. If $\mathcal{F} \subseteq I^X$ is a Moore family then the mapping $c_{\mathcal{F}} : I^X \to I^X$, defined by $c_{\mathcal{F}}(v) = \inf\{u \in I^X \mid v \leq u \text{ and } u \in \mathcal{F}\}$, is a fuzzy closure operation on X and a fuzzy set $v \in I^X$ is $c_{\mathcal{F}}$-closed if and only if $v \in \mathcal{F}$.

$c_{\mathcal{F}}$ will be called as the fuzzy closure operation on X induced by the Moore family $\mathcal{F}$.

Proposition 5.2.7. Let (X_1, c_1) and (X_2, c_2) be two fuzzy closure spaces and $\gamma : X_1 \to X_2$ be a mapping. Then $\gamma : (X_1, c_1) \to (X_2, c_2)$ is fuzzy closure preserving if and only if $\gamma^{-1}(v)$ is c_1-closed for every c_2-closed fuzzy set $v \in I^{X_2}$.

FCS will denote the category of fuzzy closure spaces and fuzzy closure preserving maps. **FCS** is a category of sets with structure, with the functor U : **FCS**$\to$ **Set** given by $U((X_1, c_1) \xrightarrow{f} (X_2, c_2)) = X_1 \xrightarrow{f} X_2$.

5.3 Sierpinski fuzzy closure space

Proposition 5.3.1. Let (X, c) be a fuzzy closure space. Let $\mathcal{F}$ be the family of all c-closed fuzzy sets. Then $c = c_{\mathcal{F}}$, where $c_{\mathcal{F}} : I^X \to I^X$ defined by $c_{\mathcal{F}}(v) = \inf\{u \in I^X \mid v \leq u \text{ and } u \in \mathcal{F}\}$, for every $v \in I^X$.

Proof. First we note that from Proposition 5.2.5, $\mathcal{F}$ is a Moore family, and so from Proposition 5.2.6, $c_{\mathcal{F}}$ is a fuzzy closure operation on the set X. Let $v \in I^X$. Then $c_{\mathcal{F}}(v) = \inf\{u \in I^X \mid v \leq u,\ u \in \mathcal{F}\}$. Now since $\mathcal{F}$ is a Moore family, $c(\inf\{u \in I^X \mid v \leq u,\ u \in \mathcal{F}\}) = \inf\{u \in I^X \mid v \leq u,\ u \in \mathcal{F}\}$, i.e., $c(c_{\mathcal{F}}(v)) = c_{\mathcal{F}}(v)$. Now as $v \leq c_{\mathcal{F}}(v)$, this implies that $c(v) \leq c(c_{\mathcal{F}}(v)) = c_{\mathcal{F}}(v)$. So we have $c(v) \leq c_{\mathcal{F}}(v)$. Also since $v \leq c(v)$ and $c(v)$ is c-closed fuzzy set, $c_{\mathcal{F}}(v) \leq c(v)$. Thus $c(v) = c_{\mathcal{F}}(v)$, for every $v \in I^X$ and therefore $c = c_{\mathcal{F}}$. □

Let $\mathcal{S} \subseteq I^X$. Consider the family $B \cup \{f_a \mid a \in I\} \cup \{v \cap f_a \mid a \in I,\ v \in B\}$, where $B = \{v \mid v$ is an arbitrary intersection of the complements of members of $\mathcal{S}\}$ and for each $a \in I$, $f_a : X \to I$ is defined as, $f_a(x) = a$, for every $x \in X$. Then this is a Moore family and we denote it by $\langle \mathcal{S} \rangle$ and $c_{\langle \mathcal{S} \rangle}$ will denote the fuzzy closure operation on X induced by the Moore family $\langle \mathcal{S} \rangle$. Note that if $\mathcal{S} \subseteq I^X$ is closed under suprema and contains all constant fuzzy sets, then $\langle \mathcal{S} \rangle$ is precisely the set consisting of the complements of all fuzzy sets in $\mathcal{S}$.

Proposition 5.3.2. Let (X_1, c_1) and (X_2, c_2) be two fuzzy closure spaces. Let $c_2 = c_{\langle \mathcal{S} \rangle}$, where $\mathcal{S} \subseteq I^{X_2}$ and $g : X_1 \to X_2$ be a map. Then $g : (X_1, c_1) \to (X_2, c_2)$ is fuzzy closure preserving if and only if $g^{-1}(u')$ is c_1-closed, for every $u \in \mathcal{S}$.

Proof. We first note that $\langle \mathcal{S} \rangle = B \cup \{f_a \mid a \in I\} \cup \{v \cap f_a \mid a \in I,\ v \in B\}$, where $B = \{v \mid v$ is an arbitrary intersection of the complements of members of $\mathcal{S}\}$. A fuzzy set $w \in I^{X_2}$ is $c_{\langle \mathcal{S} \rangle}$-closed if and only if $w \in \langle \mathcal{S} \rangle$. Clearly if $g : (X_1, c_1) \to (X_2, c_2)$ is fuzzy closure preserving, then $g^{-1}(u')$ is c_1-closed, for every $u \in \mathcal{S}$.

Conversely, suppose that $g^{-1}(u')$ is c_1-closed, for every $u \in \mathcal{S}$. Let $v \in B$ and assume that $v = \cap\{u'_j \mid j \in J\}$, $u_j \in \mathcal{S}$, for every $j \in J$. Then $g^{-1}(v) = g^{-1}(\cap u'_j) = \cap g^{-1}(u'_j)$. Now since $g^{-1}(u'_j)$ is c_1-closed for every $j \in J$ and the set of all c_1-closed fuzzy sets is closed under infima, $g^{-1}(v) = \cap g^{-1}(u'_j)$ is c_1-closed fuzzy set. Also since $g^{-1}(f_a)$ is a constant fuzzy set in X_1, for every $a \in I$ and each constant fuzzy set in X_1 is c_1-closed, $g^{-1}(f_a)$ is c_1-closed, for every $a \in I$. Now let $v \in B$ and $a \in I$. Then $g^{-1}(v \cap f_a) = g^{-1}(v) \cap g^{-1}(f_a)$ is c_1-closed. Hence $g^{-1}(w)$ is c_1-closed for every $w \in \langle \mathcal{S} \rangle$, i.e., $g^{-1}(w)$ is c_1-closed fuzzy set for every $c_{\langle \mathcal{S} \rangle}$-closed fuzzy set $w \in I^{X_2}$. Therefore by Proposition 5.2.7, $g : (X_1, c_1) \to (X_2, c_2)$ is fuzzy closure preserving.

□

In ([40], Proposition 2.5), the authors have proved that the category **FCS** has initial structures. In the following proposition, we give another description of the initial structures.

Proposition 5.3.3. A source $\{f_j : (X, c) \to (X_j, c_j) \mid j \in J\}$ in the category **FCS** is initial if and only if $c = c_{\langle \mathcal{S} \rangle}$, where $\mathcal{S} = \{f_j^{-1}(u') \mid u \in F_j,\ j \in J\}$ and F_j is the family of all c_j-closed fuzzy sets.

Proof. Let F be the family of all c-closed fuzzy sets. First suppose that the source $\{f_j : (X, c) \to (X_j, c_j) \mid j \in J\}$ is initial. Now from Proposition 5.3.1, to show that $c = c_{\langle \mathcal{S} \rangle}$, it is sufficient to show that $F = \langle \mathcal{S} \rangle$. Since each f_j is fuzzy closure preserving, by Proposition 5.2.7, $f_j^{-1}(u) \in F$ for every $u \in F_j$, $j \in J$, i.e., $(f_j^{-1}(u'))' = f_j^{-1}(u) \in F$, for every $u \in F_j$, $j \in J$. Note that $\langle \mathcal{S} \rangle = B \cup \{f_a \mid a \in I\} \cup \{v \cap f_a \mid a \in I,\ v \in B\}$, where $B = \{v \mid v$ is an arbitrary intersection of the complements of members of $\mathcal{S}\}$. Now since $w' \in F$, for every $w \in \mathcal{S}$ and F is a Moore family, $\langle \mathcal{S} \rangle \subseteq F$. Now by Proposition 5.2.7, the map $f_j \circ id_X : (X, c_{\langle \mathcal{S} \rangle}) \to (X_j, c_j)$ is fuzzy closure preserving for each $j \in J$ and the source $\{f_j : (X, c) \to (X_j, c_j) \mid j \in J\}$ is initial, so $id_X : (X, c_{\langle \mathcal{S} \rangle}) \to (X, c)$ is fuzzy closure preserving. Then from Proposition 5.2.7, $id_X^{-1}(w) = w \in \langle \mathcal{S} \rangle$, for every $w \in F$, which implies that $F \subseteq \langle \mathcal{S} \rangle$. Therefore $F = \langle \mathcal{S} \rangle$.

Conversely, assume that $c = c_{\langle \mathcal{S} \rangle}$. Let (Y, c_Y) be a fuzzy closure space and $g : Y \to X$ be a map such that $f_j \circ g : (Y, c_Y) \to (X_j, c_j)$ is fuzzy closure preserving for each j. Then to show that $g : (Y, c_Y) \to (X, c)$ is fuzzy closure preserving. Let $j \in J$, $u \in F_j$. Note that $(f_j^{-1}(u'))' = f_j^{-1}(u)$. Now $g^{-1}(f_j^{-1}(u)) = (g^{-1} \circ f_j^{-1})(u) = (f_j \circ g)^{-1}(u)$ is c_Y-closed, as $f_j \circ g : (Y, c_Y) \to (X_j, c_j)$ is fuzzy closure preserving. Hence $g^{-1}(w')$ is c_Y-closed, for every $w \in \mathcal{S}$. Thus by Proposition 5.3.2, $g : (Y, c_Y) \to (X, c)$ is fuzzy closure preserving. Therefore the source $\{f_j : (X, c) \to (X_j, c_j) \mid j \in J\}$ is initial. □

Definition 5.3.4. The fuzzy closure space $(I, c_{\langle\{id_I\}\rangle})$ will be called as the **Sierpinski fuzzy closure space**.

Theorem 5.3.5. $(I, c_{\langle\{id_I\}\rangle})$ is a Sierpinski object in the category **FCS**.

Proof. Let (X, c) be a fuzzy closure space and $\mathcal{S} = \{f \mid f : (X, c) \to (I, c_{\langle\{id_I\}\rangle})$ is fuzzy closure preserving$\}$. Let c_1 be any other fuzzy closure operation on the set X such that $f : (X, c_1) \to (I, c_{\langle\{id_I\}\rangle})$ is fuzzy closure preserving, for every $f \in \mathcal{S}$. Let F and F_1 be the family of all c-closed and c_1-closed fuzzy sets respectively. From Proposition 5.3.2, we note that a map $f : (X, c) \to (I, c_{\langle\{id_I\}\rangle})$ is fuzzy closure preserving if and only if f' belongs to F. Let $u \in F$. Then $u' : (X, c) \to (I, c_{\langle\{id_I\}\rangle})$ is fuzzy closure preserving. So, $u' \in \mathcal{S}$ and hence $u' : (X, c_1) \to (I, c_{\langle\{id_I\}\rangle})$ is fuzzy closure preserving. Then $(u')' = u \in F_1$ and thus $F \subseteq F_1$. If we take $c_1 = c_{\langle\mathcal{S}\rangle}$, then $F \subseteq \langle\mathcal{S}\rangle$. Also since $f : (X, c) \to (I, c_{\langle\{id_I\}\rangle})$ is fuzzy closure preserving map, for every $f \in \mathcal{S}$, $f' \in F$ for every $f \in \mathcal{S}$ and since the family of all c-closed fuzzy sets is closed under infima and contains all constant fuzzy sets, $\langle\mathcal{S}\rangle \subseteq F$. Hence $F = \langle\mathcal{S}\rangle$. Thus $c = c_F = c_{\langle\mathcal{S}\rangle}$. Hence using Proposition 5.3.3, $\mathcal{S}$ is an initial source in **FCS**. Therefore $(I, c_{\langle\{id_I\}\rangle})$ is a Sierpinski object in the category **FCS**. □

5.4 Characterization of FCS

Theorem 5.4.1. Let $\mathbf{C} = (\mathbf{C}, F)$ be a category of sets with structure. Then $\mathbf{C}$ is isomorphic to the category **FCS** if and only if there exists a $\mathbf{C}$-structured set (I, r) in $\mathbf{C}$ with underlying set $I = [0, 1]$, satisfying the following conditions:

1. Every F-structured source $\{f_j : X \to F(I, r) \mid j \in J\}$ has an initial lift.

2. The map $\sup : (I, r)^T \to (I, r)$ is a $\mathbf{C}$-morphism, for every set T $(\neq \emptyset)$. (Here $(I, r)^T = (I^T, r_0)$, where $r_0 \in \mathbf{C}(I^T)$ such that, $\{p_t : (I^T, r_0) \to (I, r) \mid$

$t \in T\}$ is the initial lift of the F-structured source $\{p_t : I^T \to F(I,r) \mid t \in T\}$, $p_t : I^T \to F(I,r)$ are projection maps).

3. (I,r) is a Sierpinski object in **C**.

   Let (X,s) be a **C**-structured set, we say that $g \in I^X$ is open in (X,s) if $g : (X,s) \to (I,r)$ is a C-morphism. A family $\{g_j \mid j \in J\} \subseteq I^X$ with each g_j open in (X,s) is called a subbase of (X,s) if the source $\{g_j : (X,s) \to (I,r) \mid j \in J\}$ in **C** is initial.

4. If ζ is a subbase of (X,s) and $g \in I^X$ is open in (X,s), then $g' \in \langle \mathcal{S} \rangle$, where $\mathcal{S} = \{f^{-1}(u') \mid f \in \zeta,\ u \in \langle \{id_I\} \rangle \}$

5. Given any set X and $s \in C(X)$, the map $f_a : (X,s) \to (I,r)$ is a **C**-morphism, for every $a \in I$.

Proof. First we show that the category **FCS** satisfies the conditions (1)-(5).
(1) Every U-structured source $\{f_j : X \to U(I, c_{\langle \{id_I\} \rangle}) \mid j \in J\}$ has the unique initial lift $\{f_j : (X, c_{\langle \mathcal{S} \rangle}) \to (I, c_{\langle \{id_I\} \rangle}) \mid j \in J\}$, where $c_{\langle \mathcal{S} \rangle}$ is the fuzzy closure operation on the set X such that $\mathcal{S} = \{f_j^{-1}(u') \mid u \in \langle \{id_I\} \rangle,\ j \in J\}$.
(2) Let $T\ (\neq \emptyset)$ be a set. To show that the map $\sup : (I, c_{\langle \{id_I\} \rangle})^T \to (I, c_{\langle \{id_I\} \rangle})$ is fuzzy closure preserving. Note that $(I, c_{\langle \{id_I\} \rangle})^T = (I^T, c_{\langle \mathcal{F} \rangle})$, where $\mathcal{F} = \{p_t^{-1}(u') \mid u \in \langle \{id_I\} \rangle,\ t \in T\}$ and p_t are projections $I^T \to I$, for every $t \in T$. Let $f \in I^T$ and consider $\sup(f) = \sup\{f(t) \mid t \in T\} = \sup\{p_t(f) \mid t \in T\} = (\cup\{p_t \mid t \in T\})(f)$. This implies that $\sup = \cup\{p_t \mid t \in T\}$ and so $(\sup)' = \cap\{p_t' \mid t \in T\}$. Now since $(id_I)' \in \langle \{id_I\} \rangle$, $p_t^{-1}(((id_I)')') = p_t^{-1}(id_I) = p_t \in \mathcal{F}$, for every $t \in T$ and so $p_t' \in \langle \mathcal{F} \rangle$, for every $t \in T$ and since $\mathcal{F}$ is a Moore family, $(\sup)' = \cap\{p_t' \mid t \in T\} \in \langle \mathcal{F} \rangle$. Therefore from Proposition 5.3.2, the map $\sup : (I, c_{\langle \{id_I\} \rangle})^T \to (I, c_{\langle \{id_I\} \rangle})$ is fuzzy closure preserving.
(3) By Theorem 5.3.5, $(I, c_{\langle \{id_I\} \rangle})$ is a Sierpinski object in the category **FCS**.
(4) Let $g \in I^X$ be open in (X,c), i.e., $g : (X,c) \to (I, c_{\langle \{id_I\} \rangle})$ is fuzzy closure preserving. This implies that g' is c-closed. Now since ζ is a subbase of (X,c), the source $\{f : (X,c) \to (I, c_{\langle \{id_I\} \rangle}) \mid f \in \zeta\}$ is initial. Then by Proposition 5.3.3, $c = c_{\langle \mathcal{S} \rangle}$, where $\mathcal{S} = \{f^{-1}(u') \mid f \in \zeta,\ u \in \langle \{id_I\} \rangle \}$. Now since g' is c-closed, $g' \in \langle \mathcal{S} \rangle$.
(5) Let $a \in I$ and (X,c) be a fuzzy closure space. Then the map $f_a : (X,c) \to (I, c_{\langle \{id_I\} \rangle})$ is fuzzy closure preserving as $f_a^{-1}((id_I)') = (f_a)'$ is a constant fuzzy set in X and hence c-closed.

Let X be a set and $S(X)$ denote the collection of all fuzzy closure operations on X. Now suppose that $\mathbf{C}$ is isomorphic to $\mathbf{FCS}$. Then for each set X there exists a bijection $\Psi_X : C(X) \to S(X)$ with the property that for all s in $C(X)$, p in $C(Y)$ and maps $f : X \to Y$, we have that $f : (X, s) \to (Y, p)$ is a $\mathbf{C}$-morphism if and only if $f : (X, \Psi_X(s)) \to (Y, \Psi_Y(p))$ is fuzzy closure preserving. Let $r = \Psi_I^{-1}(c_{\langle\{id_I\}\rangle})$. Then (I, r) is a Sierpinski object in $\mathbf{C}$ and satisfies the conditions (1)-(5).

Conversely, let $\mathbf{C} = (\mathbf{C}, F)$ be a category of sets with structure and assume that there exist a $\mathbf{C}$-structured set (I, r) in the category $\mathbf{C}$ with underlying set $I = [0, 1]$, satisfying the conditions (1)-(5). Now we have to show that $\mathbf{C}$ is isomorphic to the category $\mathbf{FCS}$. Define a map $\Psi_X : C(X) \to S(X)$ such that $\Psi_X(s) = c_{\mathcal{F}_s}$, where $\mathcal{F}_s = \{h' \mid h : (X, s) \to (I, r)$ is a C-morphism$\}$ and $c_{\mathcal{F}_s} : I^X \to I^X$ is defined as $c_{\mathcal{F}_s}(v) = \inf\{u \in I^X \mid v \leq u$ and $u \in \mathcal{F}_s\}$. Then by the condition (5), all the constant fuzzy sets belong to $\mathcal{F}_s$. Now let $\{h_t : X \to I \mid t \in T\}$ be a family such that $h_t : (X, s) \to (I, r)$ is a C-morphism, for every $t \in T$. Define $\gamma : X \to I^T$ as $\gamma(x)(t) = h_t(x)$. Then $p_t \circ \gamma = h_t$, for every $t \in T$. Now since the source $\{p_t : (I^T, r_0) \to (I, r) \mid t \in T\}$ is initial and $p_t \circ \gamma = h_t : (X, s) \to (I, r)$ is a C-morphism, for every $t \in T$, $\gamma : (X, s) \to (I^T, r_0)$ is a C-morphism. Now consider $(\sup \circ \gamma)(x) = \sup(\gamma(x)) = \sup\{\gamma(x)(t) \mid t \in T\} = \sup\{h_t(x) \mid t \in T\} = (\cup\{h_t \mid t \in T\})(x)$. This implies that $\cup\{h_t \mid t \in T\} = \sup \circ \gamma$ and since $\gamma : (X, s) \to (I^T, r_0)$ and $\sup : (I^T, r_0) \to (I, r)$ are C-morphisms, $\cup h_t : (X, s) \to (I, r)$ is a C-morphism. Hence $(\cup\{h_t \mid t \in T\})' = \cap\{h_t' \mid t \in T\} \in \mathcal{F}_s$. Therefore $\mathcal{F}_s$ is a Moore family and hence from Proposition 5.2.6, $c_{\mathcal{F}_s}$ is a fuzzy closure operation on the set X.

Now we show that the map Ψ_X is a bijection. Let $s_1, s_2 \in C(X)$ such that $\Psi_X(s_1) = \Psi_X(s_2)$, i.e., $c_{\mathcal{F}_{s_1}} = c_{\mathcal{F}_{s_2}}$, where $\mathcal{F}_{s_1} = \{h' \mid h : (X, s_1) \to (I, r)$ is a C-morphism$\}$ and $\mathcal{F}_{s_2} = \{h' \mid h : (X, s_2) \to (I, r)$ is a C-morphism$\}$. Then since $\mathcal{F}_{s_1}$ and $\mathcal{F}_{s_2}$ are precisely the family of all $c_{\mathcal{F}_{s_1}}$-closed and $c_{\mathcal{F}_{s_2}}$-closed fuzzy sets respectively and $c_{\mathcal{F}_{s_1}} = c_{\mathcal{F}_{s_2}}$, $\mathcal{F}_{s_1} = \mathcal{F}_{s_2}$. Suppose $\mathcal{F}_{s_1} = \mathcal{F}_{s_2} = \mathcal{F}$. Now since (I, r) is a Sierpinski object in $\mathbf{C}$, the source $\mathcal{S} = \{h : (X, s_1) \to (I, r) : h' \in \mathcal{F}\}$ in $\mathbf{C}$ is initial. We note that the map $h \circ id_X : (X, s_2) \to (I, r)$ is a $\mathbf{C}$-morphism for every $h \in \mathcal{S}$ and the source $\mathcal{S} = \{h : (X, s_1) \to (I, r) : h' \in \mathcal{F}\}$ is initial, so $id_X : (X, s_2) \to (X, s_1)$ is a $\mathbf{C}$-morphism. Similarly $id_X^{-1} = id_X : (X, s_1) \to (X, s_2)$ is a $\mathbf{C}$-morphism. We also note that $id_X : (X, s_1) \to (X, s_1)$ is a $\mathbf{C}$-morphism. Hence by the definition of a category of sets with structure, $s_1 = s_2$.

Now let $c \in S(X)$, i.e., c is a fuzzy closure operation on X and suppose that $\mathcal{F} = \{h_k \mid k \in K\}$ be the family of all c-closed fuzzy sets. Then the F-structured source $\{h'_k : X \to F(I, r) \mid k \in K\}$ will have an initial lift, say $\{h'_k : (X, s) \to (I, r) \mid k \in K\}$. Then clearly all maps $h'_k : (X, s) \to (I, r)$, $k \in K$ are **C**-morphisms. This implies that $h_k = (h'_k)' \in \mathcal{F}_s$, for every $k \in K$ and so $\mathcal{F} \subseteq \mathcal{F}_s$. Now let $f \in \mathcal{F}_s$, i.e., $f' : (X, s) \to (I, r)$ is a **C**-morphism. This implies that f' is open in (X, s). Since $\{h'_k : (X, s) \to (I, r) \mid k \in K\}$ is an initial source, $\{h'_k \mid k \in K\}$ is a subbase for (X, s). So $f = (f')' \in \langle \mathcal{S} \rangle$, where $\mathcal{S} = \{(h'_k)^{-1}(u') \mid \ u \in \langle \{id_I\} \rangle, k \in K\}$. Note that $\langle \mathcal{S} \rangle = \langle \{h'_k \mid k \in K\} \rangle$. Now since $\{h_k \mid k \in K\}$ is the family of all c-closed fuzzy sets, it is closed under infima and contains all constant fuzzy sets. So the family $\{h'_k \mid k \in K\}$ is closed under suprema and contains all constant fuzzy sets. Hence $\langle \mathcal{S} \rangle = \langle \{h'_k \mid k \in K\} \rangle = \{h_k \mid k \in K\}$. Thus $f \in \{h_k \mid k \in K\} = \mathcal{F}$. This implies that $\mathcal{F}_s \subseteq \mathcal{F}$ and hence $\mathcal{F}_s = \mathcal{F}$. So, $c = c_{\mathcal{F}_s} = \Psi_X(s)$. Therefore the map $\Psi_X : C(X) \to S(X)$ is a bijection.

Now it remains to show that a map $f : (X, s) \to (Y, p)$ is a **C**-morphism if and only if $f : (X, c_{\mathcal{F}_s}) \to (Y, c_{\mathcal{F}_p})$ is fuzzy closure preserving. First suppose that $f : (X, s) \to (Y, p)$ is a **C**-morphism and let $u \in \mathcal{F}_p$, i.e., $u' : (Y, p) \to (I, r)$ is a **C**-morphism. Then $u' \circ f : (X, s) \to (I, r)$ will be a **C**-morphism. Note that $u' \circ f = f^{-1}(u')$. So, $f^{-1}(u') : (X, s) \to (I, r)$ is C-morphism. Thus $f^{-1}(u) = (f^{-1}(u'))' \in \mathcal{F}_s$. Therefore the map $f : (X, c_{\mathcal{F}_s}) \to (Y, c_{\mathcal{F}_p})$ is fuzzy closure preserving. Conversely, let $f : (X, c_{\mathcal{F}_s}) \to (Y, c_{\mathcal{F}_p})$ be fuzzy closure preserving, i.e., for every $u \in \mathcal{F}_p$, $f^{-1}(u) \in \mathcal{F}_s$. This means that for every $u \in \mathcal{F}_p$, $u' \circ f : (X, s) \to (I, r)$ is a **C**-morphism. Now since $\{u' : (Y, p) \to (I, r) \mid u \in \mathcal{F}_p\}$ is an initial source, $f : (X, s) \to (Y, p)$ is a **C**-morphism.

□

5.5 Conclusion

Srivastava et al. [40] introduced and studied the category **FCS** of fuzzy closure spaces and fuzzy closure preserving maps. In this chapter, we have introduced the Sierpinski fuzzy closure space and proved that it is a Sierpinski object in the category **FCS**. Further, we have given a characterization of the category **FCS** with the help of the Sierpinski fuzzy closure space.