Chapter 4

On coreflective hulls in Str-Q-TOP

4.1 Introduction

Solovyov [36] introduced stratified Q-topological spaces. Singh and Srivastava [31] introduced the Q-topological space $(Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle)$ which is a Sierpinski object in the category **Str**-Q-**TOP** of stratified Q-topological spaces.

Singh and Srivastava [32] obtained the epireflective hull of $(Q, \langle\{id_Q\}\rangle)$ in the category Q-**TOP** of Q-topological spaces. Along with reflective subcategories, coreflective subcategories have also received much attention and have been studied extensively by many authors (cf., e.g., Herrlich and Strecker [17] and [18]). Singh [34] determined the coreflective hull of the L-Sierpinski space in the category L-**TOP** of L-topological spaces. Hoffmann [19] determined the coreflective hulls of the category of discrete topological spaces and the category of indiscrete topological spaces in the category **TOP** of topological spaces. Singh and Srivastava [35] determined the coreflective hull of the category of stratified indiscrete fuzzy topological spaces in the category **Str-FTOP** of stratified fuzzy topological spaces.

In this chapter, we determine the coreflective hull of $(Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle)$ in the category **Str**-Q-**TOP**. We also determine the coreflective hulls of the categories **Str-Dis**-Q-**TOP** of discrete Q-topological spaces and **Str-Ind**-Q-**TOP** of stratified indiscrete Q-topological spaces in the category **Str**-Q-**TOP**.

4.2 The category Str-Q-TOP

First we recall the following definition of a stratified Q-topological space from chapter 1 of the thesis.

Definition 4.2.1. [36] A Q-topological space (X, τ) is said to be **stratified** if $\underline{q} \in \tau$, for every $q \in Q$, where $\underline{q} : X \to Q$ is defined as $\underline{q}(x) = q$, for every $x \in X$.

Str-Q-TOP will denote the category of stratified Q-topological spaces and Q-continuous maps. **Str-Q-TOP** is a construct via the obvious forgetful functor V : **Str-Q-TOP**$\to$ **Set**.

The following Proposition 4.2.2 can be easily verified:

Proposition 4.2.2. Let $\{h_k : Z \to V(Z_k, \eta_k) \mid k \in K\}$ be a V-structured source (where Z is a set and (Z_k, η_k) is a stratified Q-topological space for each k). Then $\{h_k : (Z, \eta) \to (Z_k, \eta_k) \mid k \in K\}$, where $\eta = \langle\{\alpha_k \circ h_k \mid \alpha_k \in \eta_k, k \in K\}\rangle$, is the initial lift of the source $\{h_k : Z \to V(Z_k, \eta_k) \mid k \in K\}$ in **Str-Q-TOP**.

Remark 4.2.3. By Proposition 4.2.2, it follows that **Str-Q-TOP** is a topological category over **Set**.

In view of the Remark 4.2.3 and Proposition 1.2.33, we have the following result:

Proposition 4.2.4. In the category **Str-Q-TOP**, quotient morphisms are precisely extremal epimorphisms.

Proposition 4.2.5. Let (X, τ) and (Y, σ) be stratified Q-topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ be a Q-continuous map. Then,

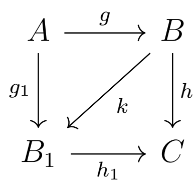
1. $f : (X, \tau) \to (Y, \sigma)$ is final in **Str-Q-TOP** if and only if $\sigma = \{v \in Q^Y \mid v \circ f \in \tau\}$,
2. $f : (X, \tau) \to (Y, \sigma)$ is a quotient morphism in **Str-Q-TOP** if and only if it is final in **Str-Q-TOP** and f is onto.

Singh and Srivastava [31] introduced the Q-topological space $(Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle)$. It can be easily verified that $(Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle)$ is a Sierpinski object in the category **Str-Q-TOP**.

Definition 4.2.6. [17] Let **C** be a category and let $f : A \to C$ be a morphism in **C**. Then f is said to have

1. an (extremal) **epi-mono factorization** provided that for some (extremal) epimorphism $g : A \to B$ in **C** and some monomorphism $h : B \to C$ in **C**, $f = h \circ g$,

2. a **unique extremal epi-mono factorization** provided that it has an extremal epi-mono factorization $f = h \circ g$ and if $f = h_1 \circ g_1$ is an extremal epi-mono factorization, then there exists an isomorphism $k : B \to B_1$ in **C**, such that the diagram



(4.2.1)

commutes.

Definition 4.2.7. [17] A category **C** is said to have

1. the (extremal) **epi-mono factorization property** provided that each of its morphisms has an (extremal) epi-mono factorization,

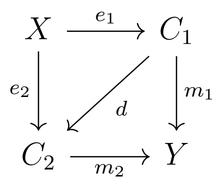
2. the **unique extremal epi-mono factorization property** provided that each of its morphisms has a unique extremal epi-mono factorization,

3. the **strong unique extremal epi-mono factorization property** provided that it has the unique extremal epi-mono factorization property and, in **C**, the composition of extremal epimorphisms is an extremal epimorphism.

Proposition 4.2.8. The category **Str**-Q-**TOP** have the strong unique extremal epi-mono factorization property.

Proof. Let (X, τ) and (Y, σ) be stratified Q-topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ be a Q-continuous map. Let $f(X) = \{f(x) \mid x \in X\}$. Then $f = m \circ e$, where $m : f(X) \to Y$ is defined as $m(f(x)) = f(x)$ and $e : X \to f(X)$ is defined as $e(x) = f(x)$. Next let $\eta = \{v \in Q^{f(X)} \mid v \circ e \in \tau\}$. We note here that since (X, τ) is a stratified Q-topological space, $(f(X), \eta)$ is a stratified Q-topological space and also by Propositions 4.2.4 and 4.2.5, $e : (X, \tau) \to (f(X), \eta)$

is an extremal epimorphism in **Str**-Q-**TOP**. It can also be easily verified that $m : (f(X), \eta) \to (Y, \sigma)$ is a monomorphism in **Str**-Q-**TOP**. Next let $(X, \tau) \xrightarrow{e_1} (C_1, \tau_{C_1}) \xrightarrow{m_1} (Y, \sigma)$ and $(X, \tau) \xrightarrow{e_2} (C_2, \tau_{C_2}) \xrightarrow{m_2} (Y, \sigma)$ be extremal epi-mono factorizations of f in **Str**-Q-**TOP**. Then by Propositions 4.2.4 and 4.2.5, $e_1 : X \to C_1$ is onto and so $C_1 = e_1(X) = \{e_1(x) \mid x \in X\}$. Define $d : C_1 \to C_2$ as $d(e_1(x)) = e_2(x)$. Now we will first prove that the map d is well-defined. Let $e_1(x_1) = e_1(x_2)$, then $(m_1 \circ e_1)(x_1) = (m_1 \circ e_1)(x_2) = (m_2 \circ e_2)(x_2) = m_2(e_2(x_2))$. Also $(m_1 \circ e_1)(x_1) = (m_2 \circ e_2)(x_1) = m_2(e_2(x_1))$. Thus $m_2(e_2(x_1)) = m_2(e_2(x_2))$ and since $m_2 : (C_2, \tau_{C_2}) \to (Y, \sigma)$ is a monomorphism in **Str**-Q-**TOP**, m_2 is one-one and so $e_2(x_1) = e_2(x_2)$. Hence the map d is well-defined. Also it can be easily proved that the diagram



(4.2.2)

commutes. Now we will prove that the map $d : (C_1, \tau_{C_1}) \to (C_2, \tau_{C_2})$ is an isomorphism in **Str**-Q-**TOP**. Let $\alpha \in \tau_{C_2}$, then $\alpha \circ d \circ e_1 = \alpha \circ e_2 \in \tau$ as $e_2 : (X, \tau) \to (C_2, \tau_{C_2})$ is Q-continuous. Now since $e_1 : (X, \tau) \to (C_1, \tau_{C_1})$ is an extremal epimorphism in **Str**-Q-**TOP** and $\alpha \circ d \circ e_1 \in \tau$, by Propositions 4.2.4 and 4.2.5, it follows that $\alpha \circ d \in \tau_{C_1}$. Hence $d : (C_1, \tau_{C_1}) \to (C_2, \tau_{C_2})$ is Q-continuous. Next let $v \in Q^{C_2}$ such that $v \circ d \in \tau_{C_1}$. Then since $e_1 : (X, \tau) \to (C_1, \tau_{C_1})$ is Q-continuous, $v \circ d \circ e_1 \in \tau$. This implies that $v \circ e_2 \in \tau$. Now since $e_2 : (X, \tau) \to (C_2, \tau_{C_2})$ is an extremal epimorphism in **Str**-Q-**TOP** and $v \circ e_2 \in \tau$, by Propositions 4.2.4 and 4.2.5, it follows that $v \in \tau_{C_2}$. Thus by Proposition 4.2.5, $d : (C_1, \tau_{C_1}) \to (C_2, \tau_{C_2})$ is a final morphism in **Str**-Q-**TOP**. Now we will show that d is bijective. Let $d(e_1(x_1)) = d(e_1(x_2))$. This implies that $e_2(x_1) = e_2(x_2) \Rightarrow (m_2 \circ e_2)(x_1) = (m_2 \circ e_2)(x_2) \Rightarrow (m_1 \circ e_1)(x_1) = (m_1 \circ e_1)(x_2) \Rightarrow m_1(e_1(x_1)) = m_1(e_1(x_2))$ and since $m_1 : (C_1, \tau_{C_1}) \to (B, \tau_B)$ is a monomorphism in **Str**-Q-**TOP**, m_1 is one-one and so $e_1(x_1) = e_1(x_2)$ and hence d is one-one. Now let $c_2 \in C_2$. Since by Propositions 4.2.4 and 4.2.5, e_2 is onto, there exists $x \in X$ such that $e_2(x) = c_2$ and then $d(e_1(x)) = c_2$ and so d is onto. Thus $d : (C_1, \tau_{C_1}) \to (C_2, \tau_{C_2})$ is a final morphism in **Str**-Q-**TOP** and d is bijective and so by Proposition 1.2.23, $d : (C_1, \tau_{C_1}) \to (C_2, \tau_{C_2})$ is an isomorphism in **Str**-Q-**TOP**. Now by Proposition 1.2.22 and Proposition 4.2.4, composition of extremal epimorphisms in **Str-Q-TOP** is an extremal epimorphism in **Str-Q-TOP**. Therefore **Str-Q-TOP** have the strong unique extremal epi-mono factorization property.

□

Definition 4.2.9. [36] Let (Z, η) be a stratified Q-topological space and let $M \subseteq Z$. Then $\eta_M = \{\alpha|_M | \ \alpha \in \eta\}$, where $\alpha|_M \colon M \to Q$ is defined as $\alpha|_M(m) = \alpha(m)$, for every $m \in M$, is a stratified Q-topology on M, called as the subspace Q-topology on M and (M, η_M) is called a subspace of (Z, η).

Definition 4.2.10. [32] Let (Z, η) be a stratified Q-topological space, X be a set and $g : Z \to X$ be an onto map. Then $\tau = \{\alpha \in Q^X \mid \alpha \circ g \in \eta\}$ is a stratified Q-topology on X, called as the quotient Q-topology on X with respect to (Z, η) and g. The Q-continuous map $g : (Z, \eta) \to (X, \tau)$ is called a quotient map and (X, τ) is called as the quotient Q-topological space of (Z, η) with respect to the quotient map g.

Remark 4.2.11. We mention here that quotient morphisms (in category theoretic sense) in **Str-Q-TOP** are precisely quotient maps.

([31]) Let Z be a set. Q^Z is clearly the largest Q-topology on Z and called as the discrete Q-topology on Z and the stratified Q-topological space (Z, Q^Z) is called a discrete Q-topological space. The Q-topology $\eta = \{\underline{q} \mid q \in Q\}$ is called as the stratified indiscrete Q-topology on Z and (Z, η) is called a stratified indiscrete Q-topological space. We mention here that discrete and indiscrete objects in the category **Str-Q-TOP** are respectively the discrete and stratified indiscrete Q-topological spaces.

Str-Dis-Q-TOP will denote the full subcategory of **Str-Q-TOP** consisting of discrete Q-topological spaces and **Str-Ind-Q-TOP** will denote the full subcategory of **Str-Q-TOP** consisting of stratified indiscrete Q-topological spaces.

4.3 Coreflective Hull of $(Q, \langle \{id_Q\} \cup \{\underline{q} \mid q \in Q\} \rangle)$ in Str-Q-TOP

From now onwards, in this chapter subcategories are always assumed to be full and isomorphism-closed.

Definition 4.3.1. [1] Let $\mathbf{W}$ be a subcategory of a category $\mathbf{C}$ and let B be a $\mathbf{C}$-object. A **W-coreflection** (or **W-coreflection arrow**) for B is a $\mathbf{C}$-morphism $f : A \to B$ from a $\mathbf{W}$-object A to B with the following universal property:

for any $\mathbf{C}$-morphism $g : \hat{A} \to B$ from some $\mathbf{W}$-object $\hat{A}$ to B, there exists a unique $\mathbf{W}$-morphism $\hat{g} : \hat{A} \to A$ such that the triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \hat{g}}\uparrow & \nearrow{\scriptstyle g} & \\ \hat{A} & & \end{array} \tag{4.3.1}$$

commutes.

Definition 4.3.2. Let $\mathbf{C}$ be a category and let E be a class of $\mathbf{C}$-morphisms.

1. [1] A subcategory $\mathbf{W}$ of $\mathbf{C}$ is called E**-coreflective subcategory** of $\mathbf{C}$ if for each $\mathbf{C}$-object B, there exists a $\mathbf{W}$-coreflection arrow in E. In particular, we use the terms **coreflective** (resp. **epicoreflective**, **mono-coreflective**) in case E is the class of all morphisms (resp. epimorphisms, monomorphisms) of $\mathbf{C}$.

2. [17] Let $\mathcal{A}$ be a class of $\mathbf{C}$-objects. The smallest E-coreflective subcategory of $\mathbf{C}$ containing $\mathcal{A}$ is called the E**-coreflective hull** of $\mathcal{A}$ in the category $\mathbf{C}$. In particular, we use the terms **coreflective hull** (resp. **epicoreflective hull** , **mono-coreflective hull**) in case E is the class of all morphisms (resp. epimorphisms, monomorphisms) of $\mathbf{C}$.

Theorem 4.3.3. [17] Let $\mathbf{C}$ be a well-powered category which has coproducts and the extremal epi-mono factorization property. Then the mono-coreflective hull in $\mathbf{C}$ of a class $\mathcal{A}$ of $\mathbf{C}$ objects exists. Furthermore, if $\mathbf{C}$ has the strong unique extremal epi-mono factorization property, then the objects of this mono-coreflective hull of $\mathcal{A}$ are exactly all the extremal quotients of coproducts of objects in $\mathcal{A}$.

Theorem 4.3.4. [17] Let $\mathbf{C}$ be a well-powered category which has coproducts and the extremal epi-mono factorization property. Let $\mathbf{W}$ be a subcategory of $\mathbf{C}$. Then $\mathbf{W}$ is a monocoreflective subcategory of $\mathbf{C}$ if and only if $\mathbf{W}$ is closed under the formation of coproducts and extremal quotient objects.

Proposition 4.3.5. [17] Let **C** be a category and **W** be an epicoreflective subcategory of **C**. Then **W** is a monocoreflective subcategory of **C**.

Remark 4.3.6. The empty set $\emptyset$ has a unique stratified Q-topological structure $\{\emptyset\}$ which is simultaneously discrete and indiscrete. The stratified Q-topological space $(\emptyset, \{\emptyset\})$ will be denoted by $\mathbb{I}$ and it can be easily verified that $\mathbb{I}$ is the initial object in the category **Str**-Q-**TOP**. We also mention here that $\{\mathbb{I}\}$ is a coreflective subcategory of **Str**-Q-**TOP**. Now if **W** is any coreflective subcategory of **Str**-Q-**TOP**, then there exists a **W**-coreflection $c_W : (X_W, \tau_W) \to \mathbb{I}$ for $\mathbb{I}$, but then (X_W, τ_W) must be equal to $\mathbb{I}$ and hence $\mathbb{I} \in \mathbf{W}$. Thus every coreflective subcategory of **Str**-Q-**TOP** contains $\mathbb{I}$ and $\{\mathbb{I}\}$ is the smallest coreflective subcategory of **Str**-Q-**TOP**.

The following Proposition 4.3.7 is concerned with the extension of Proposition 3.4 in [35], for the category **Str**-Q-**TOP**.

Proposition 4.3.7. Let **W** be a coreflective subcategory of **Str**-Q-**TOP**. Then **W** is a monocoreflective subcategory of **Str**-Q-**TOP**.

Proof. If $\mathbf{W} = \{\mathbb{I}\}$, then it can be easily proved that it is a monocoreflective subcategory of **Str**-Q-**TOP**. Now suppose that $\mathbf{W} \neq \{\mathbb{I}\}$. We will first prove that **W** is an epicoreflective subcategory of **Str**-Q-**TOP**. Let (X, τ) be a stratified Q-topological space with non-empty underlying set and let $c_W : (X_W, \tau_W) \to (X, \tau)$ be its **W**-coreflection. Let (Y, σ) be a stratified Q-topological space and let $h, g : (X, \tau) \to (Y, \sigma)$ be distinct Q-continuous maps. Then there exists $x \in X$ such that $h(x) \neq g(x)$. Consider the inclusion map $i_x : (\{x\}, \tau_d) \to (X, \tau)$, where τ_d is the discrete Q-topology on $\{x\}$. Clearly, $i_x : (\{x\}, \tau_d) \to (X, \tau)$ is Q-continuous and $h \circ i_x \neq g \circ i_x$. Now since $\mathbf{W} \neq \{\mathbb{I}\}$, **W** contains a non-empty stratified Q-topological space, say (Z, η). Let $f : (Z, \eta) \to (\{x\}, \tau_d)$ be the constant map, which is clearly Q-continuous. Also $h \circ i_x \circ f \neq g \circ i_x \circ f$. Now since $c_W : (X_W, \tau_W) \to (X, \tau)$ is a coreflection, there exists a unique Q-continuous map $l : (Z, \eta) \to (X_W, \tau_W)$ such that $c_W \circ l = i_x \circ f$. Then, $h \circ c_W \circ l = h \circ i_x \circ f \neq g \circ i_x \circ f = g \circ c_W \circ l$. This implies that $h \circ c_W \neq g \circ c_W$. Thus $c_W : (X_W, \tau_W) \to (X, \tau)$ is an epimorphism and so **W** is an epicoreflective subcategory of **Str**-Q-**TOP**. Therefore by Proposition 4.3.5, **W** is a monocoreflective subcategory of **Str**-Q-**TOP**. □

In view of Proposition 4.2.2, **Str**-Q-**TOP** is a topological category over **Set** and since **Set** is a well-powered category and has coproducts, **Str**-Q-**TOP** is a well-powered category and has coproducts (cf. Theorem 21.16, Corollary 21.17 in [1]). We mention here that by Proposition 4.2.4 and Remark 4.2.11, extremal epimorphisms in **Str-Q-TOP** are precisely quotient maps and for a given stratified Q-topological space (Z, η), the extremal quotients of (Z, η) in **Str-Q-TOP** are precisely the quotient Q-topological spaces of (Z, η). Also by Proposition 4.2.8, **Str-Q-TOP** has strong unique extremal epi-mono factorization property. Thus from Theorem 4.3.3 and Proposition 4.3.7, we have the following result:

Theorem 4.3.8. Let (Z, η) be a stratified Q-topological space. Then the coreflective hull of (Z, η) exists in **Str-Q-TOP**. Moreover, its objects are precisely the quotient Q-topological spaces of coproducts of copies of (Z, η).

By Theorem 4.3.4 and Proposition 4.3.7, we have the following result:

Theorem 4.3.9. Let **W** be a subcategory of **Str-Q-TOP**. Then **W** is a coreflective subcategory of **Str-Q-TOP** if and only if it is closed under the formation of coproducts and quotient Q-topological spaces.

Let (Z, η) be a stratified Q-topological space and $\{k\}$ be a fixed singleton set. Consider the set $Z \times \{k\}$. Let $\eta_k^* = \{\alpha^* \mid \alpha \in \eta\}$, where $\alpha^* : Z \times \{k\} \to Q$ is defined as $\alpha^*(z, k) = \alpha(z)$. Then η_k^* is a stratified Q-topology on $Z \times \{k\}$.

Let $\{(Z_k, \eta_k) \mid k \in K\}$ be a non-empty family of stratified Q-topological spaces. Let $Z = \bigcup_{k \in K} Z_k \times \{k\}$ and $\eta^+ = \{v \in Q^Z \mid v|_{Z_k \times \{k\}} \in \eta_k^*, \forall k \in K\}$. Then η^+ is a stratified Q-topology on Z and $\{in_k : (Z_k, \eta_k) \to (Z, \eta^+) \mid k \in K\}$ is a coproduct of $\{(Z_k, \eta_k) \mid k \in K\}$ in **Str-Q-TOP**, where $in_k : Z_k \to Z$ is defined by $in_k(z_k) = (z_k, k)$, for every $z_k \in Z_k$.

Lemma 4.3.10. Let $g : Z \to Y$ be a map. Let $\beta : Y \to Q$ be a map and $\alpha \in \langle \{\beta\} \cup \{\underline{q} \mid q \in Q\} \rangle$, then $\alpha \circ g \in \langle \{\beta \circ g\} \cup \{\underline{q} \mid q \in Q\} \rangle$.

Proof. Consider the map $g : (Z, \langle \{\beta \circ g\} \cup \{\underline{q} \mid q \in Q\} \rangle) \to (Y, \langle \{\beta\} \cup \{\underline{q} \mid q \in Q\} \rangle)$. By Proposition 1.2.40, this map is Q-continuous. Therefore if $\alpha \in \langle \{\beta\} \cup \{\underline{q} \mid q \in Q\} \rangle$, then $\alpha \circ g \in \langle \{\beta \circ g\} \cup \{\underline{q} \mid q \in Q\} \rangle$. □

For a given stratified Q-topological space (Z, η), CH((Z, η)) will denote the coreflective hull of (Z, η) in the category **Str-Q-TOP**.

Lemma 4.3.11. Let (Z,η) and (Y,σ) be stratified Q-topological spaces and let $(Y,\sigma) \in \mathrm{CH}((Z,\eta))$. Then either (Y,σ) is $\mathbb{I}$ or there exists a non-empty index set K such that (Y,σ) is a quotient Q-topological space of the coproduct $(\bigcup_{k\in K} Z_k \times \{k\}, \eta^+)$ of $\{(Z_k,\eta_k) \mid k \in K\}$ in **Str-Q-TOP**, where for each $k \in K$, (Z_k,η_k) is (Z,η).

Proof. By Theorem 4.3.8, there exists an index set K, a stratified Q-topological space (W,θ) which is a coproduct of K copies of (Z,η) and a quotient map $f : (W,\theta) \to (Y,\sigma)$. If $K = \emptyset$, then since coproduct of empty family of stratified Q-topological spaces in **Str-Q-TOP** is $\mathbb{I}$ and quotient Q-topological space of $\mathbb{I}$ is $\mathbb{I}$, (Y,σ) is $\mathbb{I}$. Now suppose that $K \neq \emptyset$ and consider the family $\{(Z_k,\eta_k) \mid k \in K\}$ of stratified Q-topological spaces, where each (Z_k,η_k) is (Z,η). Then since $(\bigcup_{k\in K} Z_k \times \{k\}, \eta^+)$ is also a coproduct of K copies of (Z,η) in **Str-Q-TOP**, there exists an isomorphism $g : (\bigcup_{k\in K} Z_k \times \{k\}, \eta^+) \to (W,\theta)$. Next, since an isomorphism in **Str-Q-TOP** is a quotient map and composition of quotient maps is a quotient map, $f \circ g : (\bigcup_{k\in K} Z_k \times \{k\}, \eta^+) \to (Y,\sigma)$ is a quotient map. Hence (Y,σ) is a quotient Q-topological space of $(\bigcup_{k\in K} Z_k \times \{k\}, \eta^+)$. □

The following Theorem 4.3.12 is concerned with the extension of Proposition 6.1 in [35], for the category **Str-Q-TOP**.

Theorem 4.3.12. Let (Z,η) and (Y,σ) be stratified Q-topological spaces. Then $(Y,\sigma) \in \mathrm{CH}((Z,\eta))$ if and only if either (Y,σ) is $\mathbb{I}$ or there exists a non-empty family $\{(Y_k,\sigma_k) \mid k \in K\}$ of subspaces of (Y,σ) such that

1. $Y = \bigcup_{k\in K} Y_k$
2. For each $v \in Q^Y$, $v \in \sigma$ if and only if $v|_{Y_k} \in \sigma_k$, for every $k \in K$.
3. (Y_k,σ_k) is a quotient Q-topological space of (Z,η), for every $k \in K$.

Proof. Let $(Y,\sigma) \in \mathrm{CH}((Z,\eta))$. Then by Lemma 4.3.11, either (Y,σ) is $\mathbb{I}$ or there exists a non-empty index set K such that (Y,σ) is a quotient Q-topological space of the coproduct $(\bigcup_{k\in K} Z_k \times \{k\}, \eta^+)$ of $\{(Z_k,\eta_k) \mid k \in K\}$ in **Str-Q-TOP**, where for each $k \in K$, (Z_k,η_k) is (Z,η). Let $f : (\bigcup_{k\in K} Z_k \times \{k\}, \eta^+) \to (Y,\sigma)$ be a quotient map. Next, let $Y_k = f(Z_k \times \{k\})$ and let σ_k be the subspace Q-topology on Y_k. Then as f is onto, $Y = \bigcup_{k\in K} Y_k$. Next, let $v \in \sigma$. Then, $v|_{Y_k} \in \sigma_k$, for every $k \in K$, as σ_k is the subspace Q-topology on Y_k, for every

$k \in K$. Now let $v \in Q^Y$ such that $v|_{Y_k} \in \sigma_k$, for every $k \in K$. Consider the map $f|_{Z_k \times \{k\}}\colon (Z_k \times \{k\}, \eta_k^*) \to (Y_k, \sigma_k)$. This map is Q-continuous as $u|_{Y_k} \circ f|_{Z_k \times \{k\}} = (u \circ f)|_{Z_k \times \{k\}} \in \eta_k^*$, for every $u \in \sigma$. Then, since for every $k \in K$, $v|_{Y_k} \in \sigma_k$, $(v \circ f)|_{Z_k \times \{k\}} = v|_{Y_k} \circ f|_{Z_k \times \{k\}} \in \eta_k^*$, for every $k \in K$. This implies that $v \circ f \in \eta^+$ and since $f : (\bigcup_{k \in K} Z_k \times \{k\}, \eta^+) \to (Y, \sigma)$ is a quotient map, $v \in \sigma$. Next, we have to prove that (Y_k, σ_k) is a quotient Q-topological space of (Z, η), for every $k \in K$. Fix $k_0 \in K$. First we will show that $f|_{Z_{k_0} \times \{k_0\}}\colon (Z_{k_0} \times \{k_0\}, \eta_{k_0}^*) \to (Y_{k_0}, \sigma_{k_0})$ is a quotient map. Clearly it is onto and Q-continuous. Now let $\alpha \in Q^{Y_{k_0}}$ such that $\alpha \circ f|_{Z_{k_0} \times \{k_0\}} \in \eta_{k_0}^*$. Let $v \in Q^Y$ be a map such that $v|_{Y_k} \in \sigma_k$, for every $k(\neq k_0) \in K$ and $v|_{Y_{k_0}} = \alpha$. Then $(v \circ f)|_{Z_k \times \{k\}} = v|_{Y_k} \circ f|_{Z_k \times \{k\}} \in \eta_k^*$, for every $k(\neq k_0) \in K$ as $f|_{Z_k \times \{k\}}\colon (Z_k \times \{k\}, \eta_k^*) \to (Y_k, \sigma_k)$ is Q-continuous, for every $k(\neq k_0) \in K$. Now $(v \circ f)|_{Z_{k_0} \times \{k_0\}} = v|_{Y_{k_0}} \circ f|_{Z_{k_0} \times \{k_0\}} = \alpha \circ f|_{Z_{k_0} \times \{k_0\}} \in \eta_{k_0}^*$. Thus $(v \circ f)|_{Z_k \times \{k\}} \in \eta_k^*$, for every $k \in K$. This implies that $v \circ f \in \eta^+$. Hence $v \in \sigma$ and so $v|_{Y_{k_0}} = \alpha \in \sigma_{k_0}$. So $f|_{Z_{k_0} \times \{k_0\}}\colon (Z_{k_0} \times \{k_0\}, \eta_{k_0}^*) \to (Y_{k_0}, \sigma_{k_0})$ is a quotient map. Next, consider the map $l_{k_0} : Z_{k_0} \to Z_{k_0} \times \{k_0\}$ defined as $l_{k_0}(z_{k_0}) = (z_{k_0}, k_0)$, for every $z_{k_0} \in Z_{k_0}$. We will show that $l_{k_0} : (Z_{k_0}, \eta_{k_0}) \to (Z_{k_0} \times \{k_0\}, \eta_{k_0}^*)$ is a quotient map. Clearly this map is onto. Now let $\alpha^* \in \eta_{k_0}^*$. Then, $\alpha^* \circ l_{k_0} = \alpha \in \eta_{k_0}$. Next, let $\beta^* \in Q^{Z_{k_0} \times \{k_0\}}$ such that $\beta^* \circ l_{k_0} \in \eta_{k_0}$. Then, $\beta^* = (\beta^* \circ l_{k_0})^* \in \eta_{k_0}^*$. Hence $l_{k_0} : (Z_{k_0}, \eta_{k_0}) \to (Z_{k_0} \times \{k_0\}, \eta_{k_0}^*)$ is a quotient map. Since composition of quotient maps is a quotient map, $f|_{Z_{k_0} \times \{k_0\}} \circ l_{k_0} : (Z_{k_0}, \eta_{k_0}) \to (Y_{k_0}, \sigma_{k_0})$ is a quotient map and so (Y_{k_0}, σ_{k_0}) is a quotient Q-topological space of (Z_{k_0}, η_{k_0}). Thus (Y_k, σ_k) is a quotient Q-topological space of (Z_k, η_k), for every $k \in K$, i.e. (Y_k, σ_k) is a quotient Q-topological space of (Z, η), for every $k \in K$.

Conversely, if (Y, σ) is $\mathbb{I}$, then by Remark 4.3.6, $(Y, \sigma) \in \mathrm{CH}((Z, \eta))$. Now assume that there exists a non-empty family $\{(Y_k, \sigma_k) \mid k \in K\}$ of subspaces of (Y, σ) satisfying (1), (2) and (3). Then by (3), there exists a quotient map $f_k : (Z_k, \eta_k) \to (Y_k, \sigma_k)$, for every $k \in K$ (where for each $k \in K$, (Z_k, η_k) is (Z, η)). Define a map $f : \bigcup_{k \in K} Z_k \times \{k\} \to Y$ as $f(z, k) = f_k(z)$, for every $(z, k) \in \bigcup_{k \in K} Z_k \times \{k\}$. Then clearly f is onto. We will show that $f : (\bigcup_{k \in K} Z_k \times \{k\}, \eta^+) \to (Y, \sigma)$ is a quotient map. Let $v \in \sigma$. Then $v|_{Y_k} \in \sigma_k$ and since $f_k : (Z_k, \eta_k) \to (Y_k, \sigma_k)$ is Q-continuous, $v|_{Y_k} \circ f_k \in \eta_k$. So, $(v|_{Y_k} \circ f_k)^* \in \eta_k^*$ and since $(v|_{Y_k} \circ f_k)^* = v|_{Y_k} \circ f|_{Z_k \times \{k\}} = (v \circ f)|_{Z_k \times \{k\}}$, $(v \circ f)|_{Z_k \times \{k\}} \in \eta_k^*$. Hence $(v \circ f)|_{Z_k \times \{k\}} \in \eta_k^*$, for every $k \in K$. This implies that $v \circ f \in \eta^+$. Next, let $v \in Q^Y$ such that $v \circ f \in \eta^+$, i.e., $(v \circ f)|_{Z_k \times \{k\}} \in \eta_k^*$, for every $k \in K$. This implies that $(v|_{Y_k} \circ f_k)^* \in \eta_k^*$, for every $k \in K$. Thus $v|_{Y_k} \circ f_k \in \eta_k$, for every $k \in K$. Thus $v|_{Y_k} \in \sigma_k$, for every $k \in K$ as $f_k : (Z_k, \eta_k) \to (Y_k, \sigma_k)$ is a quotient map, for every

$k \in K$. Then by (2), $v \in \sigma$. Therefore $f : (\bigcup_{k\in J} Z_k \times \{k\}, \eta^+) \to (Y, \sigma)$ is a quotient map and by Theorem 4.3.8, $(Y, \sigma) \in \mathrm{CH}((Z, \eta))$.

□

Proposition 4.3.13. Let a stratified Q-topological space (Z, η) be a quotient Q-topological space of $(Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle)$ via a quotient map $g : (Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle) \to (Z, \eta)$. Then $|Z| \leq |Q|$ and $\eta = \langle\{g^{-1}\} \cup \{\underline{q} \mid q \in Q\}\rangle$, if g is bijective.

Further, if (Z, η) is a stratified Q-topological space such that $|Z| = |Q|$ and $\eta = \langle\{\beta\} \cup \{\underline{q} \mid q \in Q\}\rangle$, where $\beta : Z \to Q$ is a bijection, then (Z, η) is a quotient Q-topological space of $(Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle)$ via the quotient map $\beta^{-1} : (Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle) \to (Z, \eta)$.

Proof. Let a stratified Q-topological space (Z, η) be a quotient Q-topological space of $(Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle)$ with respect to a quotient map $g : (Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle) \to (Z, \eta)$. Then as g is onto, $|Z| \leq |Q|$. Now suppose that g is bijective. Let $\alpha \in \eta$, then $\alpha \circ g \in \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle$. Then by Lemma 4.3.10, $\alpha \circ g \circ g^{-1} \in \langle\{id_Q \circ g^{-1}\} \cup \{\underline{q} \mid q \in Q\}\rangle$, i.e. $\alpha \in \langle\{g^{-1}\} \cup \{\underline{q} \mid q \in Q\}\rangle$. Hence $\eta \subseteq \langle\{g^{-1}\} \cup \{\underline{q} \mid q \in Q\}\rangle$. Now since the map $g : (Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle) \to (Z, \langle\{g^{-1}\} \cup \{\underline{q} \mid q \in Q\}\rangle)$ is Q-continuous, $\alpha \circ g \in \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle$, for every $\alpha \in \langle\{g^{-1}\} \cup \{\underline{q} \mid q \in Q\}\rangle$. Then since $g : (Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle) \to (Z, \eta)$ is a quotient map, $\alpha \in \eta$, for every $\alpha \in \langle\{g^{-1}\} \cup \{\underline{q} \mid q \in Q\}\rangle$. This implies that $\langle\{g^{-1}\} \cup \{\underline{q} \mid q \in Q\}\rangle \subseteq \eta$. Therefore $\eta = \langle\{g^{-1}\} \cup \{\underline{q} \mid q \in Q\}\rangle$.

Now let (Z, η) be a stratified Q-topological space such that $|Z| = |Q|$ and $\eta = \langle\{\beta\} \cup \{\underline{q} \mid q \in Q\}\rangle$, where $\beta : Z \to Q$ is a bijection. We have to prove that $\beta^{-1} : (Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle) \to (Z, \eta)$ is a quotient map. Clearly it is Q-continuous. Now let $\alpha \in Q^Z$ such that $\alpha \circ \beta^{-1} \in \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle$. Then by Lemma 4.3.10, $\alpha \circ \beta^{-1} \circ \beta \in \langle\{\beta\} \cup \{\underline{q} \mid q \in Q\}\rangle$, i.e., $\alpha \in \langle\{\beta\} \cup \{\underline{q} \mid q \in Q\}\rangle = \eta$. Therefore $\beta^{-1} : (Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle) \to (Z, \eta)$ is a quotient map. □

Now by Theorem 4.3.12 and Proposition 4.3.13, we have the following result:

Theorem 4.3.14. Let (Z, η) be a stratified Q-topological space. Then $(Z, \eta) \in \mathrm{CH}((Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle))$ if and only if either (Z, η) is $\mathbb{I}$ or there exists a non-empty family $\{(Z_k, \eta_k) \mid k \in K\}$ of subspaces of (Z, η) satisfying the following conditions:

1. $Z = \bigcup_{k\in K} Z_k$ such that for each $k \in K$, there exists a quotient map $g_k : (Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle) \to (Z_k, \eta_k)$. Furthermore, if g_k is bijective, then $\eta_k = \langle\{g_k^{-1}\} \cup \{\underline{q} \mid q \in Q\}\rangle$.

2. For each $v \in Q^Z$, $v \in \eta$ if and only if $v|_{Z_k} \in \eta_k$, for every $k \in K$.

4.4 Coreflective hull of Str-Dis-Q-TOP and Str-Ind-Q-TOP in the category Str-Q-TOP

Proposition 4.4.1. Let (Z, η) be a discrete Q-topological space with $|Z| \geq 1$. Then $\mathrm{CH}((Z, \eta)) =$ **Str-Dis-Q-TOP**.

Proof. First suppose that (Y, σ) belongs to $\mathrm{CH}((Z, \eta))$. If (Y, σ) is $\mathbb{I}$, then clearly $(Y, \sigma) \in$ **Str-Dis-Q-TOP**. Now suppose that (Y, σ) is a stratified Q-topological space with non-empty underlying set. Then by Theorem 4.3.12, there exists a non-empty family $\{(Y_k, \sigma_k) \mid k \in K\}$ of subspaces of (Y, σ), where each (Y_k, σ_k) is quotient Q-topological space of (Z, η), such that $Y = \bigcup_{k\in K} Y_k$ and for each $v \in Q^Y$, $v \in \sigma$ if and only if $v|_{Y_k} \in \sigma_k$, for every $k \in K$. Now since (Z, η) is a discrete Q-topological space and each (Y_k, σ_k) is quotient Q-topological space of (Z, η), each (Y_k, σ_k) is a discrete Q-topological space. Now let $v \in Q^Y$, then $v|_{Y_k} \in \sigma_k$, for every $k \in K$ (since σ_k is discrete). Thus $v \in \sigma$. Therefore (Y, σ) is a discrete Q-topological space and so $\mathrm{CH}((Z, \eta)) \subseteq$ **Str-Dis-Q-TOP**. Next let (Y, σ) be a discrete Q-topological space. Let $Y = \{y_k \mid k \in K\}$. Consider the family $\{(\{y_k\}, \sigma_k) \mid k \in K\}$ of stratified Q-topological spaces, where σ_k is the discrete Q-topology on $\{y_k\}$, for every $k \in K$. Clearly each $(\{y_k\}, \sigma_k)$ is a subspace of (Y, σ) and $Y = \bigcup_{k\in K} Y_k$, where $Y_k = \{y_k\}$. Also for each $v \in Q^Y$, $v \in \sigma$ if and only if $v|_{Y_k} \in \sigma_k$, for every $k \in K$ and each (Y_k, σ_k) is a quotient Q-topological space of (Z, η). Thus by Theorem 4.3.12, (Y, σ) belongs to $\mathrm{CH}((Z, \eta))$ and so **Str-Dis-Q-TOP**$\subseteq \mathrm{CH}((Z, \eta))$. Therefore $\mathrm{CH}((Z, \eta)) =$ **Str-Dis-Q-TOP**.

□

By Proposition 4.4.1, it is clear that **Str-Dis-Q-TOP** is a coreflective subcategory of **Str-Q-TOP**. Thus we have the following result:

Theorem 4.4.2. The coreflective hull of **Str-Dis-Q-TOP** in the category **Str-Q-TOP** is **Str-Dis-Q-TOP**.

Str-IG-Q-**TOP** will denote the full subcategory of **Str**-Q-**TOP** consisting of coproducts of stratified indiscrete Q-topological spaces.

Lemma 4.4.3. **Str-IG**-Q-**TOP** is an isomorphism closed subcategory of **Str**-Q-**TOP**.

Proof. Let $(Y,\sigma) \in$ **Str-IG**-Q-**TOP** and let $\{u_k : (Y_k,\sigma_k) \to (Y,\sigma) \mid k \in K\}$ be a coproduct in **Str**-Q-**TOP**, where each (Y_k,σ_k) is a stratified indiscrete Q-topological space. Next let $h : (Y,\sigma) \to (Z,\eta)$ be an isomorphism in **Str**-Q-**TOP**. Then it is easy to prove that $\{h \circ u_k : (Y_k,\sigma_k) \to (Z,\eta) \mid k \in K\}$ is a coproduct in **Str**-Q-**TOP** and so $(Z,\eta) \in$ **Str-IG**-Q-**TOP**. Therefore **Str-IG**-Q-**TOP** is an isomorphism closed subcategory of **Str**-Q-**TOP**.

□

Lemma 4.4.4. Let $\{(Z_k,\eta_k) \mid k \in K\}$ be a non-empty family of stratified indiscrete Q-topological spaces and let (Z,η) be a stratified Q-topological space such that $Z = \bigcup_{k\in K} Z_k$ and for every $\alpha \in Q^Z, \alpha \in \eta$ if and only if $\alpha|_{Z_k} \in \eta_k$, for every $k \in K$. Then $(Z,\eta) \in$ **Str-IG**-Q-**TOP**.

Proof. If Z_k's, $k \in K$ are mutually disjoint then it can be easily proved that $\{l_k : (Z_k,\eta_k) \to (Z,\eta) \mid k \in K\}$, where $l_k : Z_k \to Z$ is the inclusion map, is a coproduct in **Str**-Q-**TOP** and hence $(Z,\eta) \in$ **Str-IG**-Q-**TOP**. Now if some Z_k's, $k \in K$ are not mutually disjoint, then we will first prove that there exists at least one maximal sized subset J of K, $|J| \geq 2$, such that for every $\alpha \in \eta$, there exist $q_\alpha \in Q$ and $\alpha|_{Z_j} = \underline{q_\alpha}$, for every $j \in J$. Since Z_k's, $k \in K$ are not mutually disjoint, there exist $k_1, k_2 (k_1 \neq k_2) \in K$ such that $Z_{k_1} \cap Z_{k_2} \neq \emptyset$. Let $\alpha \in \eta$. Then $\alpha|_{Z_{k_1}} \in \eta_{k_1}$ and $\alpha|_{Z_{k_2}} \in \eta_{k_2}$. Now since (Z_{k_1},η_{k_1}) and (Z_{k_2},η_{k_2}) both are stratified indiscrete Q-topological spaces, $\alpha|_{Z_{k_1}}$ and $\alpha|_{Z_{k_2}}$ both are constant maps and since $Z_{k_1} \cap Z_{k_2} \neq \emptyset$, $\alpha|_{Z_{k_1}} = \alpha|_{Z_{k_2}}$. So, there exists $q_\alpha \in Q$ such that $\alpha|_{Z_{k_1}} = \alpha|_{Z_{k_2}} = \underline{q_\alpha}$. Thus for every $\alpha \in \eta$, there exist $q_\alpha \in Q$ such that $\alpha|_{Z_{k_1}} = \alpha|_{Z_{k_2}} = \underline{q_\alpha}$. Now let $J = \{k \in K \mid \alpha|_{Z_k} = \underline{q_\alpha}$ for every $\alpha \in \eta\}$. Then $|J| \geq 2$ since $k_1, k_2 \in J$ and clearly J is maximal sized subset of K such that for every $\alpha \in \eta$, $\alpha|_{Z_j} = \underline{q_\alpha}$, for every $j \in J$. Let $\{J_i \mid i \in I\}$ be the set of all such maximal sized subsets of K and let $K_0 = K \setminus (\bigcup_{i\in I} J_i)$ and for each J_i, let $Z_{J_i} = \bigcup_{j_i \in J_i} Z_{j_i}$. Let η_{J_i} be the stratified indiscrete Q-topology on Z_{J_i}. Now let $T = K_0 \cup I$ and let $\{(Y_t,\sigma_t) \mid t \in T\}$ be the family of Q-topological spaces such that for $t \in K_0$, (Y_t,σ_t) is (Z_t,η_t) and for $t \in I$, (Y_t,σ_t) is (Z_{J_t},η_{J_t}). Then it is easy to prove that $\{(Y_t,\sigma_t) \mid t \in T\}$ is a family of stratified indiscrete Q-topological spaces such that Z is disjoint union of Y_t's, $t \in T$ and for $\alpha \in Q^Z$, $\alpha \in \eta$ if and only if $\alpha|_{Y_t} \in \sigma_t$, for every $t \in T$. Then $\{l_t : (Y_t, \sigma_t) \to (Z, \eta) \mid t \in T\}$, where $l_t : Y_t \to Z$ is the inclusion map, is a coproduct in **Str-Q-TOP** and hence $(Z, \eta) \in$ **Str-IG-Q-TOP**. □

Proposition 4.4.5. The category **Str-IG-Q-TOP** is a coreflective subcategory of **Str-Q-TOP**.

Proof. By Theorem 4.3.9, to show that **Str-IG-Q-TOP** is a coreflective subcategory of **Str-Q-TOP**, it is sufficient to show that it is closed under the formation of coproducts and quotient Q-topological spaces. We first note that the coproduct of empty family of stratified Q-topological spaces in **Str-Q-TOP** is $\mathbb{I}$ and $\mathbb{I} \in$ **Str-IG-Q-TOP**. Now Let $\{(Y_k, \sigma_k) \mid k \in K\} \subseteq$ **Str-IG-Q-TOP**, $K \neq \emptyset$ and let $\{m_k : (Y_k, \sigma_k) \to (Y, \sigma) \mid k \in K\}$ be a coproduct of $\{(Y_k, \sigma_k) \mid k \in K\}$ in **Str-Q-TOP**. For each $k \in K$, let $\{u_k^j : (Y_k^j, \sigma_k^j) \to (Y_k, \sigma_k) \mid j \in J_k\}$ be a coproduct in the category **Str-Q-TOP**, where (Y_k^j, σ_k^j) is a stratified indiscrete Q-topological space, for every $j \in J_k$. Let $\{w_k^j : (Y_k^j, \sigma_k^j) \to (Z, \eta) \mid j \in J_k, k \in K\}$ be a coproduct of $\{(Y_k^j, \sigma_k^j) \mid j \in J_k, k \in K\}$ in **Str-Q-TOP**. Then by the definition of coproduct, for each $k \in K$, there exists a Q-continuous map $g_k : (Y_k, \sigma_k) \to (Z, \eta)$ such that $w_k^j = g_k \circ u_k^j$. Now we will prove that $\{g_k : (Y_k, \sigma_k) \to (Z, \eta) \mid k \in K\}$ is a coproduct in **Str-Q-TOP**. Let (X, τ) be a stratified Q-topological space and let $h_k : (Y_k, \sigma_k) \to (X, \tau)$ be Q-continuous maps. Now since $\{w_k^j : (Y_k^j, \sigma_k^j) \to (Z, \eta) \mid j \in J_k, k \in K\}$ is a coproduct in **Str-Q-TOP**, there exists a unique Q-continuous map $f : (Z, \eta) \to (X, \tau)$ such that $f \circ w_k^j = h_k \circ u_k^j$. This implies that $(f \circ g_k) \circ u_k^j = h_k \circ u_k^j$, for every $j \in J_k, k \in K$ (since $w_k^j = g_k \circ u_k^j$). Then since for each $k \in K$, $\{u_k^j : (Y_k^j, \sigma_k^j) \to (Y_k, \sigma_k) \mid j \in J_k\}$ is a coproduct in **Str-Q-TOP**, $f \circ g_k = h_k$, for every $k \in K$. Further, we can easily prove that the map $f : (Z, \eta) \to (X, \tau)$ is the unique Q-continuous map such that $f \circ g_k = h_k$, for every $k \in K$. Hence $\{g_k : (Y_k, \sigma_k) \to (Z, \eta) \mid k \in K\}$ is a coproduct in **Str-Q-TOP**. Now since $\{m_k : (Y_k, \sigma_k) \to (Y, \sigma) \mid k \in K\}$ and $\{g_k : (Y_k, \sigma_k) \to (Z, \eta) \mid k \in K\}$ both are coproducts in **Str-Q-TOP**, there exists an isomorphism $h : (Z, \eta) \to (Y, \sigma)$. Now since the category **Str-IG-Q-TOP** is an isomorphism closed subcategory of **Str-Q-TOP** and $(Z, \eta) \in$ **Str-IG-Q-TOP**, $(Y, \sigma) \in$ **Str-IG-Q-TOP**. Next, we will show that **Str-IG-Q-TOP** is closed under the quotient Q-topological spaces. Let $(Y, \sigma) \in$ **Str-IG-Q-TOP** and let $f : (Y, \sigma) \to (Z, \eta)$ be a quotient map. We have to show that $(Z, \eta) \in$ **Str-IG-Q-TOP**. If (Y, σ) is $\mathbb{I}$, then since quotient

Q-topological space of $\mathbb{I}$ is $\mathbb{I}$, (Z, η) is $\mathbb{I}$ and in this case (Z, η) belongs to **Str-IG-**Q-**TOP**. Now suppose that $(Y, \sigma) \neq \mathbb{I}$ and let $\{u_k : (Y_k, \sigma_k) \to (Y, \sigma) \mid k \in K\}$ be a coproduct in **Str**-Q-**TOP**, where $K \neq \emptyset$ and each (Y_k, σ_k) is an stratified indiscrete Q-topological space. Now since $\{in_k : (Y_k, \sigma_k) \to (\bigcup_{k\in K} Y_k \times \{k\}, \sigma^+) \mid k \in K\}$ is also a coproduct of $\{(Y_k, \sigma_k) \mid k \in K\}$ in **Str**Q-**TOP**, there exists an isomorphism $g : (\bigcup_{k\in K} Y_k \times \{k\}, \sigma^+) \to (Y, \sigma)$. Let $h = f \circ g$. Thus we have a quotient map $h : (\bigcup_{k\in K} Y_k \times \{k\}, \sigma^+) \to (Z, \eta)$. Let $Z_k = h(Y_k \times \{k\})$ and let η_k be the stratified indiscrete Q-topology on Z_k, for every $k \in K$. Then since h is onto, $Z = \bigcup_{k\in K} Z_k$. Now, let $\alpha \in \eta$, then $\alpha \circ h \in \sigma^+$. This implies that $(\alpha \circ h)|_{Y_k \times \{k\}} \in \sigma_k^*$, for every $k \in K$, i.e. $(\alpha \circ h)|_{Y_k \times \{k\}}$ is a constant map, for every $k \in K$. Thus $\alpha|_{Z_k}$ is a constant map, for every $k \in K$ and hence $\alpha|_{Z_k} \in \eta_k$, for every $k \in K$. Next, let $\alpha \in Q^Z$ such that $\alpha|_{Z_k} \in \eta_k$, for every $k \in K$. Then it is easy to prove that $\alpha \circ h \in \sigma^+$ and hence $\alpha \in \eta$. Thus we have a non-empty family $\{(Z_k, \eta_k) \mid k \in K\}$ of stratified indiscrete Q-topological spaces such that $Z = \bigcup_{k\in K} Z_k$ and for $\alpha \in Q^Z, \alpha \in \eta$ if and only if $\alpha|_{Z_k} \in \eta_k$, for every $k \in K$. Then by Lemma 4.4.4, $(Z, \eta) \in$ **Str-IG-**Q-**TOP**. Hence **Str-IG-**Q-**TOP** is a coreflective subcategory of **Str**-Q-**TOP**. □

Theorem 4.4.6. The coreflective hull of **Str-Ind-**Q-**TOP** in the category **Str-**Q-**TOP** is **Str-IG-**Q-**TOP**.

Proof. By Proposition 4.4.5, **Str-IG-**Q-**TOP** is a coreflective subcategory of **Str-**Q-**TOP** containing **Str-Ind-**Q-**TOP**. Hence the coreflective hull of **Str-Ind-**Q-**TOP** in **Str-**Q-**TOP** is a subcategory of **Str-IG-**Q-**TOP**. Now since the coreflective hull of **Str-Ind-**Q-**TOP** in **Str-**Q-**TOP** is a coreflective subcategory of **Str-**Q-**TOP**, by Theorem 4.3.9, it contains coproducts of indiscrete Q-topological spaces. Hence **Str-IG-**Q-**TOP** is a subcategory of the coreflective hull of **Str-Ind-**Q-**TOP** in **Str-**Q-**TOP**. Therefore **Str-IG-**Q-**TOP** is the coreflective hull of **Str-Ind-**Q-**TOP** in the category **Str-**Q-**TOP**.

□

4.5 Conclusion

In this chapter, we have determined the coreflective hull of $(Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle)$ in the category **Str-**Q-**TOP** of stratified Q-topological spaces. We have also determined the coreflective hulls of the categories **Str-Dis-**Q-**TOP** of discrete

Q-topological spaces and **Str-Ind**-Q-**TOP** of stratified indiscrete Q-topological spaces in the category **Str**-Q-**TOP**.