Chapter 3

On injective objects and existence of injective hulls in Q-TOP$/(Y, \sigma)$

3.1 Introduction

Cagliari and Mantovani [7] gave a characterization of injective objects (with respect to the class of embeddings in the category **TOP** of topological spaces) in the comma category **TOP**$/B$. In [9], we can find a result related to the existence of an injective hull of an object in the comma category $\mathbf{TOP}_0/B$ ($B \in \mathbf{TOP}_0$ and $\mathbf{TOP}_0$ is the category of T_0-topological spaces).

It is well known that the category $\mathbf{TOP}_0$ is a reflective subcategory of the category **TOP**. In [8] (see also [42]), Cagliari and Mantovani considered the reflector $\pi : \mathbf{TOP} \to \mathbf{TOP}_0$ and they mentioned that for any topological space Y, by virtue of the reflector π, corresponding to each object (X, f) of the comma category **TOP**$/Y$ we have the object (X_0, f_0), where $X_0 = \pi(X)$ and $f_0 = \pi(f)$, of the category $\mathbf{TOP}_0/Y_0$, which is called as the T_0-reflection of (X, f). Cagliari and Mantovani [8] gave a characterization of injective objects (with respect to the class of embeddings in **TOP**) in the comma category **TOP**$/Y$ with the help of their T_0-reflection. Cagliari and Mantovani [8] also proved that the existence of an injective hull of (X, f) in the comma category **TOP**$/Y$ is equivalent to the existence of an injective hull of its T_0-reflection (X_0, f_0) in the comma category $\mathbf{TOP}/Y_0$ (and in the comma category $\mathbf{TOP}_0/Y_0$).

As mentioned in the first chapter of the thesis, Solovyov [36] introduced the notion of Q-topological spaces and Q-continuous maps and studied the category Q-**TOP** of Q-topological spaces (where Q is a fixed member of a fixed variety of Ω-algebras). Solovyov [36] also introduced the concept of stratified Q-topological spaces and T_0-Q-topological spaces. Singh and Srivastava [32] proved that the category Q-$\mathbf{TOP_0}$ of T_0-Q-topological spaces is a reflective subcategory of Q-**TOP**. In [32], for a given Q-topological space (X,τ), Singh and Srivastava defined an equivalence relation $\sim$ on X as, for every $x_1, x_2 \in X$, $x_1 \sim x_2$ if $\alpha(x_1) = \alpha(x_2)$, for every $\alpha \in \tau$. By taking $\tilde{X} = X/\sim$, the set of equivalence classes, and $\tilde{\tau}$ to be the corresponding quotient Q-topology on $\tilde{X}$ induced by the quotient map $q_X : X \to \tilde{X}$, $q_X(x) = [x]$ (where $[x]$ is the equivalence class of x), and τ, they proved that $q_X : (X,\tau) \to (\tilde{X},\tilde{\tau})$ is a Q-$\mathbf{TOP_0}$-reflection for (X,τ) and as a result of this Q-$\mathbf{TOP_0}$ is a reflective subcategory of Q-**TOP** (cf. Theorem 4.1 in [32]). Consequently, we have the reflector (cf. Proposition 4.22 and Definition 4.23 in [1]) $R : Q$-**TOP**$\to Q$-$\mathbf{TOP_0}$ give by $R((X,\tau)) = (\tilde{X},\tilde{\tau})$ and if $f : (X,\tau) \to (Y,\sigma)$ is a Q-continuous map, then $R(f) = \tilde{f}$, where $\tilde{f} : (\tilde{X},\tilde{\tau}) \to (\tilde{Y},\tilde{\sigma})$ is the unique Q-continuous map such that $q_Y \circ f = \tilde{f} \circ q_X$. Thus for a given Q-topological space (Y,σ), corresponding to each object $((X,\tau), f)$ of the comma category Q-**TOP**$/(Y,\sigma)$, we have the object $((\tilde{X},\tilde{\tau}),\tilde{f})$ of the comma category Q-$\mathbf{TOP_0}/(\tilde{Y},\tilde{\sigma})$, which is called as the T_0-reflection of $((X,\tau), f)$.

Motivated by Cagliari and Mantovani [8], in this chapter, we have obtained a characterization of injective objects (with respect to the class of embeddings in Q-**TOP**) in the comma category Q-**TOP**$/(Y,\sigma)$, when (Y,σ) is a stratified Q-topological space, with the help of their T_0-reflection. Further, we have proved that for any Q-topological space (Y,σ), the existence of an injective hull of $((X,\tau), f)$ in the comma category Q-**TOP**$/(Y,\sigma)$ is equivalent to the existence of an injective hull of its T_0-reflection $((\tilde{X},\tilde{\tau}),\tilde{f})$ in the comma category Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$ (and in the comma category Q-$\mathbf{TOP_0}/(\tilde{Y},\tilde{\sigma})$).

3.2 Comma category, injective objects and injective hulls

Definition 3.2.1. [1] Let **C** be a category and B be an object of **C**. Then objects of the **comma category C**$/B$ are pairs (X, f), where X is a **C**-object and $f : X \to B$ is a **C**-morphism. Given any two objects (X, f) and (Y, g) of $\mathbf{C}/B$, a **$\mathbf{C}/B$-morphism** $h : (X, f) \to (Y, g)$ is a **C**-morphism $h : X \to Y$ such that $g \circ h = f$.

Let $\mathcal{H}$ be a class of morphisms in a category **C**.

Definition 3.2.2. [8] An object I is **$\mathcal{H}$-injective** if for all $h : X \to Y$ in $\mathcal{H}$ and a morphism $f : X \to I$, there exists a morphism $g : Y \to I$ such that $g \circ h = f$.

Definition 3.2.3. [8] A morphism $h : X \to I$ in $\mathcal{H}$ is **$\mathcal{H}$-essential** if for every morphism $k : I \to Y$, the composite $k \circ h : X \to Y$ lies in $\mathcal{H}$ only if $k : I \to Y$ does; if, in addition, I is $\mathcal{H}$-injective, then $h : X \to I$ is an **$\mathcal{H}$-injective hull** of X.

Definition 3.2.4. [8] An object (X, f) of the comma category $\mathbf{C}/B$ is said to be **$\mathcal{H}$-injective** if for any commutative diagram in **C**

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[v]{} & B \end{array} \tag{3.2.1}$$

with $h : U \to V$ in $\mathcal{H}$, there exists a morphism $s : V \to X$ for which the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle f} \\ V & \xrightarrow[v]{} & B \end{array} \tag{3.2.2}$$

Definition 3.2.5. [8] A $\mathbf{C}/B$-morphism $j : (Y, g) \to (X, f)$ with $j : Y \to X$ in $\mathcal{H}$ is said to be **$\mathcal{H}$-essential** if for any $\mathbf{C}/B$-morphism $k : (X, f) \to (Z, h)$ such that $k \circ j : Y \to Z$ is in $\mathcal{H}$, necessarily $k : X \to Z$ is in $\mathcal{H}$ follows; if in addition (X, f) is $\mathcal{H}$-injective, then $j : (Y, g) \to (X, f)$ is said to be an **$\mathcal{H}$-injective hull** of (Y, g) in $\mathbf{C}/B$.

From now onwards, injective, essential, injective hull in Q-**TOP** (Q-$\mathbf{TOP}_0$) and in any comma category Q-$\mathbf{TOP}/(Y, \sigma)$ (Q-$\mathbf{TOP}_0/(Z, \eta)$) will denote respectively $\mathcal{H}$-injective, $\mathcal{H}$-essential and $\mathcal{H}$-injective hull for $\mathcal{H}$ the class of embeddings in Q-**TOP** (Q-$\mathbf{TOP}_0$).

3.3 T_0-reflection

Let (X, τ) be a Q-topological space. Singh and Srivastava [32] defined a relation $\sim$ on X as, for every $x_1, x_2 \in X$, $x_1 \sim x_2$ if $\alpha(x_1) = \alpha(x_2)$, for every $\alpha \in \tau$. Then it can be easily proved that $\sim$ is an equivalence relation on X. Let $\tilde{X} = X/\sim$, the set of equivalence classes, and let $q_X : X \to \tilde{X}$ be defined as, $q_X(x) = [x]$, for every $x \in X$, where $[x]$ is the equivalence class of x. Let $\tilde{\tau} = \{\beta \in Q^{\tilde{X}} \mid \beta \circ q_X \in \tau\}$. Then $(\tilde{X}, \tilde{\tau})$ is a T_0-Q-topological space. It can also be easily verified that for a given T_0-Q-topological space (Z, η) and a Q-continuous map $f : (X, \tau) \to (Z, \eta)$, there exists a unique Q-continuous map $f' : (\tilde{X}, \tilde{\tau}) \to (Z, \eta)$ such that $f' \circ q_X = f$. Hence $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ is a Q-$\mathbf{TOP}_0$-reflection for (X, τ) and as a result of this Q-$\mathbf{TOP}_0$ is a reflective subcategory of Q-**TOP** (cf. Theorem 4.1 in [32]). Consequently, we have the reflector (cf. Proposition 4.22 and Definition 4.23 in [1]) $R : Q$-**TOP**$\to Q$-$\mathbf{TOP}_0$ give by $R((X, \tau)) = (\tilde{X}, \tilde{\tau})$ and if $f : (X, \tau) \to (Y, \sigma)$ is a Q-continuous map, then $R(f) = \tilde{f}$, where $\tilde{f} : (\tilde{X}, \tilde{\tau}) \to (\tilde{Y}, \tilde{\sigma})$ is the unique Q-continuous map such that the following diagram commutes:

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{f} & (Y,\sigma) \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ (\tilde{X},\tilde{\tau}) & \xrightarrow[\tilde{f}]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{3.3.1}$$

Thus corresponding to each object $((X, \tau), f)$ of the category Q-**TOP**$/(Y, \sigma)$, we have the object $((\tilde{X}, \tilde{\tau}), \tilde{f})$ of the category Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$. $((\tilde{X}, \tilde{\tau}), \tilde{f})$ is called the T_0-reflection of $((X, \tau), f)$.

We mention here that if (X, τ) is a T_0-Q-topological space, then $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ is an isomorphism in Q-**TOP**.

Proposition 3.3.1. Let (X, τ) be a Q-topological space. Then $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ is initial and final in Q-**TOP**.

Proof. By the definition of $\tilde{\tau}$, it follows that $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ is final. Now let $\alpha \in \tau$. Define $\beta : \tilde{X} \to Q$ as $\beta([x]) = \alpha(x)$. Then it can be easily proved that β is well defined and $\beta \circ q_X = \alpha$. Thus $\beta \circ q_X \in \tau$ and this implies that $\beta \in \tilde{\tau}$. Thus $\alpha = \beta \circ q_X$, where $\beta \in \tilde{\tau}$. Therefore $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ is initial in Q-**TOP**. $\square$

Proposition 3.3.2. Let (X,τ) and (Y,σ) be Q-topological spaces. A Q-continuous map $f:(X,\tau)\to(Y,\sigma)$ is an embedding in Q-**TOP** if and only if f is one-one and $\tilde{f}:(\tilde{X},\tilde{\tau})\to(\tilde{Y},\tilde{\sigma})$ is an embedding in Q-**TOP**.

Proof. Suppose first that the map $f:(X,\tau)\to(Y,\sigma)$ is an embedding in Q-**TOP**. Then f is one-one and $f:(X,\tau)\to(Y,\sigma)$ is initial in Q-**TOP**. Now we have to prove that $\tilde{f}:(\tilde{X},\tilde{\tau})\to(\tilde{Y},\tilde{\sigma})$ is an embedding in Q-**TOP**. Let $\tilde{f}([x_1])=\tilde{f}([x_2])\Rightarrow(\tilde{f}\circ q_X)(x_1)=(\tilde{f}\circ q_X)(x_2)\Rightarrow(q_Y\circ f)(x_1)=(q_Y\circ f)(x_2)\Rightarrow[f(x_1)]=[f(x_2)]\Rightarrow u(f(x_1))=u(f(x_2))$, for every $u\in\sigma\Rightarrow(u\circ f)(x_1)=(u\circ f)(x_2)$, for every $u\in\sigma\Rightarrow[x_1]=[x_2]$ (as $f:(X,\tau)\to(Y,\sigma)$ is initial, so $\tau=\{u\circ f\mid u\in\sigma\}$) $\Rightarrow\tilde{f}$ is one-one. Now let $\beta\in\tilde{\tau}$, then $\beta\circ q_X\in\tau$ and so $\beta\circ q_X=u\circ f$, for some $u\in\sigma$. Also $u=v\circ q_Y$, for some $v\in\tilde{\sigma}$. Thus $\beta\circ q_X=u\circ f=v\circ q_Y\circ f=v\circ(q_Y\circ f)=v\circ(\tilde{f}\circ q_X)=(v\circ\tilde{f})\circ q_X\Rightarrow\beta=v\circ\tilde{f}$ (as q_X is onto). Hence $\tilde{f}:(\tilde{X},\tilde{\tau})\to(\tilde{Y},\tilde{\sigma})$ is initial in Q-**TOP**. Therefore $\tilde{f}:(\tilde{X},\tilde{\tau})\to(\tilde{Y},\tilde{\sigma})$ is an embedding in Q-**TOP**.

Conversely, suppose that f is one-one and $\tilde{f}:(\tilde{X},\tilde{\tau})\to(\tilde{Y},\tilde{\sigma})$ is an embedding in Q-**TOP**. We have to show that $f:(X,\tau)\to(Y,\sigma)$ is an embedding in Q-**TOP**. Since f is one-one, it is sufficient to show that $f:(X,\tau)\to(Y,\sigma)$ is initial. Let $\alpha\in\tau$, then $\alpha=\beta\circ q_X$, for some $\beta\in\tilde{\tau}$. Then since $\tilde{f}:(\tilde{X},\tilde{\tau})\to(\tilde{Y},\tilde{\sigma})$ is initial and $\beta\in\tilde{\tau}$, $\beta=v\circ\tilde{f}$, for some $v\in\tilde{\sigma}$. So $\alpha=\beta\circ q_X=v\circ\tilde{f}\circ q_X=v\circ(\tilde{f}\circ q_X)=v\circ(q_Y\circ f)=(v\circ q_Y)\circ f=u\circ f$, where $u=v\circ q_Y\in\sigma$. Thus $\alpha=u\circ f$, where $u\in\sigma$. Hence $f:(X,\tau)\to(Y,\sigma)$ is initial in Q-**TOP**. Therefore $f:(X,\tau)\to(Y,\sigma)$ is an embedding in Q-**TOP**.

□

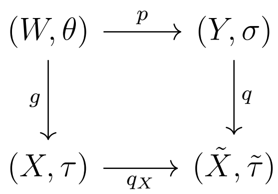
In view of Proposition 1.2.23, we have the following result:

Proposition 3.3.3. Let (X,τ) and (Y,σ) be Q-topological spaces and let $f:(X,\tau)\to(Y,\sigma)$ be an initial map in Q-**TOP** such that f is bijective. Then $f:(X,\tau)\to(Y,\sigma)$ is an isomorphism in Q-**TOP**.

Proposition 3.3.4. ([1], Proposition 11.18) In any category, monomorphisms, regular monomorphisms and retractions are pullback stable.

Remark 3.3.5. We know that Q-**TOP** is a topological category over **Set** (cf. Theorem 1.2.42) and since the category **Set** is complete, Q-**TOP** is complete (cf. Definition 12.2 and Corollary 21.17 in [1]). Hence by Theorem 12.4 in [1], it follows that Q-**TOP** has pullbacks.

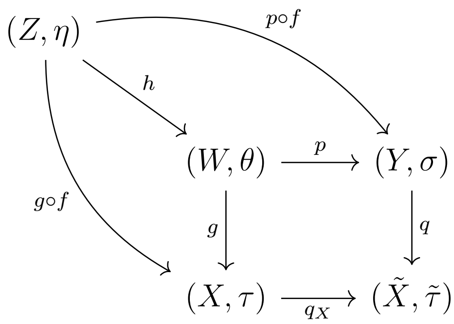
Proposition 3.3.6. Let (X,τ) and (Y,σ) be Q-topological spaces and let $q : (Y,\sigma) \to (\tilde{X},\tilde{\tau})$ be a Q-continuous map. Let $p : (W,\theta) \to (Y,\sigma)$ be a pullback of $q_X : (X,\tau) \to (\tilde{X},\tilde{\tau})$ along $q : (Y,\sigma) \to (\tilde{X},\tilde{\tau})$ in the category Q-**TOP**



(3.3.2)

Then $\tilde{p} : (\tilde{W},\tilde{\theta}) \to (\tilde{Y},\tilde{\sigma})$ is an isomorphism in Q-**TOP**.

Proof. First we will prove that the map $p : (W,\theta) \to (Y,\sigma)$ is initial in Q-**TOP**. Let (Z,η) be a Q-topological space and let $f : Z \to W$ be a map such that $p \circ f : (Z,\eta) \to (Y,\sigma)$ is Q-continuous. Then, $q_X \circ (g \circ f) = q \circ (p \circ f)$. So $q_X \circ (g \circ f) : (Z,\eta) \to (\tilde{X},\tilde{\tau})$ is Q-continuous, but since $q_X : (X,\tau) \to (\tilde{X},\tilde{\tau})$ is initial in Q-**TOP**, $g \circ f : (Z,\eta) \to (X,\tau)$ is Q-continuous. Now since the diagram 3.3.2 is a pullback, there exists a unique Q-continuous map $h : (Z,\eta) \to (W,\theta)$ such that the following diagram commutes:



(3.3.3)

Now if $h \neq f$, then if we consider the diagram 3.3.3 in the category **Set**, then in **Set** we have two maps $f, h : Z \to W$ for which the diagram 3.3.3 commutes, but this will be a contradiction because the diagram 3.3.2, if considered in the category **Set**, is a pullback square in **Set** also. Thus $h = f$ and hence $f : (Z,\eta) \to (W,\theta)$ is Q-continuous. Therefore $p : (W,\theta) \to (Y,\sigma)$ is initial in Q-**TOP**. Now consider the following commutative diagram in Q-**TOP**:

$$\begin{array}{ccc} (W,\theta) & \xrightarrow{p} & (Y,\sigma) \\ {\scriptstyle q_W}\downarrow & & \downarrow{\scriptstyle q_Y} \\ (\tilde{W},\tilde{\theta}) & \xrightarrow[\tilde{p}]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{3.3.4}$$

Since $p : (W,\theta) \to (Y,\sigma)$ is initial in Q-**TOP**, as in the proof of Proposition 3.3.2, we can prove that $\tilde{p} : (\tilde{W},\tilde{\theta}) \to (\tilde{Y},\tilde{\sigma})$ is initial in Q-**TOP**. Now since q_X is onto, i.e. q_X is a retraction in **Set** and since the diagram 3.3.2, if considered in the category **Set**, is a pullback square in **Set** also, by Proposition 3.3.4, p is a retraction in **Set**, i.e. p is onto. Since $q_Y \circ p = \tilde{p} \circ q_W$ and both p and q_Y are onto, $\tilde{p} \circ q_W$ is onto. This implies that $\tilde{p}$ is onto. Next, we will prove that $\tilde{p}$ is one-one. Let $\tilde{p}([w_1]) = \tilde{p}([w_2]) \Rightarrow (\tilde{p} \circ q_W)(w_1) = (\tilde{p} \circ q_W)(w_2) \Rightarrow (q_Y \circ p)(w_1) = (q_Y \circ p)(w_2) \Rightarrow [p(w_1)] = [p(w_2)] \Rightarrow u(p(w_1)) = u(p(w_2))$, for every $u \in \sigma \Rightarrow (u \circ p)(w_1) = (u \circ p)(w_2)$, for every $u \in \sigma \Rightarrow [w_1] = [w_2]$ (since $\theta = \{u \circ p \mid u \in \sigma\}$ as $p : (W,\theta) \to (Y,\sigma)$ is initial in Q-**TOP**). This implies that $\tilde{p}$ is one-one. Thus $\tilde{p} : (\tilde{W},\tilde{\theta}) \to (\tilde{Y},\tilde{\sigma})$ is initial in Q-**TOP** and $\tilde{p}$ is bijective. Therefore by Proposition 3.3.3, $\tilde{p} : (\tilde{W},\tilde{\theta}) \to (\tilde{Y},\tilde{\sigma})$ is an isomorphism in Q-**TOP**. □

3.4 A characterization of injective objects in Q-TOP$/(Y,\sigma)$

Proposition 3.4.1. [2] Let B be a set. Then injective objects (with respect to the class of injective maps in the category **Set**) in the comma category **Set**$/B$ are precisely surjective maps over B.

Proposition 3.4.2. Let (X,τ) and (Y,σ) be Q-topological spaces and let $f : (X,\tau) \to (Y,\sigma)$ be an initial map in Q-**TOP** such that f is onto, then $((X,\tau), f)$ is injective in Q-**TOP**$/(Y,\sigma)$.

Proof. Let following be a commutative square in Q-**TOP**:

$$\begin{array}{ccc} (W,\theta) & \xrightarrow{l} & (X,\tau) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ (Z,\eta) & \xrightarrow[g]{} & (Y,\sigma) \end{array} \tag{3.4.1}$$

where $h:(W,\theta)\to(Z,\eta)$ is an embedding in Q-**TOP**.

Now since f is onto, by Proposition 3.4.1, (X,f) is injective in the comma category **Set**$/Y$. So there exists a function $k:Z\to X$ such that $k\circ h=l$ and $f\circ k=g$. Now let $\alpha\in\tau$, then $\alpha=\beta\circ f$, for some $\beta\in\sigma$ as $f:(X,\tau)\to(Y,\sigma)$ is initial in Q-**TOP**. Then $\alpha\circ k=\beta\circ f\circ k=\beta\circ(f\circ k)=\beta\circ g\in\eta$ as $g:(Z,\eta)\to(Y,\sigma)$ is Q-continuous. Thus $k:(Z,\eta)\to(X,\tau)$ is Q-continuous. Hence we have a Q-continuous map $k:(Z,\eta)\to(X,\tau)$ such that $k\circ h=l$ and $f\circ k=g$. Therefore $((X,\tau),f)$ is injective in Q-**TOP**$/(Y,\sigma)$. □

Corollary 3.4.3. Let (X,τ) be a Q-topological space, then $((X,\tau),q_X)$ is injective in Q-**TOP**$/(\tilde{X},\tilde{\tau})$.

Proof. It immediately follows from Proposition 3.3.1 and Proposition 3.4.2. □

Proposition 3.4.4. Let $((X,\tau),f)$ be injective in Q-**TOP**$/(Y,\sigma)$ and $((Y,\sigma),g)$ be injective in Q-**TOP**$/(Z,\eta)$. Then $((X,\tau),g\circ f)$ is injective in Q-**TOP**$/(Z,\eta)$.

Proof. Let following be a commutative diagram in Q-**TOP**:

$$\begin{array}{ccc} (A,\tau_A) & \xrightarrow{\ l\ } & (X,\tau) \\ & & \downarrow f \\ k\downarrow & & (Y,\sigma) \\ & & \downarrow g \\ (B,\tau_B) & \xrightarrow[h]{} & (Z,\eta) \end{array} \tag{3.4.2}$$

where $k:(A,\tau_A)\to(B,\tau_B)$ is an embedding in Q-**TOP**.

Now since $((Y,\sigma),g)$ is injective in Q-**TOP**$/(Z,\eta)$, there exists a Q-continuous map $s:(B,\tau_B)\to(Y,\sigma)$ such that the following diagram commutes:

$$\begin{array}{ccc} (A,\tau_A) & \xrightarrow{f\circ l} & (Y,\sigma) \\ k\downarrow & \nearrow s & \downarrow g \\ (B,\tau_B) & \xrightarrow[h]{} & (Z,\eta) \end{array} \tag{3.4.3}$$

Next since $((X,\tau),f)$ is injective in Q-**TOP**$/(Y,\sigma)$, there exists a Q-continuous map $s':(B,\tau_B)\to(X,\tau)$ such that the following diagram commutes:

$$\begin{array}{ccc} (A,\tau_A) & \xrightarrow{\ l\ } & (X,\tau) \\ {\scriptstyle k}\downarrow & \overset{s'}{\nearrow} & \downarrow{\scriptstyle f} \\ (B,\tau_B) & \xrightarrow[s]{} & (Y,\sigma) \end{array} \tag{3.4.4}$$

Then $s' \circ k = l$ and $g \circ f \circ s' = g \circ (f \circ s') = g \circ s = h$. Therefore $((X,\tau), g \circ f)$ is injective in Q-**TOP**$/(Z,\eta)$. □

The following Lemma 3.4.5 can be easily verified:

Lemma 3.4.5. Let (X,τ) and (Y,σ) be T_0-Q-topological spaces and let $f : (X,\tau) \to (Y,\sigma)$ be a Q-continuous map. Then $f : (X,\tau) \to (Y,\sigma)$ is an embedding in Q-**TOP** if and only if it is an embedding in Q-$\mathbf{TOP}_0$.

Proposition 3.4.6. Let (X,τ) and (Y,σ) be T_0-Q-topological spaces and let $f : (X,\tau) \to (Y,\sigma)$ be a Q-continuous map. Then $((X,\tau), f)$ is injective in Q-**TOP**$/(Y,\sigma)$ if and only if it is injective in Q-$\mathbf{TOP}_0/(Y,\sigma)$.

Proof. Suppose first that $((X,\tau), f)$ is injective in Q-**TOP**$/(Y,\sigma)$. Let following be a commutative diagram in Q-$\mathbf{TOP}_0$:

$$\begin{array}{ccc} (A,\tau_A) & \xrightarrow{\ l\ } & (X,\tau) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ (B,\tau_B) & \xrightarrow[k]{} & (Y,\sigma) \end{array} \tag{3.4.5}$$

where $h : (A,\tau_A) \to (B,\tau_B)$ is an embedding in Q-$\mathbf{TOP}_0$.
Then by Lemma 3.4.5, $h : (A,\tau_A) \to (B,\tau_B)$ is an embedding in Q-**TOP** and then since $((X,\tau), f)$ is injective in Q-**TOP**$/(Y,\sigma)$, there exists a Q-continuous map $s : (B,\tau_B) \to (X,\tau)$ such that $s \circ h = l$ and $f \circ s = k$. Therefore $((X,\tau), f)$ is injective in Q-$\mathbf{TOP}_0/(Y,\sigma)$.

Conversely, assume that $((X,\tau), f)$ is injective in Q-$\mathbf{TOP}_0/(Y,\sigma)$. Let following be a commutative diagram in Q-**TOP**:

$$\begin{array}{ccc} (W,\theta) & \xrightarrow{\ g\ } & (X,\tau) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ (Z,\eta) & \xrightarrow[k]{} & (Y,\sigma) \end{array} \tag{3.4.6}$$

where $h : (W, \theta) \to (Z, \eta)$ is an embedding in Q-**TOP**.

We note that since (X, τ) and (Y, σ) are T_0-Q-topological spaces, $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ and $q_Y : (Y, \sigma) \to (\tilde{Y}, \tilde{\sigma})$ are isomorphisms in Q-**TOP**. Now we will first prove that $((\tilde{X}, \tilde{\tau}), \tilde{f})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$. Let following be a commutative diagram in Q-$\mathbf{TOP}_0$:

$$\begin{tikzcd} (N,\tau_N) \arrow[r, "l"] \arrow[d, "m"'] & (\tilde{X},\tilde{\tau}) \arrow[d, "\tilde{f}"] \\ (M,\tau_M) \arrow[r, "n"'] & (\tilde{Y},\tilde{\sigma}) \end{tikzcd} \tag{3.4.7}$$

where $m : (M, \tau_M) \to (N, \tau_N)$ is an embedding in Q-$\mathbf{TOP}_0$.

Now since $((X, \tau), f)$ is injective in Q-$\mathbf{TOP}_0/(Y, \sigma)$, there exists a Q-continuous map $s : (M, \tau_M) \to (X, \tau)$ such that the following diagram commutes:

$$\begin{tikzcd} (N,\tau_N) \arrow[r, "l"] \arrow[d, "m"'] & (\tilde{X},\tilde{\tau}) \arrow[r, "q_X^{-1}"] & (X,\tau) \arrow[d, "f"] \\ (M,\tau_M) \arrow[r, "n"'] \arrow[urr, "s"] & (\tilde{Y},\tilde{\sigma}) \arrow[r, "q_Y^{-1}"'] & (Y,\sigma) \end{tikzcd} \tag{3.4.8}$$

Thus we have a Q-continuous map $q_X \circ s : (M, \tau_M) \to (\tilde{X}, \tilde{\tau})$ and it can be easily verified that the following is a commutative diagram in Q-$\mathbf{TOP}_0$:

$$\begin{tikzcd} (N,\tau_N) \arrow[r, "l"] \arrow[d, "m"'] & (\tilde{X},\tilde{\tau}) \arrow[d, "\tilde{f}"] \\ (M,\tau_M) \arrow[r, "n"'] \arrow[ur, "q_X \circ s"] & (\tilde{Y},\tilde{\sigma}) \end{tikzcd} \tag{3.4.9}$$

Thus $((\tilde{X}, \tilde{\tau}), \tilde{f})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$. Now since $h : (W, \theta) \to (Z, \eta)$ is an embedding in Q-**TOP**, by Proposition 3.3.2, $\tilde{h} : (\tilde{W}, \tilde{\theta}) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-**TOP** and then by Lemma 3.4.5, $\tilde{h} : (\tilde{W}, \tilde{\theta}) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-$\mathbf{TOP}_0$. Then since $((\tilde{X}, \tilde{\tau}), \tilde{f})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$, there exists a Q-continuous map $s' : (\tilde{Z}, \tilde{\eta}) \to (\tilde{X}, \tilde{\tau})$ such that the following diagram commutes:

$$\begin{array}{ccc} (\tilde{W},\tilde{\theta}) & \xrightarrow{\tilde{g}} & (\tilde{X},\tilde{\tau}) \\ {\scriptstyle \tilde{h}}\downarrow & \nearrow^{s'} & \downarrow{\scriptstyle \tilde{f}} \\ (\tilde{Z},\tilde{\eta}) & \xrightarrow[\tilde{k}]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{3.4.10}$$

Let $p = q_X^{-1} \circ s' \circ q_Z$. Then $p \circ h = q_X^{-1} \circ s' \circ q_Z \circ h = q_X^{-1} \circ s' \circ (q_Z \circ h) = q_X^{-1} \circ s' \circ (\tilde{h} \circ q_W) = q_X^{-1} \circ (s' \circ \tilde{h}) \circ q_W = q_X^{-1} \circ \tilde{g} \circ q_W = q_X^{-1} \circ (\tilde{g} \circ q_W) = q_X^{-1} \circ (q_X \circ g) = g$ and $f \circ p = f \circ q_X^{-1} \circ s' \circ q_Z = (f \circ q_X^{-1}) \circ s' \circ q_Z = (q_Y^{-1} \circ \tilde{f}) \circ s' \circ q_Z = q_Y^{-1} \circ (\tilde{f} \circ s') \circ q_Z = q_Y^{-1} \circ \tilde{k} \circ q_Z = q_Y^{-1} \circ (\tilde{k} \circ q_Z) = q_Y^{-1} \circ (q_Y \circ k) = k$. Thus we have a Q-continuous map $p : (Z, \eta) \to (X, \tau)$ such that $p \circ h = g$ and $f \circ p = k$. Therefore $((X, \tau), f)$ is injective in Q-**TOP**$/(Y, \sigma)$. □

Proposition 3.4.7. Let (X, τ) be a Q-topological space and (Y, σ) be a stratified Q-topological space. If $((X, \tau), f)$ is injective in Q-**TOP**$/(Y, \sigma)$, then $f : (X, \tau) \to (Y, \sigma)$ is a retraction in Q-**TOP**. In particular, for any $x \in X$ there exists a section $s_x : (Y, \sigma) \to (X, \tau)$ of $f : (X, \tau) \to (Y, \sigma)$ with $s_x(f(x)) = x$.

Proof. Consider the following commutative diagram in Q-**TOP**:

$$\begin{array}{ccc} (\{x\},\delta) & \xrightarrow{i_x} & (X,\tau) \\ {\scriptstyle f_x}\downarrow & & \downarrow{\scriptstyle f} \\ (Y,\sigma) & \xrightarrow[id_Y]{} & (Y,\sigma) \end{array} \tag{3.4.11}$$

where $\delta = \{\underline{q} \mid q \in Q\}$, $i_x : \{x\} \to X$ is the inclusion map and $f_x : \{x\} \to Y$ is defined as $f_x(x) = f(x)$.

It can be easily seen that $f_x : (\{x\}, \delta) \to (Y, \sigma)$ is an embedding in Q-**TOP**. Then since $((X, \tau), f)$ is injective in Q-**TOP**$/(Y, \sigma)$, there exists a Q-continuous map $s_x : (Y, \sigma) \to (X, \tau)$ such that $s_x \circ f_x = i_x$ and $f \circ s_x = id_Y$. Therefore $s_x(f(x)) = x$ and $f : (X, \tau) \to (Y, \sigma)$ is a retraction in Q-**TOP**.

□

Let (X, τ) be a Q-topological space and let $x \in X$. Consider the subset $\{x' \in X \mid x' \sim x\}$ of X (note that $\sim$ is the equivalence relation on X defined in the starting of the section 3). Clearly this subset gives the equivalence class $[x]$ of x. We will denote this subset of X by C_x where it is considered as a subset of X

and it will be denoted by $[x]$ where it is considered as an element of $\tilde{X}$, to avoid confusions.

Proposition 3.4.8. Let (X,τ) and (Y,σ) be Q-topological spaces and let $f : (X,\tau) \to (Y,\sigma)$ be a Q-continuous map. Then $f(C_x) \subseteq C_{f(x)}$, for every $x \in X$.

Proof. Let $y \in f(C_x)$, then $y = f(x')$, for some $x' \in C_x$. Now let $\beta \in \sigma$. Then since $f : (X,\tau) \to (Y,\sigma)$ is Q-continuous, $\beta \circ f \in \tau$ and then since $x' \sim x$, $(\beta \circ f)(x') = (\beta \circ f)(x) \Rightarrow \beta(f(x')) = \beta(f(x))$. Thus $f(x') \sim f(x)$. This implies that $y = f(x') \in C_{f(x)}$. Therefore $f(C_x) \subseteq C_{f(x)}$.

□

Proposition 3.4.9. ([1], Proposition 9.5) Every retract of an injective object is injective.

Proposition 3.4.10. Let (X,τ) and (Y,σ) be Q-topological spaces. If $((X,\tau),f)$ is injective in Q-$\mathbf{TOP}/(Y,\sigma)$, then $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$.

Proof. We know that in the category **Set**, retractions are precisely onto maps and since q_X is onto, q_X is a retraction in **Set**. So there exists a map $g : \tilde{X} \to X$ such that $q_X \circ g = id_{\tilde{X}}$. This implies that $(q_X \circ g)([x]) = [x]$, for every $x \in X \Rightarrow [g([x])] = [x]$, for every $x \in X \Rightarrow \alpha(g[x]) = \alpha(x)$, for every $\alpha \in \tau$ and for every $x \in X$. Now we will show that $g : (\tilde{X},\tilde{\tau}) \to (X,\tau)$ is Q-continuous. Let $\alpha \in \tau$. Then $((\alpha \circ g) \circ q_X)(x) = \alpha(g([x])) = \alpha(x)$, for every $x \in X$. Thus $(\alpha \circ g) \circ q_X = \alpha \in \tau$. Hence $\alpha \circ g \in \tilde{\tau}$. So $g : (\tilde{X},\tilde{\tau}) \to (X,\tau)$ is Q-continuous. Now $q_Y \circ f \circ g = (q_Y \circ f) \circ g = (\tilde{f} \circ q_X) \circ g = \tilde{f} \circ (q_X \circ g) = \tilde{f} \circ id_{\tilde{X}} = \tilde{f}$. Thus we have morphisms $g : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((X,\tau), q_Y \circ f)$ and $q_X : ((X,\tau), q_Y \circ f) \to ((\tilde{X},\tilde{\tau}),\tilde{f})$ in Q-$\mathbf{TOP}/(\tilde{Y},\tilde{\sigma})$ such that $q_X \circ g = id_{\tilde{X}}$. Thus $((\tilde{X},\tilde{\tau}),\tilde{f})$ is a retract of $((X,\tau), q_Y \circ f)$ in Q-$\mathbf{TOP}/(\tilde{Y},\tilde{\sigma})$. By Proposition 3.4.4 and Corollary 3.4.3, $((X,\tau), q_Y \circ f)$ is injective in Q-$\mathbf{TOP}/(\tilde{Y},\tilde{\sigma})$ and then by Proposition 3.4.9, $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-$\mathbf{TOP}/(\tilde{Y},\tilde{\sigma})$. Therefore by Proposition 3.4.6, $((\tilde{X},\tilde{\tau}),\tilde{f})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$.

□

The following Theorem 3.4.11 is concerned with the extension of Theorem 2.6 of [8], in the category Q-$\mathbf{TOP}/(Y,\sigma)$.

Theorem 3.4.11. Let (X, τ) be a Q-topological space and (Y, σ) be a stratified Q-topological space. Then $((X, \tau), f)$ is injective in Q-**TOP**$/(Y, \sigma)$ if and only if

1. $f(C_x) = C_{f(x)}$, for every $x \in X$.
2. Its T_0-reflection $((\tilde{X}, \tilde{\tau}), \tilde{f})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$.

Proof. Let $((X, \tau), f)$ be injective in Q-**TOP**$/(Y, \sigma)$.

(1) Let $x \in X$. Then by Proposition 3.4.8, $f(C_x) \subseteq C_{f(x)}$. Now we have to show that $C_{f(x)} \subseteq f(C_x)$. By Proposition 3.4.7, there exists a section $s_x : (Y, \sigma) \to (X, \tau)$ of $f : (X, \tau) \to (Y, \sigma)$ such that $s_x(f(x)) = x$ and $f \circ s_x = id_Y$. Then since by Proposition 3.4.8, $s_x(C_{f(x)}) \subseteq C_{(s_x \circ f)(x)}$, $s_x(C_{f(x)}) \subseteq C_x$ (as $s_x(f(x)) = x$). Then $f(s_x(C_{f(x)})) \subseteq f(C_x) \Rightarrow (f \circ s_x)(C_{f(x)}) \subseteq f(C_x) \Rightarrow id_Y(C_{f(x)}) \subseteq f(C_x) \Rightarrow C_{f(x)} \subseteq f(C_x)$. Therefore $f(C_x) = C_{f(x)}$.

(2) Follows from Proposition 3.4.10

Conversely, assume that (1) and (2) hold. Let following be a commutative diagram in Q-**TOP**:

$$\begin{array}{ccc} (W, \theta) & \xrightarrow{m} & (X, \tau) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ (Z, \eta) & \xrightarrow[n]{} & (Y, \sigma) \end{array} \tag{3.4.12}$$

where $h : (W, \theta) \to (Z, \eta)$ is an embedding in Q-**TOP**.
Now since $((\tilde{X}, \tilde{\tau}), \tilde{f})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$, by Proposition 3.4.6, $((\tilde{X}, \tilde{\tau}), \tilde{f})$ is injective in Q-**TOP**$/(\tilde{Y}, \tilde{\sigma})$. So there exists a Q-continuous map $s : (Z, \eta) \to (\tilde{X}, \tilde{f})$ such that the following diagram commutes:

$$\begin{array}{ccccc} (W, \theta) & \xrightarrow{m} & (X, \tau) & \xrightarrow{q_X} & (\tilde{X}, \tilde{\tau}) \\ {\scriptstyle h}\downarrow & & \overset{s}{\nearrow} & & \downarrow{\scriptstyle \tilde{f}} \\ (Z, \eta) & \xrightarrow[n]{} & (Y, \sigma) & \xrightarrow[q_Y]{} & (\tilde{Y}, \tilde{\sigma}) \end{array} \tag{3.4.13}$$

Let $\tilde{X} = \{[x_j] \mid j \in J\}$. Now let $w \in m^{-1}(C_{x_j})$, then $(s \circ h)(w) = (q_X \circ m)(w) = [m(w)] = [x_j]$ (as $m(w) \in C_{x_j}$, $[m(w)] = [x_j]$). So $h(w) \in s^{-1}(\{[x_j]\})$. Thus for each $j \in J$, we have a map $h_j : m^{-1}(C_{x_j}) \to s^{-1}(\{[x_j]\})$ defined as $h_j(w) = h(w)$, for every $w \in m^{-1}(C_{x_j})$. Next let $z \in s^{-1}(\{[x_j]\})$, then $(q_Y \circ n)(z) = (\tilde{f} \circ s)(z) = \tilde{f}(s(z)) = \tilde{f}([x_j]) = (\tilde{f} \circ q_X)(x_j) = (q_Y \circ f)(x_j)$. This implies that $[n(z)] = [f(x_j)]$. So $n(z) \sim f(x_j)$ and hence $n(z) \in C_{f(x_j)}$. Thus for each $j \in J$, we have a map $n_j : s^{-1}(\{[x_j]\}) \to C_{f(x_j)}$ defined as $n_j(z) = n(z)$, for every $z \in s^{-1}(\{[x_j]\})$. Now consider the following commutative diagram in category **Set**:

$$\begin{array}{ccc} m^{-1}(C_{x_j}) & \xrightarrow{m_j} & C_{x_j} \\ {\scriptstyle h_j}\downarrow & & \downarrow{\scriptstyle f_j} \\ s^{-1}(\{[x_j]\}) & \xrightarrow[n_j]{} & C_{f(x_j)} \end{array} \tag{3.4.14}$$

where $f_j : C_{x_j} \to C_{f(x_j)}$ is defined as $f_j(x) = f(x)$, for every $x \in C_{x_j}$ and $m_j : m^{-1}(C_{x_j}) \to C_{x_j}$ is defined as $m_j(w) = m(w)$, for every $w \in m^{-1}(C_{x_j})$. Now by (1), $f_j : C_{x_j} \to C_{f(x_j)}$ is onto and so by Proposition 3.4.1, (C_{x_j}, f_j) is injective in the comma category $\mathbf{Set}/C_{f(x_j)}$. Also since h is one-one, h_j is one-one. So there exists a map $g_j : s^{-1}(\{[x_j]\}) \to C_{x_j}$ such that $g_j \circ h_j = m_j$ and $f_j \circ g_j = n_j$. Thus for each $j \in J$, we have a map $g_j : s^{-1}(\{[x_j]\}) \to C_{x_j}$ such that $g_j \circ h_j = m_j$ and $f_j \circ g_j = n_j$. Note that since $\tilde{X} = \bigcup_{j\in J}\{[x_j]\}$, $Z = \bigcup_{j\in J} s^{-1}(\{[x_j]\})$. Thus we can define a map $g : Z \to X$ as $g(z) = g_j(z)$, if $z \in s^{-1}(\{[x_j]\})$. Now we have to prove that $f \circ g = n$, $g \circ h = m$ and $g : (Z, \eta) \to (X, \tau)$ is Q-continuous. Let $z \in Z$, then there exists a unique $j \in J$ such that $z \in s^{-1}(\{[x_j]\})$. Then $(f \circ g)(z) = f(g(z)) = f(g_j(z)) = f_j(g_j(z)) = (f_j \circ g_j)(z) = n_j(z) = n(z)$. This implies that $f \circ g = n$. Now consider $(g \circ h)(w) = g(h(w)) = g_j(h(w))$, if $h(w) \in s^{-1}(\{[x_j]\})$. Now if $h(w) \in s^{-1}(\{[x_j]\})$, then $(s \circ h)(w) = [x_j] \Rightarrow (q_X \circ m)(w) = [x_j] \Rightarrow [m(w)] = [x_j] \Rightarrow m(w) \sim x_j \Rightarrow m(w) \in C_{x_j} \Rightarrow w \in m^{-1}(C_{x_j})$ and so $h(w) = h_j(w)$. Thus if $h(w) \in s^{-1}(\{[x_j]\})$, then $(g \circ h)(w) = g_j(h(w)) = g_j(h_j(w)) = (g_j \circ h_j)(w) = m_j(w) = m(w)$. Thus $g \circ h = m$. Now to show that $g : (Z, \eta) \to (X, \tau)$ is Q-continuous. Let $\alpha \in \tau$, then $\alpha = \beta \circ q_X$, for some $\beta \in \tilde{\tau}$. Now let $z \in Z$, then there exists a unique $j \in J$ such that $z \in s^{-1}(\{[x_j]\})$. Now consider $(\alpha \circ g)(z) = (\beta \circ q_X \circ g)(z) = (\beta \circ q_X)(g(z)) = (\beta \circ q_X)(g_j(z)) = \beta([g_j(z)]) = \beta([x_j])$ (since $g_j(z) \in C_{x_j}$, $[g_j(z)] = [x_j]$). Thus $(\alpha \circ g)(z) = \beta([x_j]) = \beta(s(z)) = (\beta \circ s)(z)$. Thus $\alpha \circ g = \beta \circ s \in \eta$ as $s : (Z, \eta) \to (\tilde{X}, \tilde{\tau})$ is Q-continuous. Hence $g : (Z, \eta) \to (X, \tau)$ is Q-continuous. Therefore $((X, \tau), f)$ is injective in Q-**TOP**$/(Y, \sigma)$.

□

3.5 Existence of injective hulls in Q-TOP$/(Y,\sigma)$

Proposition 3.5.1. Let (X,τ) and (Y,σ) be T_0-Q-topological spaces. Then $((X,\tau),f)$ has an injective hull in Q-$\mathbf{TOP}_0/(Y,\sigma)$ if and only if it has an injective hull in Q-$\mathbf{TOP}/(Y,\sigma)$ and in this case injective hulls coincide.

Proof. Suppose first that $((X,\tau),f)$ has an injective hull $j:((X,\tau),f)\to((Z,\eta),g)$ in Q-$\mathbf{TOP}_0/(Y,\sigma)$. Then $((Z,\eta),g)$ is injective in Q-$\mathbf{TOP}_0/(Y,\sigma)$ and then by Proposition 3.4.6, $((Z,\eta),g)$ is injective in Q-$\mathbf{TOP}/(Y,\sigma)$. Furthermore, $j:(X,\tau)\to(Z,\eta)$ is an embedding in Q-$\mathbf{TOP}_0$ and then by Lemma 3.4.5, $j:(X,\tau)\to(Z,\eta)$ is an embedding in Q-$\mathbf{TOP}$. Now we have to prove that $j:((X,\tau),f)\to((Z,\eta),g)$ is essential in Q-$\mathbf{TOP}/(Y,\sigma)$. We note that since $j:(X,\tau)\to(Z,\eta)$ is an embedding in Q-$\mathbf{TOP}$, by Proposition 3.3.2, $\tilde{j}:(\tilde{X},\tilde{\tau})\to(\tilde{Z},\tilde{\eta})$ is an embedding in Q-$\mathbf{TOP}$ and hence $\tilde{j}:(\tilde{X},\tilde{\tau})\to(\tilde{Z},\tilde{\eta})$ is an embedding in Q-$\mathbf{TOP}_0$ by Lemma 3.4.5. Now we will first prove that $\tilde{j}:((\tilde{X},\tilde{\tau}),\tilde{f})\to((\tilde{Z},\tilde{\eta}),\tilde{g})$ is essential in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$. Let $h:((\tilde{Z},\tilde{\eta}),\tilde{g})\to((A,\tau_A),m)$ be a morphism in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$ such that $h\circ\tilde{j}:(\tilde{X},\tilde{\tau})\to(A,\tau_A)$ is an embedding in Q-$\mathbf{TOP}_0$. Since composition of embeddings is an embedding, $h\circ\tilde{j}\circ q_X:(X,\tau)\to(A,\tau_A)$ is an embedding in Q-$\mathbf{TOP}_0$ and then since $h\circ\tilde{j}\circ q_X=(h\circ q_Z)\circ j$, $(h\circ q_Z)\circ j:(X,\tau)\to(A,\tau_A)$ is an embedding in Q-$\mathbf{TOP}_0$. Thus we have a morphism $h\circ q_Z:((Z,\eta),g)\to((A,\tau_A),q_Y^{-1}\circ m)$ in Q-$\mathbf{TOP}_0/(Y,\sigma)$ such that $(h\circ q_Z)\circ j:(X,\tau)\to(A,\tau_A)$ is an embedding in Q-$\mathbf{TOP}_0$. Then since $j:((X,\tau),f)\to((Z,\eta),g)$ is essential in Q-$\mathbf{TOP}_0/(Y,\sigma)$, $h\circ q_Z:(Z,\eta)\to(A,\tau_A)$ is an embedding in Q-$\mathbf{TOP}_0$. Now since $(Z,\eta)\in Q$-$\mathbf{TOP}_0$, $q_Z:(Z,\eta)\to(\tilde{Z},\tilde{\eta})$ is an isomorphism in Q-$\mathbf{TOP}_0$ and so by Proposition 1.2.20(1), it is essential in Q-$\mathbf{TOP}_0$ and then since $h\circ q_Z:(Z,\eta)\to(A,\tau_A)$ is an embedding in Q-$\mathbf{TOP}_0$, $h:(\tilde{Z},\tilde{\eta})\to(A,\tau_A)$ is an embedding in Q-$\mathbf{TOP}_0$. Thus $\tilde{j}:((\tilde{X},\tilde{\tau}),\tilde{f})\to((\tilde{Z},\tilde{\eta}),\tilde{g})$ is essential in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$. Next let $k:((Z,\eta),g)\to((W,\theta),l)$ be a morphism in Q-$\mathbf{TOP}/(Y,\sigma)$ such that $k\circ j:(X,\tau)\to(W,\theta)$ is an embedding in Q-$\mathbf{TOP}$. Then by Proposition 3.3.2, $\tilde{k}\circ\tilde{j}:(\tilde{X},\tilde{\tau})\to(\tilde{W},\tilde{\theta})$ is an embedding in Q-$\mathbf{TOP}$ and then by Lemma 3.4.5, it is an embedding in Q-$\mathbf{TOP}_0$. Thus we have a morphism $\tilde{k}:((\tilde{Z},\tilde{\eta}),\tilde{g})\to((\tilde{W},\tilde{\theta}),\tilde{l})$ in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$ such that $\tilde{k}\circ\tilde{j}:(\tilde{X},\tilde{\tau})\to(\tilde{W},\tilde{\theta})$ is an embedding in Q-$\mathbf{TOP}_0$. Then since $\tilde{j}:((\tilde{X},\tilde{\tau}),\tilde{f})\to((\tilde{Z},\tilde{\eta}),\tilde{g})$ is essential in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$, $\tilde{k}:(\tilde{Z},\tilde{\eta})\to(\tilde{W},\tilde{l})$ is an embedding in Q-$\mathbf{TOP}_0$ and hence it is an embedding in Q-$\mathbf{TOP}$ by Lemma 3.4.5. Now we will prove that k is one-one. Let $k(z_1)=k(z_2)\Rightarrow(q_W\circ k)(z_1)=(q_W\circ k)(z_2)\Rightarrow(\tilde{k}\circ q_Z)(z_1)=$

$(\tilde{k} \circ q_Z)(z_2) \Rightarrow z_1 = z_2$ (since $\tilde{k} \circ q_Z$ is one-one as both $\tilde{k}$ and q_Z are one-one). Hence k is one-one. Thus by Proposition 3.3.2, $k : (Z, \eta) \to (W, \theta)$ is an embedding in Q-**TOP**. Hence $j : ((X, \tau), f) \to ((Z, \eta), g)$ is essential in Q-**TOP**$/(Y, \sigma)$. Therefore $j : ((X, \tau), f) \to ((Z, \eta), g)$ is an injective hull of $((X, \tau), f)$ in Q-**TOP**$/(Y, \sigma)$.

Conversely, let $((X, \tau), f)$ have an injective hull $j : ((X, \tau), f) \to ((Z, \eta), g)$ in Q-**TOP**$/(Y, \sigma)$. Now since $(X, \tau) \in Q$-**TOP**$_0$, $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ is an isomorphism in Q-**TOP** and hence $q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})$ is an embedding in Q-**TOP**. Also since $j : (X, \tau) \to (Z, \eta)$ is an embedding in Q-**TOP**, by Proposition 3.3.2, $\tilde{j} : (\tilde{X}, \tilde{\tau}) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-**TOP**. Thus $\tilde{j} \circ q_X : (X, \tau) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-**TOP** and since $\tilde{j} \circ q_X = q_Z \circ j$, $q_Z \circ j : (X, \tau) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-**TOP**. Thus we have a morphism $q_Z : ((Z, \eta), g) \to ((\tilde{Z}, \tilde{\eta}), q_Y^{-1} \circ \tilde{g})$ in Q-**TOP**$/(Y, \sigma)$ such that $q_Z \circ j : (X, \tau) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-**TOP** and then since $j : ((X, \tau), f) \to ((Z, \eta), g)$ is essential in Q-**TOP**$/(Y, \sigma)$, $q_Z : (Z, \eta) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-**TOP**. Hence $q_Z : (Z, \eta) \to (\tilde{Z}, \tilde{\eta})$ is initial in Q-**TOP** and q_Z is bijective and thus by Proposition 3.3.3, $q_Z : (Z, \eta) \to (\tilde{Z}, \tilde{\eta})$ is an isomorphism in Q-**TOP**. Now since Q-**TOP**$_0$ is an isomorphism closed subcategory of Q-**TOP**, $(Z, \eta) \in Q$-**TOP**$_0$. Thus $j : ((X, \tau), f) \to ((Z, \eta), g)$ is a morphism in Q-**TOP**$_0/(Y, \sigma)$. Now since $j : (X, \tau) \to (Z, \eta)$ is an embedding in Q-**TOP**, by Lemma 3.4.5, $j : (X, \tau) \to (Z, \eta)$ is an embedding in Q-**TOP**$_0$. It can also be easily verified that $j : ((X, \tau), f) \to ((Z, \eta), g)$ is essential in Q-**TOP**$_0/(Y, \sigma)$. We also note that since $((Z, \eta), g)$ is injective in Q-**TOP**$/(Y, \sigma)$ by Proposition 3.4.6, it is injective in Q-**TOP**$_0/(Y, \sigma)$. Therefore $j : ((X, \tau), f) \to ((Z, \eta), g)$ is an injective hull of $((X, \tau), f)$ in Q-**TOP**$_0/(Y, \sigma)$. □

Definition 3.5.2. [41] Let $m : U \to B$ and $e : A \to U$ be morphisms in a category **C**. Then a **pullback complement of the pair** (m, e) in the category **C** is a pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & U \\ {\scriptstyle \bar{m}}\downarrow & & \downarrow{\scriptstyle m} \\ P & \xrightarrow[\bar{e}]{} & B \end{array} \tag{3.5.1}$$

such that, given any pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{d} & U \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle m} \\ Y & \xrightarrow[g]{} & B \end{array} \tag{3.5.2}$$

and a morphism $h : X \to A$ with $e \circ h = d$, there is a unique morphism $h' : Y \to P$ with $\bar{e} \circ h' = g$ and $h' \circ k = \bar{m} \circ h$.

([41], [8]) In the category **Set**, pullback complement of the pair (m, e), where $m : Z \to Y$ is one-one and $e : X \to Z$ is a map, always exists and given by

$$\begin{array}{ccc} X & \xrightarrow{e} & Z \\ {\scriptstyle \bar{m}}\downarrow & & \downarrow{\scriptstyle m} \\ (Y \setminus m(Z)) + X & \xrightarrow[\bar{e}]{} & Y \end{array} \tag{3.5.3}$$

where $(Y \setminus m(Z)) + X = \{(y, 1) \mid y \in Y \setminus m(Z)\} \cup \{(x, 2) \mid x \in X\}$, $\bar{m} : X \to (Y \setminus m(Z)) + X$ is defined as $\bar{m}(x) = (x, 2)$ and $\bar{e} : (Y \setminus m(Z)) + X \to Y$ is defined as $\bar{e}(y, 1) = y$, $\bar{e}(x, 2) = (m \circ e)(x)$. It can also be easily verified that if e is onto, then $\bar{e}$ is onto.

Proposition 3.5.3. Let $m : (Z, \eta) \to (Y, \sigma)$ be an embedding in Q-**TOP** and let $e : (X, \tau) \to (Z, \eta)$ be initial in Q-**TOP**. Then there exists a pullback complement of (m, e) in Q-**TOP**:

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{e} & (Z,\eta) \\ {\scriptstyle \bar{m}}\downarrow & & \downarrow{\scriptstyle m} \\ (W,\theta) & \xrightarrow[\bar{e}]{} & (Y,\sigma) \end{array} \tag{3.5.4}$$

where $\bar{e} : (W, \theta) \to (Y, \sigma)$ is initial in Q-**TOP**.

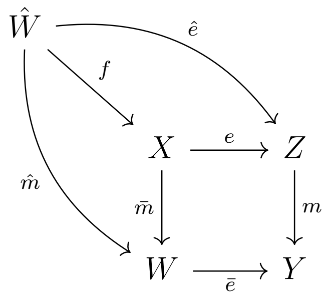
Proof. Let us consider the pullback complement of the pair (m, e) in **Set** given by the following:

$$\begin{CD} X @>{e}>> Z \\ @V{\bar{m}}VV @VV{m}V \\ W @>>{\bar{e}}> Y \end{CD} \tag{3.5.5}$$

where $W = (Y\setminus m(Z))+X$, $\bar{m} : X \to (Y\setminus m(Z))+X$ is defined as $\bar{m}(x) = (x, 2)$ and $\bar{e} : (Y \setminus m(Z)) + X \to Y$ is defined as $\bar{e}(y, 1) = y$, $\bar{e}(x, 2) = (m \circ e)(x)$. Now let $\theta = \{\beta \circ \bar{e} \mid \beta \in \sigma\}$. Let $\beta \circ \bar{e} \in \theta$, where $\beta \in \sigma$. Then $\beta \circ \bar{e} \circ \bar{m} = \beta \circ m \circ e = \beta \circ (m \circ e) \in \tau$ (as $m \circ e : (X, \tau) \to (Y, \sigma)$ is Q-continuous). Thus $\bar{m} : (X, \tau) \to (W, \theta)$ is Q-continuous. Next let following be a commutative diagram in Q-**TOP**:

$$\begin{CD} (\hat{W},\hat{\theta}) @>{\hat{e}}>> (Z,\eta) \\ @V{\hat{m}}VV @VV{m}V \\ (W,\theta) @>>{\bar{e}}> (Y,\sigma) \end{CD} \tag{3.5.6}$$

Since the diagram 3.5.5 is a pullback square in **Set**, there exists a unique map $f : \hat{W} \to X$ such that the following diagram commutes:



(3.5.7)

Now we will show that $f : (\hat{W}, \hat{\theta}) \to (X, \tau)$ is Q-continuous. Let $\beta \circ e \in \tau$, where $\beta \in \eta$. Then $\beta \circ e \circ f = \beta \circ \hat{e} \in \hat{\theta}$ (as $\hat{e} : (\hat{W}, \hat{\theta}) \to (Z, \eta)$ is Q-continuous). Thus $f : (\hat{W}, \hat{\theta}) \to (X, \tau)$ is Q-continuous. Hence the diagram 3.5.4 is a pullback square in Q-**TOP**. Now let following be a pullback square in Q-**TOP**:

$$\begin{CD} (A,\tau_A) @>{d}>> (Z,\eta) \\ @V{k}VV @VV{m}V \\ (B,\tau_B) @>>{g}> (Y,\sigma) \end{CD} \tag{3.5.8}$$

and let $h : (A, \tau_A) \to (X, \tau)$ be a Q-continuous map such that $e \circ h = d$. Then if we consider the diagram 3.5.8 in **Set**, then it is a pullback square in **Set** also and since the diagram 3.5.5 is a pullback complement diagram in **Set**, there exists a unique map $h' : B \to W$ such that $\bar{e} \circ h' = g$ and $h' \circ k = \bar{m} \circ h$. So now it is sufficient to show that $h' : (B, \tau_B) \to (W, \theta)$ is Q-continuous. Let $\beta \circ \bar{e} \in \theta$, where $\beta \in \sigma$. Then $\beta \circ \bar{e} \circ h' = \beta \circ g \in \tau_B$ (as $g : (B, \tau_B) \to (Y, \sigma)$ is Q-continuous). Thus $h' : (B, \tau_B) \to (W, \theta)$ is Q-continuous. Therefore the diagram 3.5.4 gives a pullback complement of (m, e) in Q-**TOP**. □

Proposition 3.5.4. [7] Let **C** be a category and let following be a pullback square in **C**

$$\begin{array}{ccc} W & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array} \tag{3.5.9}$$

If (X, f) is injective in $\mathbf{C}/Y$, then (W, q) is injective in $\mathbf{C}/Z$.

The following Theorem 3.5.5 is concerned with the extension of Theorem 2.11 of [8], in the category Q-**TOP**$/(Y, \sigma)$.

Theorem 3.5.5. Let (X, τ) and (Y, σ) be Q-topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ be a Q-continuous map. Then following statements are equivalent:

1. $((X, \tau), f)$ has an injective hull in Q-**TOP**$/(Y, \sigma)$.

2. $((\tilde{X}, \tilde{\tau}), \tilde{f})$ has an injective hull in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$.

3. $((\tilde{X}, \tilde{\tau}), \tilde{f})$ has an injective hull in Q-**TOP**$/(\tilde{Y}, \tilde{\sigma})$.

Proof. (1) ⇒ (2) Let $((X, \tau), f)$ have an injective hull $j : ((X, \tau), f) \to ((Z, \eta), g)$ in Q-**TOP**$/(Y, \sigma)$. We will show that $\tilde{j} : ((\tilde{X}, \tilde{\tau}), \tilde{f}) \to ((\tilde{Z}, \tilde{\eta}), \tilde{g})$ is an injective hull of $((\tilde{X}, \tilde{\tau}), \tilde{f})$ in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$. Now since $((Z, \eta), g)$ is injective in Q-**TOP**$/(Y, \sigma)$, by Proposition 3.4.10 $((\tilde{Z}, \tilde{\eta}), \tilde{g})$ is injective in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$. Next since $j : (X, \tau) \to (Y, \sigma)$ is an embedding in Q-**TOP**, $\tilde{j} : (\tilde{X}, \tilde{\tau}) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-**TOP** and then by Lemma 3.4.5, $\tilde{j} : (\tilde{X}, \tilde{\tau}) \to (\tilde{Z}, \tilde{\eta})$ is an embedding in Q-$\mathbf{TOP}_0$. Next let $k : ((\tilde{Z}, \tilde{\eta}), \tilde{g}) \to ((W, \theta), l)$ be a morphism in Q-$\mathbf{TOP}_0/(\tilde{Y}, \tilde{\sigma})$ such that $k \circ \tilde{j} : (\tilde{X}, \tilde{\tau}) \to (W, \theta)$ is an embedding in Q-$\mathbf{TOP}_0$. We have to prove that $k : (\tilde{Z}, \tilde{\eta}) \to (W, \theta)$ is an embedding in Q-$\mathbf{TOP}_0$. Consider the Q-topology $\{\alpha \circ p_1 \mid \alpha \in \theta\}$ on $W \times Z$, where $p_1 : W \times Z \to W$ is the first projection map. Let $T = W \times Z$ and $\rho = \{\alpha \circ p_1 \mid \alpha \in \theta\}$. Now we will prove that $(\tilde{T}, \tilde{\rho})$ and (W, θ) are isomorphic. Define a map $h : \tilde{T} \to W$ as $h([(w,z)]) = w$. Let $[(w_1, z_1)] = [(w_2, z_2)] \Rightarrow (\alpha \circ p_1)(w_1, z_1) = (\alpha \circ p_1)(w_2, z_2)$, for every $\alpha \in \theta \Rightarrow \alpha(w_1) = \alpha(w_2)$, for every $\alpha \in \theta \Rightarrow w_1 = w_2$ (since $(W, \theta) \in Q$-$\mathbf{TOP}_0$ and if $w_1 \neq w_2$, then there exists $\alpha \in \theta$ such that $\alpha(w_1) \neq \alpha(w_2)$). Thus the map h is well-defined. Now let $h([(w_1, z_1)]) = h([(w_2, z_2)]) \Rightarrow w_1 = w_2 \Rightarrow \alpha(w_1) = \alpha(w_2)$, for every $\alpha \in \theta \Rightarrow (\alpha \circ p_1)(w_1, z_1) = (\alpha \circ p_1)(w_2, z_2)$, for every $\alpha \in \theta \Rightarrow [(w_1, z_1)] = [(w_2, z_2)]$. Thus h is one-one and hence h is bijective. Now let $\alpha \in \theta$. Then $(\alpha \circ h \circ q_{W \times Z})(w, z) = \alpha(w) = (\alpha \circ p_1)(w, z) \Rightarrow \alpha \circ h \circ q_{W \times Z} = \alpha \circ p_1 \in \rho \Rightarrow \alpha \circ h \in \tilde{\rho}$. Thus $h : (\tilde{T}, \tilde{\rho}) \to (W, \theta)$ is Q-continuous. Now let $\beta \in \tilde{\rho}$, then $\beta \circ q_{W \times Z} \in \rho$ and so $\beta \circ q_{W \times Z} = \alpha \circ p_1$, for some $\alpha \in \theta$. Then $(\beta \circ q_{W \times Z})(w, z) = (\alpha \circ p_1)(w, z) = \alpha(w) = \alpha(h([(w, z)])) = (\alpha \circ h)([(w, z)]) = (\alpha \circ h \circ q_{W \times Z})(w, z) \Rightarrow \beta \circ q_{W \times Z} = \alpha \circ h \circ q_{W \times Z}$ and so $\beta = \alpha \circ h$ (as $q_{W \times Z}$ is onto). Thus $h : (\tilde{T}, \tilde{\rho}) \to (W, \theta)$ is initial in Q-$\mathbf{TOP}$ and also h is bijective. Hence by Proposition 3.3.3, $h : (\tilde{T}, \tilde{\rho}) \to (W, \theta)$ is an isomorphism in Q-$\mathbf{TOP}$. Now define a map $k' = (k \circ q_Z, id_Z) : (Z, \eta) \to (W \times Z, \rho)$ as $k'(z) = ((k \circ q_Z)(z), z)$. Let $\alpha \circ p_1 \in \rho$ ($\alpha \in \theta$). Then $(\alpha \circ p_1 \circ k')(z) = (\alpha \circ p_1)(k'(z)) = (\alpha \circ p_1)((k \circ q_Z)(z), z) = (\alpha \circ k \circ q_Z)(z)$. This implies that $\alpha \circ p_1 \circ k' = \alpha \circ k \circ q_Z \in \eta$ (as $k \circ q_Z : (Z, \eta) \to (W, \theta)$ is Q-continuous and $\alpha \in \theta$). Thus $k' : (Z, \eta) \to (W \times Z, \rho)$ is Q-continuous. Now consider $(l \circ h \circ q_{W \times Z} \circ k')(z) = (l \circ h \circ q_{W \times Z})(k'(z)) = (l \circ h \circ q_{W \times Z})((k \circ q_Z)(z), z) = (l \circ h)([((k \circ q_Z)(z), z)]) = l((k \circ q_Z)(z)) = (l \circ k \circ q_Z)(z) = (\tilde{g} \circ q_Z)(z) = (q_Y \circ g)(z)$. This implies that $l \circ h \circ q_{W \times Z} \circ k' = q_Y \circ g$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} (Z,\eta) & \xrightarrow{k'} & (W \times Z, \rho) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle l \circ h \circ q_{W \times Z}} \\ (Y,\sigma) & \xrightarrow[q_Y]{} & (\tilde{Y}, \tilde{\sigma}) \end{array} \tag{3.5.10}$$

Now since q_Y is onto, by Proposition 3.4.1, (Y, q_Y) is injective in the comma category $\mathbf{Set}/\tilde{Y}$ and since k' is one-one, there exists a map $m : W \times Z \to Y$ such that $m \circ k' = g$ and $q_Y \circ m = l \circ h \circ q_{W \times Z}$. Let $v \circ q_Y \in \sigma$, where $v \in \tilde{\sigma}$, then $v \circ q_Y \circ m = v \circ l \circ h \circ q_{W \times Z} \in \rho$ (as $l \circ h \circ q_{W \times Z} : (W \times Z, \rho) \to (\tilde{Y}, \tilde{\sigma})$ is Q-continuous). Thus $m : (W \times Z, \rho) \to (Y, \sigma)$ is Q-continuous. Hence $((W \times Z, \rho), m) \in Q$-$\mathbf{TOP}/(Y, \sigma)$. Now it can be easily verified that the following diagram commutes:

$$\begin{array}{ccc} (Z,\eta) & \xrightarrow{k'} & (W\times Z,\rho) \\ {\scriptstyle q_Z}\downarrow & & \downarrow{\scriptstyle q_{W\times Z}} \\ (\tilde{Z},\tilde{\eta}) & \xrightarrow[h^{-1}\circ k]{} & (\tilde{T},\tilde{\rho}) \end{array} \tag{3.5.11}$$

Thus $\tilde{k'} = h^{-1}\circ k$. Let $p = k'\circ j$. Then $\tilde{p} = \tilde{k'}\circ\tilde{j} = (h^{-1}\circ k)\circ\tilde{j} = h^{-1}\circ(k\circ\tilde{j})$. Now since $h^{-1} : (W,\theta) \to (\tilde{T},\tilde{\rho})$ is an isomorphism in Q-**TOP**, by Proposition 1.2.20, it is an embedding in Q-**TOP**. By Proposition 1.2.18(1), we know that composition of embeddings is an embedding and since $\tilde{p} = h^{-1}\circ(k\circ\tilde{j})$, $\tilde{p} : (\tilde{X},\tilde{\tau}) \to (\tilde{T},\tilde{\rho})$ is an embedding in Q-**TOP**. Also $p = k'\circ j$ is one-one. Thus by Proposition 3.3.2, $p = k'\circ j : (X,\tau) \to (W\times Z,\rho)$ is an embedding in Q-**TOP**. Thus $k' : ((Z,\eta),g) \to ((W\times Z,\rho),m)$ is a morphism in Q-**TOP**$/(Y,\sigma)$ such that $k'\circ j : (X,\tau) \to (W\times Z,\rho)$ is an embedding in Q-**TOP** and then since $j : ((X,\tau),f) \to ((Z,\eta),g)$ is essential in Q-**TOP**$/(Y,\sigma)$, $k' : (Z,\eta) \to (W\times Z,\rho)$ is an embedding in Q-**TOP**. Then by Proposition 3.3.2, $\tilde{k'} : (\tilde{Z},\tilde{\eta}) \to (\tilde{T},\tilde{\rho})$ is an embedding in Q-**TOP**. Now since $\tilde{k'} = h^{-1}\circ k$, $h^{-1}\circ k : (\tilde{Z},\tilde{\eta}) \to (\tilde{T},\tilde{\rho})$ is an embedding in Q-**TOP**. Then by Proposition 1.2.18(2), $k : (\tilde{Z},\tilde{\eta}) \to (W,\theta)$ is an embedding in Q-**TOP** and then by Lemma 3.4.5, $k : (\tilde{Z},\tilde{\eta}) \to (W,\theta)$ is an embedding in Q-$\mathbf{TOP}_0$. Thus $\tilde{j} : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((\tilde{Z},\tilde{\eta}),\tilde{g})$ is essential in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$. Therefore $\tilde{j} : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((\tilde{Z},\tilde{\eta}),\tilde{g})$ is an injective hull of $((\tilde{X},\tilde{\tau}),\tilde{f})$ in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$.

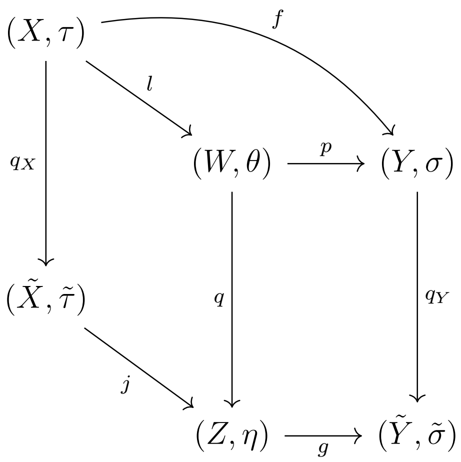
(2) $\Rightarrow$ (3) Follows from Proposition 3.5.1.

(3) $\Rightarrow$ (1) Let $j : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((Z,\eta),g)$ be an injective hull of $((\tilde{X},\tilde{\tau}),\tilde{f})$ in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$. Then by Proposition 3.5.1, $j : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((Z,\eta),g)$ is an injective hull of $(\tilde{X},\tilde{\tau}),\tilde{f})$ in Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$. Thus clearly $(Z,\eta) \in Q$-$\mathbf{TOP}_0$. Now let $q : (W,\theta) \to (Z,\eta)$ be a pullback of $q_Y : (Y,\sigma) \to (\tilde{Y},\tilde{\sigma})$ along $g : (Z,\eta) \to (\tilde{Y},\tilde{\sigma})$ in Q-**TOP**:

$$\begin{array}{ccc} (W,\theta) & \xrightarrow{q} & (Z,\eta) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ (Y,\sigma) & \xrightarrow[q_Y]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{3.5.12}$$

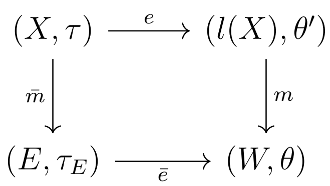
We note that by Proposition 3.3.6, $\tilde{q} : (\tilde{W},\tilde{\theta}) \to (\tilde{Z},\tilde{\eta})$ is an isomorphism in Q-**TOP**. Now since $g\circ j\circ q_X = (g\circ j)\circ q_X = \tilde{f}\circ q_X = q_Y\circ f$ and the diagram

3.5.12 is a pullback, there exists a unique Q-continuous map $l:(X,\tau)\to(W,\theta)$ for which the following diagram commutes:



(3.5.13)

Now since $q_X:(X,\tau)\to(\tilde{X},\tilde{\tau})$ and $j:(\tilde{X},\tilde{\tau})\to(Z,\eta)$ both are initial in Q-**TOP** and composition of initial maps is initial, $j\circ q_X:(X,\tau)\to(Z,\eta)$ is initial in Q-**TOP** and since $j\circ q_X=q\circ l$, $q\circ l:(X,\tau)\to(Z,\eta)$ is initial in Q-**TOP** and then by Proposition 1.2.18(2), $l:(X,\tau)\to(W,\theta)$ is initial in Q-**TOP**. Now let $e:X\to l(X)$ defined as $e(x)=l(x)$ and let $m:l(X)\to W$ be the inclusion map. Next if we take the Q-topology $\theta'=\{\beta\circ m\mid\beta\in\theta\}$ on $l(X)$, then we have a factorization of $l:(X,\tau)\to(W,\theta)$ given by $l=m\circ e$ such that $e:(X,\tau)\to(l(X),\theta')$ is onto and $m:(l(X),\theta')\to(W,\theta)$ is an embedding in Q-**TOP**. Then since $l:(X,\tau)\to(W,\theta)$ is initial in Q-**TOP** and $l=m\circ e$, $e:(X,\tau)\to(l(X),\theta')$ is initial in Q-**TOP**. Thus by Proposition 3.5.3, there exists a pullback complement of (m,e) in Q-**TOP** given by:



(3.5.14)

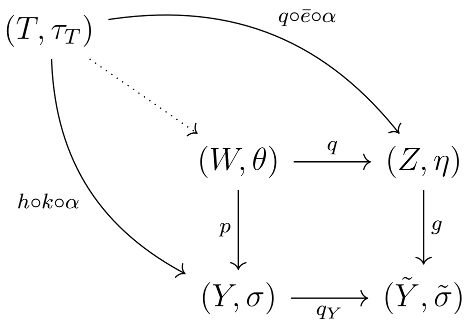
where $E=(W\setminus m(l(X)))+X$, $\bar{m}:X\to E$ is defined as $\bar{m}(x)=(x,2)$, $\bar{e}:E\to W$ is defined as $\bar{e}(w,1)=w$, $\bar{e}(x,2)=(m\circ e)(x)$ and $\bar{e}:(E,\tau_E)\to(W,\theta)$ is initial in Q-**TOP**. Also since e is onto, $\bar{e}$ is onto. Now since $\bar{e}$ is onto and $\bar{e}:(E,\tau_E)\to(W,\theta)$ is initial in Q-**TOP**, as in the proof of the Proposition 3.3.6, we can prove that $\tilde{\bar{e}}$ is bijective and $\tilde{\bar{e}}:(\tilde{E},\tilde{\tau_E})\to(\tilde{W},\tilde{\theta})$ is initial in Q-**TOP** and hence by Proposition 3.3.3, $\tilde{\bar{e}}:(\tilde{E},\tilde{\tau_E})\to(\tilde{W},\tilde{\theta})$ is an isomorphism in Q-**TOP**. Also by Proposition 3.4.2, $((E,\tau_E),\bar{e})$ is injective in Q-**TOP**$/(W,\theta)$. Now since $((Z,\eta),g)$ is injective in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$ and the diagram 3.5.12 is a pullback, by Proposition 3.5.4, $((W,\theta),p)$ is injective in Q-**TOP**$/(Y,\sigma)$. Thus by Proposition 3.4.4, $((E,\tau_E),p\circ\bar{e})$ is injective in Q-**TOP**$/(Y,\sigma)$. Now since $p\circ\bar{e}\circ\bar{m} = p\circ l = f$, $\bar{m} : ((X,\tau),f) \to ((E,\tau_E),p\circ\bar{e})$ is a morphism in Q-**TOP**$/(Y,\sigma)$. Now we will prove that $\bar{m} : ((X,\tau),f) \to ((E,\tau_E),p\circ\bar{e})$ is an injective hull of $((X,\tau),f)$ in Q-**TOP**$/(Y,\sigma)$. We know that by the Theorem 1.2.42, Q-**TOP** is a topological category over **Set** and so by Proposition 1.2.33, regular monomorphisms in Q-**TOP** are precisely embeddings in Q-**TOP**. Then since the diagram 3.5.14 is a pullback and $m : (l(X),\theta') \to (W,\theta)$ is an embedding in Q-**TOP**, by Proposition 3.3.4, $\bar{m} : (X,\tau) \to (E,\tau_E)$ is an embedding in Q-**TOP**. Next let $k : ((E,\tau_E),p\circ\bar{e}) \to ((G,\tau_G),h)$ be a morphism in Q-**TOP**$/(Y,\sigma)$ such that $k\circ\bar{m} : (X,\tau) \to (G,\tau_G)$ is an embedding in Q-**TOP**. We have to show that $k : (E,\tau_E) \to (G,\tau_G)$ is an embedding in Q-**TOP**. Now since $k\circ\bar{m} : (X,\tau) \to (G,\tau_G)$ is an embedding in Q-**TOP**, by Proposition 3.3.2, $\tilde{k}\circ\tilde{\bar{m}} : (\tilde{X},\tilde{\tau}) \to (\tilde{G},\tilde{\tau_G})$ is an embedding in Q-**TOP**. Now $(\tilde{q})^{-1}\circ q_Z\circ j\circ q_X = (\tilde{q})^{-1}\circ q_Z\circ(j\circ q_X) = (\tilde{q})^{-1}\circ q_Z\circ(q\circ l) = (\tilde{q})^{-1}\circ(q_Z\circ q)\circ l = (\tilde{q})^{-1}\circ(\tilde{q}\circ q_W)\circ l = q_W\circ l$. Thus the following diagram commutes:

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{\;l\;} & (W,\theta) \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_W} \\ (\tilde{X},\tilde{\tau}) & \xrightarrow[(\tilde{q})^{-1}\circ q_Z\circ j]{} & (\tilde{W},\tilde{\theta}) \end{array} \tag{3.5.15}$$

and hence $\tilde{l} = (\tilde{q})^{-1}\circ q_Z\circ j$. We also have $g\circ q_Z^{-1}\circ\tilde{q}\circ q_W = g\circ q_Z^{-1}\circ(\tilde{q}\circ q_W) = g\circ q_Z^{-1}\circ(q_Z\circ q) = g\circ q = q_Y\circ p$. Thus the following diagram commutes:

$$\begin{array}{ccc} (W,\theta) & \xrightarrow{\;p\;} & (Y,\sigma) \\ {\scriptstyle q_W}\downarrow & & \downarrow{\scriptstyle q_Y} \\ (\tilde{W},\tilde{\theta}) & \xrightarrow[g\circ q_Z^{-1}\circ\tilde{q}]{} & (\tilde{Y},\tilde{\sigma}) \end{array} \tag{3.5.16}$$

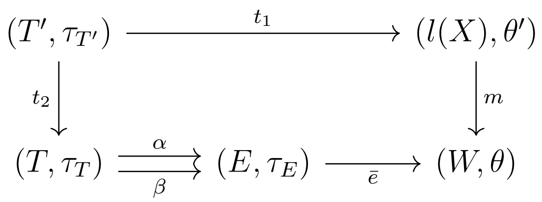
and so $\tilde{p} = g\circ q_Z^{-1}\circ\tilde{q}$. Now since $(Z,\eta)\in Q$-**TOP**$_0$, $q_Z : (Z,\eta) \to (\tilde{Z},\tilde{\eta})$ is an isomorphism in Q-**TOP** and since $(\tilde{q})^{-1} : (\tilde{Z},\tilde{\eta}) \to (\tilde{W},\tilde{\theta})$ is also an isomorphism in Q-**TOP**, $(\tilde{q})^{-1} \circ q_Z : (Z,\eta) \to (\tilde{W},\tilde{\theta})$ is an isomorphism in Q-**TOP** and so by Proposition 1.2.20(1), it is an essential embedding in Q-**TOP** and then it can be easily seen that $(\tilde{q})^{-1} \circ q_Z : ((Z,\eta), g) \to ((\tilde{W},\tilde{\theta}),\tilde{p})$ is a morphism in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$ which is essential in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$. Thus $\tilde{l} = (\tilde{q})^{-1} \circ q_Z \circ j : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((\tilde{W},\tilde{\theta}),\tilde{p})$ is essential in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$. Now since $l = \bar{e} \circ \bar{m}$, $\tilde{l} = \tilde{\bar{e}} \circ \tilde{\bar{m}}$ and so $\tilde{\bar{m}} = (\tilde{\bar{e}})^{-1} \circ \tilde{l}$. Now since $(\tilde{\bar{e}})^{-1} : (\tilde{W},\tilde{\theta}) \to (\tilde{E},\tilde{\tau_E})$ is an isomorphism in Q-**TOP**, by Proposition 1.2.20(1), it is an essential embedding in Q-**TOP**. Thus we have a morphism $(\tilde{\bar{e}})^{-1} : ((\tilde{W},\tilde{\theta}),\tilde{p}) \to ((\tilde{E},\tilde{\tau_E}),\tilde{p} \circ \tilde{\bar{e}})$ in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$ which is essential in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$. Then since $\tilde{\bar{m}} = (\tilde{\bar{e}})^{-1} \circ \tilde{l}$, $\tilde{\bar{m}} : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((\tilde{E},\tilde{\tau_E}),\tilde{p} \circ \tilde{\bar{e}})$ is essential in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$. Now since $\tilde{k} : ((\tilde{E},\tilde{\tau_E}),\tilde{p} \circ \tilde{\bar{e}}) \to ((\tilde{G},\tilde{\tau_G}),\tilde{h})$ is a morphism in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$ such that $\tilde{k} \circ \tilde{\bar{m}} : (\tilde{X},\tilde{\tau}) \to (\tilde{G},\tilde{\tau_G})$ is an embedding in Q-**TOP**, and $\tilde{\bar{m}} : ((\tilde{X},\tilde{\tau}),\tilde{f}) \to ((\tilde{E},\tilde{\tau_E}),\tilde{p}\circ\tilde{\bar{e}})$ is essential in Q-**TOP**$/(\tilde{Y},\tilde{\sigma})$, $\tilde{k} : (\tilde{E},\tilde{\tau_E}) \to (\tilde{G},\tilde{\tau_G})$ is an embedding in Q-**TOP**. Thus by Proposition 3.3.2, to prove that $k : (E,\tau_E) \to (G,\tau_G)$ is an embedding in Q-**TOP**, now it is sufficient to prove that k is one-one. Let (T,τ_T) be a Q-topological space and let $\alpha,\beta : (T,\tau_T) \to (E,\tau_E)$ be Q-continuous maps such that $k\circ\alpha = k\circ\beta$. This implies that $q_G \circ k \circ \alpha = q_G \circ k \circ \beta \Rightarrow (q_G \circ k) \circ \alpha = (q_G \circ k) \circ \beta = (\tilde{k} \circ q_E) \circ \alpha = (\tilde{k} \circ q_E) \circ \beta \Rightarrow \tilde{k} \circ (q_E \circ \alpha) = \tilde{k} \circ (q_E \circ \beta)$ and since $\tilde{k}$ is one-one, $q_E \circ \alpha = q_E \circ \beta$. Now since $g \circ q \circ \bar{e} \circ \alpha = q_Y \circ p \circ \bar{e} \circ \alpha = q_Y \circ h \circ k \circ \alpha$ and the diagram 3.5.12 is a pullback, there exists a unique Q-continuous map from (T,τ_T) to (W,θ) making the following diagram commutative



(3.5.17)

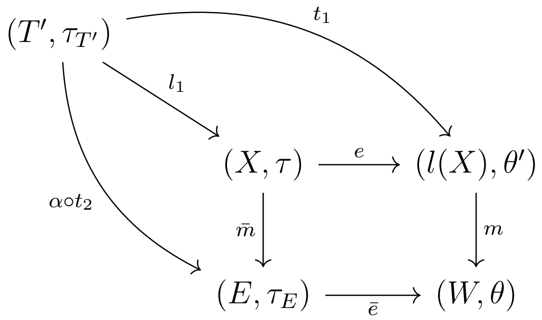
Now we have $p \circ \bar{e} \circ \beta = (p \circ \bar{e}) \circ \beta = (h \circ k) \circ \beta = h \circ (k \circ \beta) = h \circ k \circ \alpha$ and $q \circ \bar{e} \circ \beta = q_Z^{-1} \circ \tilde{q} \circ q_W \circ \bar{e} \circ \beta = q_Z^{-1} \circ \tilde{q} \circ (q_W \circ \bar{e}) \circ \beta = q_Z^{-1} \circ \tilde{q} \circ (\tilde{\bar{e}} \circ q_E) \circ \beta = q_Z^{-1}\circ\tilde{q}\circ\tilde{\bar{e}}\circ(q_E\circ\beta) = q_Z^{-1}\circ\tilde{q}\circ\tilde{\bar{e}}\circ(q_E\circ\alpha) = q_Z^{-1}\circ\tilde{q}\circ(\tilde{\bar{e}}\circ q_E)\circ\alpha = q_Z^{-1}\circ\tilde{q}\circ(q_W\circ\bar{e})\circ\alpha = (q_Z^{-1} \circ \tilde{q} \circ q_W) \circ \bar{e} \circ \alpha = q \circ \bar{e} \circ \alpha$. Also we have $p \circ \bar{e} \circ \alpha = h \circ k \circ \alpha$. Thus here we have two Q-continuous maps $\bar{e} \circ \alpha, \bar{e} \circ \beta : (T,\tau_T) \to (W,\theta)$ making the diagram 3.5.17 commutative. So $\bar{e} \circ \alpha = \bar{e} \circ \beta$. Now consider a pullback of

$\bar{e}\circ\alpha = \bar{e}\circ\beta : (T,\tau_T) \to (W,\theta)$ along $m : (l(X),\theta') \to (W,\theta)$ in Q-**TOP** given by the following:



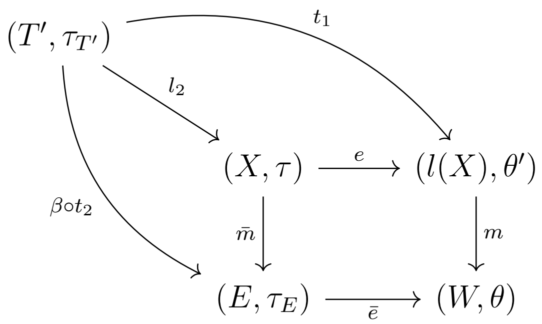
(3.5.18)

Now since the diagram 3.5.14 is a pullback square, there exists a unique Q-continuous map $l_1 : (T',\tau'_T) \to (X,\tau)$ for which the following diagram commutes:



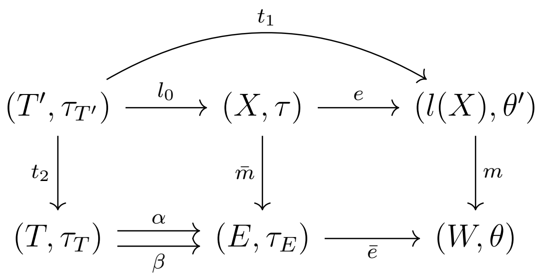
(3.5.19)

Similarly we have a unique Q-continuous map $l_2 : (T',\tau'_T) \to (X,\tau)$ for which the following diagram commutes:



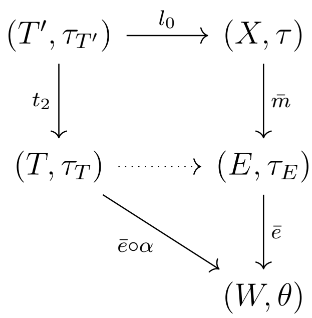
(3.5.20)

Then we have $\beta\circ t_2 = \bar{m}\circ l_2 \Rightarrow k\circ\beta\circ t_2 = k\circ\bar{m}\circ l_2 \Rightarrow k\circ\alpha\circ t_2 = k\circ\bar{m}\circ l_2$ and $\bar{m}\circ l_1 = \alpha\circ t_2 \Rightarrow k\circ\bar{m}\circ l_1 = k\circ\alpha\circ t_2$. Thus $k\circ\bar{m}\circ l_1 = k\circ\bar{m}\circ l_2$ and since $k\circ\bar{m}$ is one-one, $l_1 = l_2$. Let $l_1 = l_2 = l_0$. Thus the following diagram commutes:



(3.5.21)

Now since the diagram 3.5.14 is a pullback complement diagram, with respect to the pullback diagram 3.5.18 and Q-continuous map $l_0 : (T', \tau_{T'}) \to (X, \tau)$, there exists a unique Q-continuous map from (T, τ_T) to (E, τ_E) making the following diagram commutative



(3.5.22)

But here we have two Q-continuous maps $\alpha, \beta : (T, \tau_T) \to (E, \tau_E)$ making the diagram 3.5.22 commutative. So $\alpha = \beta$ and hence k is one-one. Therefore $\bar{m} : ((X, \tau), f) \to ((E, \tau_E), p\circ\bar{e})$ is an injective hull of $((X, \tau), f)$ in Q-**TOP**$/(Y, \sigma)$.

$\square$

3.6 Conclusion

In this chapter, we have obtained a characterization of injective objects (with respect to the class of embeddings in the category Q-**TOP** of Q-topological spaces) in the comma category Q-**TOP**$/(Y, \sigma)$, when (Y, σ) is a stratified Q-topological space, with the help of their T_0-reflection. Further, we have proved that for any Q-topological space (Y, σ), the existence of an injective hull of $((X, \tau), f)$ in the comma category Q-**TOP**$/(Y, \sigma)$ is equivalent to the existence of an injective hull of its T_0-reflection $((\tilde{X}, \tilde{\tau}), \tilde{f})$ in the comma category Q-**TOP**$/(\tilde{Y}, \tilde{\sigma})$ (and in the comma category Q-$\mathbf{TOP}_0/(\tilde{Y},\tilde{\sigma})$, where Q-$\mathbf{TOP}_0$ denotes the category of T_0-Q-topological spaces).