Chapter 2

Exponential Q-topological spaces

2.1 Introduction

In [13], Escardo and Heckmann had proved that a topological space $(Y, \mathcal{T})$ in the category **TOP** of topological spaces is exponential if and only if there exists an splitting-conjoining topology on $C((Y, \mathcal{T}), \mathbb{S})$ which is the set of all continuous functions from $(Y, \mathcal{T})$ to $\mathbb{S}$, where $\mathbb{S}$ is the Sierpinski topological space with two points 1 and 0 such that $\{1\}$ is open but $\{0\}$ is not. In this chapter, we have extended this characterization to the category Q-**TOP** of Q-topological spaces introduced by Solovyov [36].

An object A of a category **C** with finite products is called exponential if the functor $A \times - : \mathbf{C} \to \mathbf{C}$ has a right adjoint (cf. [29]). In 1983, Schwarz [29] gave a characterization of exponential objects in an initially structured category. He proved that an object A of an initially structured category **C** is exponential if and only if for every object B of **C**, there exists a proper admissible **C**-structure on $\mathbf{C}(A, B)$, where $\mathbf{C}(A, B)$ is the set of all morphisms from A to B. Further he had shown that we can restrict the B's to initially dense classes, i.e., he proved that an object A of an initially structured category **C** is exponential if and only if for

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every $B \in \mathbf{D}$, there exists a proper admissible $\mathbf{C}$-structure on $\mathbf{C}(A, B)$, where $\mathbf{D}$ is initially dense in $\mathbf{C}$.

In 1988, Alderton [5] defined the power-set $[A, B]$ of two objects A and B of a monotopological category $\mathbf{C}$. He also defined splitting and conjoining $\mathbf{C}$-structures on $[A, B]$ and gave a similar characterization (as given by Schwarz [29] in an initially structured category) of exponential objects in a monotopological category with splitting objects which says that an object A of a monotopological category $\mathbf{C}$ with splitting objects is exponential if and only if for every object B of $\mathbf{C}$, there exists a splitting-conjoining $\mathbf{C}$-structure on $[A, B]$. We mention here that the terminologies splitting, conjoining and splitting-conjoining are same as proper, admissible and proper-admissible respectively, in an initially structured category if $[A, B]$ is replaced by $\mathbf{C}(A, B)$ (cf. [29]). We also mention that in 1985, Alderton [4] has pointed out that for topological categories if we consider the power-set $[A, B]$ instead of considering the set $\mathbf{C}(A, B)$, then most of the theory developed in [29] for exponential objects holds in topological categories (cf. Alderton [4], pp. 376-377). In particular, he mentioned that the Theorem 3.2 in [29] will hold in topological categories which gives us the following characterization of exponential objects in topological categories:

An object A of a topological category $\mathbf{C}$ is exponential if and only if for every object $B \in \mathbf{D}$, there exists a splitting-conjoining $\mathbf{C}$-structure on $[A, B]$, where $\mathbf{D}$ is initially dense in $\mathbf{C}$.

As mentioned in the first chapter of the thesis, Solovyov [36] in 2008 gave the concept of Q-topological spaces and Q-continuous maps between them and studied the category Q-$\mathbf{TOP}$ of Q-topological spaces (where Q is a fixed member of a fixed variety of Ω-algebras). Solovyov [36] also introduced the Q-Sierpinski space $(Q, \langle id_Q \rangle)$ and Singh and Srivastava [30] proved that $(Q, \langle id_Q \rangle)$ is a Sierpinski object in the category Q-$\mathbf{TOP}$. Hence $\{(Q, \langle id_Q \rangle)\}$ is initially dense in Q-$\mathbf{TOP}$.

We note that Q-$\mathbf{TOP}$ is a monotopological category with splitting objects in the sense of [5] and a topological category in the sense of [4] and hence we have the following characterizations of exponential objects in the category Q-$\mathbf{TOP}$:

1. A Q-topological space (Y, σ) is exponential if and only if for every Q-topological space (Z, η), there exists an splitting-conjoining Q-topology on $[(Y, \sigma), (Z, \eta)]$ (cf. Theorem 2.3, Alderton [5]).

2. A Q-topological space (Y, σ) is exponential if and only if there exists an splitting-conjoining Q-topology on $[(Y, \sigma), (Q, \langle id_Q \rangle)]$ (cf. Alderton [4], pp. 376-377).

We mention here that the above characterizations of exponential Q-topological spaces have been obtained by using the concepts of category theory. As the category **TOP** is an example of the category Q-**TOP**, the above characterizations of exponential objects hold in **TOP**.

In 2001, Escardo and Heckmann [13] obtained the characterization (2) of exponential objects in **TOP** (using the characterization (1) as a Definition of exponential objects in **TOP**) without any use of category theory. In their development, only a basic knowledge of general topology was required. It is important to mention here that *they have called such topological spaces as exponentiable instead of exponential*. Further, the terminologies weak, strong and exponential used by Escardo and Heckmann [13] are the same as splitting, conjoining and splitting-conjoining respectively (in the category **TOP**) used by Alderton [4, 5]. It is also pointed out that, for given topological spaces $(Y_1, \mathcal{T}_1)$ and $(Y_2, \mathcal{T}_2)$, Escardo and Heckmann have taken the set $C((Y_1, \mathcal{T}_1), (Y_2, \mathcal{T}_2))$ of all continuous functions from $(Y_1, \mathcal{T}_1)$ to $(Y_2, \mathcal{T}_2)$, instead of the power-set $[(Y_1, \mathcal{T}_1), (Y_2, \mathcal{T}_2)]$, but in **TOP**, $[(Y_1, \mathcal{T}_1), (Y_2, \mathcal{T}_2)] = C((Y_1, \mathcal{T}_1), (Y_2, \mathcal{T}_2))$ (cf. Alderton [6]).

Motivated by Escardo and Heckmann [13], in this chapter, we have obtained the characterization (2) of exponential objects in the category Q-**TOP**. We have not used categorical concepts in our proofs, only some basic concepts of Q-topological spaces are required. In this process, for a given Q-topological space (Y, σ), we have also obtained a relation between the splitting-conjoining Q-topologies on $[(Y, \sigma), (Q, \langle id_Q \rangle)]$ and $[(Y, \sigma), (Z, \eta)]$, for every Q-topological space (Z, η).

2.2 Monotopological and Topological categories

Remark 2.2.1. In this chapter, we will use the definition of a topological category in the sense of Alderton [5]. We mention here that the topological categories in the sense of Alderton [5] are topological categories over **Set** in the sense of [1] (cf. Definition 1.2.32)

Definition 2.2.2. [1] A source $\{f_j : A \to A_j \mid j \in J\}$ in **Set** is called **point-separating** source if for $a, b \in A$, $a \neq b$, there exists $j \in J$ such that $f_j(a) \neq f_j(b)$.

In the following definition, a concrete category over **Set** will mean a category **C** having objects as structured sets, i.e., objects of **C** are of the form (Y, s), where Y is a set and s is a **C**-structure on Y and for given two objects (Y, s) and (Z, p) of **C**, the set of all morphisms from (Y, s) to (Z, p) consists of structure compatible functions from Y to Z and their composition is the usual composition of functions.

Definition 2.2.3. [5] Let **C** be a concrete category over **Set** with forgetful functor $|-|$: $\mathbf{C} \to \mathbf{Set}$. Then **C** is called **monotopological** if it satisfies the following conditions:

1. Every point-separating $|-|$-source $\{g_k : Y \to |(Y_k, s_k)|| \ k \in K\}$ has a unique initial lift.

2. For every set Y, the class of all **C**-structures on Y is a set and there is only one **C**-structure on $\emptyset$.

If the condition (1) is replaced by the following condition:

(T) Given any $|-|$-source $\{g_k : Y \to |(Y_k, s_k)|| \ k \in K\}$, there exists a unique initial lift of the source $\{g_k : Y \to |(Y_k, s_k)|| \ k \in K\}$,
then **C** is called **topological**.

Remark 2.2.4. If we add one more condition in the Definition 2.2.3 given by,

(a) There is only one **C**-structure on each singleton,

then **C** is called an **initially structured** category (cf. [29]).

Remark 2.2.5. We note that Q-**TOP** is a topological category (in the sense of [5]) in view of Proposition 1.2.41 and the fact that the class of all Q-topologies on any given set X is a set.

2.3 Splitting-conjoining Q-topology on Power-set in Q-TOP

Definition 2.3.1. [13] Let Y, Z and X be sets and $g : X \times Y \to Z$ be a map. Define $\overline{g} : X \to Z^Y$ as

$$\overline{g}(x)(y) = g(x, y), \text{ for every } x \in X \text{ and every } y \in Y.$$

Also if we have a map $\overline{g} : X \to Z^Y$, then we can define a map $g : X \times Y \to Z$ as

$$g(x, y) = \overline{g}(x)(y), \text{ for every } x \in X \text{ and every } y \in Y.$$

Remark 2.3.2. Let (Y, σ) and (Z, η) be Q-topological spaces. Then $C((Y, \sigma), (Z, \eta))$ will denote the set of all Q-continuous maps from (Y, σ) to (Z, η) and $(Y, \sigma) \times (Z, \eta)$ will denote the product of (Y, σ) and (Z, η) in Q-**TOP**.

Q-**TOP** has finite concrete products (in view of Proposition 1.2.45), hence we can define power-set of two Q-topological spaces as a consequence of Alderton [5], Definition 1.1.

Definition 2.3.3. Let (Y, σ) and (Z, η) be Q-topological spaces. Define the **power-set** of (Y, σ) and (Z, η) by,

$$[(Y, \sigma), (Z, \eta)] = \{f : Y \to Z \mid \exists \text{ a } Q\text{-topological space } (X, \tau),\ x \in X \text{ and } g : (X, \tau) \times (Y, \sigma) \to (Z, \eta) \text{ a } Q\text{-continuous map such that } \overline{g}(x) = f\}$$

The following definition is a consequence of the Definition 2.1 in Alderton [5], as Q-**TOP** is a category with finite concrete products.

Definition 2.3.4. Let (Y, σ) and (Z, η) be Q-topological spaces. A Q-topology θ on $[(Y, \sigma), (Z, \eta)]$ is called

1. **splitting** if $\overline{g} : (X, \tau) \to ([(Y, \sigma), (Z, \eta)], \theta)$ is Q-continuous whenever $g : (X, \tau) \times (Y, \sigma) \to (Z, \eta)$ is Q-continuous.
2. **conjoining** if $g : (X, \tau) \times (Y, \sigma) \to (Z, \eta)$ is Q-continuous whenever $\overline{g} : (X, \tau) \to ([(Y, \sigma), (Z, \eta)], \theta)$ is Q-continuous.

Thus a Q-topology θ on $[(Y, \sigma), (Z, \eta)]$ is splitting-conjoining if and only if it makes the map $g \mapsto \overline{g}$ into a well defined bijection from $C((X, \tau) \times (Y, \sigma), (Z, \eta))$ to $C((X, \tau), ([(Y, \sigma), (Z, \eta)], \theta))$, for every Q-topological space (X, τ).

Proposition 2.3.5. Let $(\prod Y_k, \eta)$ be the product of the family $\{(Y_k, \sigma_k) \mid k \in K\}$ of Q-topological spaces. Let (Y, σ) be a Q-topological space and $p_k : \prod Y_k \to Y_k,\ k \in K$ be the projection maps. Then a map $g : (Y, \sigma) \to (\prod Y_k, \eta)$ is Q-continuous if and only if $p_k \circ g : (Y, \sigma) \to (Y_k, \sigma_k)$ is Q-continuous for every $k \in K$.

Proof. Suppose first that $g : (Y, \sigma) \to (\prod Y_k, \eta)$ is Q-continuous. Let $\alpha \in \sigma_k$. Then $\alpha \circ p_k \circ g = (\alpha \circ p_k) \circ g \in \sigma$ as $g : (Y, \sigma) \to (\prod Y_k, \eta)$ is Q-continuous and $\alpha \circ p_k \in \eta$. Therefore $p_k \circ g : (Y, \sigma) \to (Y_k, \sigma_k)$ is Q-continuous.

Conversely, assume that $p_k \circ g : (Y, \sigma) \to (Y_k, \sigma_k)$ is Q-continuous, for every $k \in K$. Then $\alpha \circ p_k \circ g = \alpha \circ (p_k \circ g) \in \sigma$, for every $\alpha \in \sigma_k$ as $p_k \circ g : (Y, \sigma) \to (Y_k, \sigma_k)$ is Q-continuous. Therefore from Proposition 1.2.40, $g : (Y, \sigma) \to (\prod Y_k, \eta)$ is Q-continuous. □

Proposition 2.3.6. Let $g_1 : (X_1, \tau_1) \to (Y_1, \sigma_1)$ and $g_2 : (X_2, \tau_2) \to (Y_2, \sigma_2)$ be Q-continuous maps. Then the map $g_1 \times g_2 : (X_1, \tau_1) \times (X_2, \tau_2) \to (Y_1, \sigma_1) \times (Y_2, \sigma_2)$ defined as $(g_1 \times g_2)(x_1, x_2) = (g_1(x_1), g_2(x_2))$, for every $(x_1, x_2) \in X_1 \times X_2$, is Q-continuous.

Proof. Let $p_{X_k} : X_1 \times X_2 \to X_k$ and $p_{Y_k} : Y_1 \times Y_2 \to Y_k,\ k = 1, 2$ be projection maps. Now consider the map $p_{Y_k} \circ (g_1 \times g_2) : (X_1, \tau_1) \times (X_2, \tau_2) \to (Y_k, \sigma_k)$ and let $\alpha \in \sigma_k$. Then $(\alpha \circ p_{Y_k} \circ (g_1 \times g_2))(x_1, x_2) = (\alpha \circ g_k)(x_k) = \alpha(g_k(x_k)) = \alpha((g_k \circ p_{X_k})(x_1, x_2)) = (\alpha \circ g_k \circ p_{X_k})(x_1, x_2) = ((\alpha \circ g_k) \circ p_{X_k})(x_1, x_2)$. Hence $\alpha \circ p_{Y_k} \circ (g_1 \times g_2) = (\alpha \circ g_k) \circ p_{X_k}$. Now since $g_k : (X_k, \tau_k) \to (Y_k, \sigma_k)$ is Q-continuous and $\alpha \in \sigma_k$, $\alpha \circ g_k \in \tau_k$. Hence $\alpha \circ p_{Y_k} \circ (g_1 \times g_2) = (\alpha \circ g_k) \circ p_{X_k}$ belongs to the product Q-topology of the Q-topological spaces (X_1, τ_1) and (X_2, τ_2). Thus $p_{Y_k} \circ (g_1 \times g_2) : (X_1, \tau_1) \times (X_2, \tau_2) \to (Y_k, \sigma_k)$ is Q-continuous. Therefore by Proposition 2.3.5, the map $g_1 \times g_2 : (X_1, \tau_1) \times (X_2, \tau_2) \to (Y_1, \sigma_1) \times (Y_2, \sigma_2)$ is Q-continuous. □

The proof of following proposition is on similar lines as Lemma 2.1 in [13], though using only the concepts of Q-topological spaces, we are giving here a proof of the proposition.

Proposition 2.3.7. Let (Y, σ) and (Z, η) be Q-topological spaces. A Q-topology θ on $[(Y, \sigma), (Z, \eta)]$ is conjoining if and only if the evaluation map

$$\varepsilon : ([(Y, \sigma), (Z, \eta)], \theta) \times (Y, \sigma) \to (Z, \eta)$$
$$(f, y) \mapsto f(y)$$

is Q-continuous.

Proof. Suppose first that the Q-topology θ on $[(Y,\sigma),(Z,\eta)]$ is conjoining. Let $f \in [(Y,\sigma),(Z,\eta)]$. Then $\overline{\varepsilon}(f)(y) = \varepsilon(f,y) = f(y)$. So we get that $\overline{\varepsilon}(f) = f$, for every $f \in [(Y,\sigma),(Z,\eta)]$. Now let $\alpha \in \theta$. Then $(\alpha \circ \overline{\varepsilon})(f) = \alpha(\overline{\varepsilon}(f)) = \alpha(f)$. This implies that $\alpha \circ \overline{\varepsilon} = \alpha \in \theta$. Hence the map $\overline{\varepsilon} : ([(Y,\sigma),(Z,\eta)],\theta) \to ([(Y,\sigma),(Z,\eta)],\theta)$ is Q-continuous. Now since the Q-topology θ on $[(Y,\sigma),(Z,\eta)]$ is conjoining, $\varepsilon : ([(Y,\sigma),(Z,\eta)],\theta) \times (Y,\sigma) \to (Z,\eta)$ is Q-continuous.

Conversely, assume that the evaluation map $\varepsilon : ([(Y,\sigma),(Z,\eta)],\theta) \times (Y,\sigma) \to (Z,\eta)$ is Q-continuous. We have to show that θ is conjoining. Let $\overline{g} : (X,\tau) \to ([(Y,\sigma),(Z,\eta)],\theta)$ be Q-continuous. Consider $(\varepsilon \circ (\overline{g} \times id_Y))(x,y) = \varepsilon(\overline{g}(x),y) = \overline{g}(x)(y) = g(x,y)$. This implies that $g = \varepsilon \circ (\overline{g} \times id_Y)$. Now since $\overline{g} : (X,\tau) \to ([(Y,\sigma),(Z,\eta)],\theta)$ and $id_Y : (Y,\sigma) \to (Y,\sigma)$ are Q-continuous, by Proposition 2.3.6, $\overline{g} \times id_Y : (X,\tau) \times (Y,\sigma) \to ([(Y,\sigma),(Z,\eta)],\theta) \times (Y,\sigma)$ is Q-continuous. Hence $g : (X,\tau) \times (Y,\sigma) \to (Z,\eta)$ is Q-continuous as $g = \varepsilon \circ (\overline{g} \times id_Y)$. Therefore θ is conjoining. □

Definition 2.3.8. Let Y be a set and σ_1 and σ_2 be Q-topologies on the set Y. If $\sigma_1 \subseteq \sigma_2$, then we say that σ_1 **is weaker than** σ_2. In this case we also say that σ_2 **is stronger than** σ_1.

We note that since every topological category has splitting objects (cf. Example 2.6 (i), Alderton [5]) and Q-**TOP** is a topological category, it is a monotopological category with splitting objects. Alderton [5] mentioned that the result similar to Proposition 2.6 in [29] will hold in monotopological categories with splitting objects (cf. Alderton [5], proof of the Theorem 2.3), so we have the following result, the proof of which is on similar lines as of the Proposition 2.6 in [29], though using only the concepts of Q-topological spaces, we are giving here a proof of the proposition.

Proposition 2.3.9. Let (Y,σ) and (Z,η) be Q-topological spaces. Then,

1. Any Q-topology on $[(Y,\sigma),(Z,\eta)]$ weaker than a splitting Q-topology on $[(Y,\sigma),(Z,\eta)]$ is also splitting.
2. Any Q-topology on $[(Y,\sigma),(Z,\eta)]$ stronger than a conjoining Q-topology on $[(Y,\sigma),(Z,\eta)]$ is also conjoining.

3. Any splitting Q-topology on $[(Y,\sigma),(Z,\eta)]$ is weaker than any conjoining Q-topology on $[(Y,\sigma),(Z,\eta)]$.

4. If θ is splitting-conjoining Q-topology on $[(Y,\sigma),(Z,\eta)]$, then θ is uniquely determined.

Proof. (1) and (2) are obvious.
(3) Let θ_1 and θ_2 be conjoining and splitting Q-topologies on $[(Y,\sigma),(Z,\eta)]$ respectively. We have to show that $\theta_2 \subseteq \theta_1$. Since θ_1 is conjoining, the evaluation map $\varepsilon : ([(Y,\sigma),(Z,\eta)],\theta_1) \times (Y,\sigma) \to (Z,\eta)$ is Q-continuous. Now since θ_2 is splitting, $\overline{\varepsilon} : ([(Y,\sigma),(Z,\eta)],\theta_1) \to ([(Y,\sigma),(Z,\eta)],\theta_2)$ is Q-continuous. This implies that $\alpha \circ \overline{\varepsilon} \in \theta_1$, for every $\alpha \in \theta_2$. But we have seen that $\overline{\varepsilon}(f) = f$ and so $(\alpha \circ \overline{\varepsilon})(f) = \alpha(\overline{\varepsilon}(f)) = \alpha(f)$. Hence $\alpha = \alpha \circ \overline{\varepsilon} \in \theta_1$, for every $\alpha \in \theta_2$. Therefore $\theta_2 \subseteq \theta_1$.
(4) It follows from (3). □

2.4 A characterization of exponential objects in Q-TOP

Definition 2.4.1. [1] Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and let B be a $\mathbf{D}$-object.

1. A F-**costructured arrow with codomain** B is a pair (A, f) consisting of a $\mathbf{C}$-object A and a $\mathbf{D}$-morphism $f : FA \to B$.

2. A F-costructured arrow (A, f) with codomain B is called F-**co-universal** for B if for each F-costructured arrow $(\hat{A}, g)$ with codomain B there exists a unique $\mathbf{C}$-morphism $h : \hat{A} \to A$ such that $g = f \circ Fh$, i.e., such that the triangle

$$\begin{array}{ccc} F\hat{A} & \xrightarrow{Fh} & FA \\ {\scriptstyle g}\downarrow & \swarrow{\scriptstyle f} & \\ B & & \end{array} \tag{2.4.1}$$

commutes.

Definition 2.4.2. [1] Let **C** and **D** be categories and let $F : \mathbf{C} \to \mathbf{D}$ be a functor. We say that $F : \mathbf{C} \to \mathbf{D}$ **has a right adjoint** if for every **D**-object B there exists a F-co-universal arrow with codomain B.

Definition 2.4.3. [29] Let **C** be a category with finite products. An object A of **C** is called **exponential** if the functor $A \times - : \mathbf{C} \to \mathbf{C}$ has a right adjoint.

The following result follows from the Theorem 2.3 in Alderton [5] as Q-**TOP** is a monotopological category with splitting objects.

Theorem 2.4.4. Let (Y, σ) be a Q-topological space. Then both of the following statements are equivalent:

1. The functor $(Y, \sigma) \times - : Q$-**TOP**$\to Q$-**TOP** has a right adjoint.
2. Given any Q-topological space (Z, η), there exists an splitting-conjoining Q-topology on $[(Y, \sigma), (Z, \eta)]$.

Thus in view of Definition 2.4.3 and Theorem 2.4.4, we can define exponential Q-topological spaces as follows:

Definition 2.4.5. Let (Y, σ) be a Q-topological space. Then (Y, σ) is called **exponential** if there exists an splitting-conjoining Q-topology on $[(Y, \sigma), (Z, \eta)]$, for every Q-topological space (Z, η).

From now onwards, we will follow the Definition 2.4.5 for exponential Q-topological spaces.

Lemma 2.4.6. Let (Y, σ) and (Z, η) be Q-topological spaces. Let $f \in [(Y, \sigma), (Z, \eta)]$ and $\beta \in \eta$, then $\beta \circ f \in [(Y, \sigma), (Q, \langle id_Q \rangle)]$.

Proof. We note that since $f \in [(Y, \sigma), (Z, \eta)]$, there exist a Q-topological space (X, τ), $x \in X$ and a Q-continuous map $h : (X, \tau) \times (Y, \sigma) \to (Z, \eta)$ such that $\overline{h}(x) = f$. Now consider the map $\beta \circ h : (X, \tau) \times (Y, \sigma) \to (Q, \langle id_Q \rangle)$. Since $h : (X, \tau) \times (Y, \sigma) \to (Z, \eta)$ is Q-continuous and $\beta \in \eta$, $\beta \circ h$ belongs to the product Q-topology of the Q-topological spaces (X, τ) and (Y, σ). Now since $id_Q \circ (\beta \circ h) = \beta \circ h$, $id_Q \circ (\beta \circ h)$ belongs to the product Q-topology of the Q-topological spaces (X, τ) and (Y, σ). So by Proposition 1.2.40, $\beta \circ h : (X, \tau) \times (Y, \sigma) \to (Q, \langle id_Q \rangle)$ is Q-continuous. Now consider,

$$(\overline{\beta \circ h})(x)(y) = (\beta \circ h)(x,y) = \beta(h(x,y)) = \beta(\overline{h}(x)(y)) = (\beta \circ \overline{h}(x))(y) = (\beta \circ f)(y),$$
$$\text{for every } y \in Y.$$

Hence $(\overline{\beta \circ h})(x) = \beta \circ f$ and therefore $\beta \circ f \in [(Y,\sigma),(Q,\langle id_Q\rangle)]$. □

Let (Y,σ) and (Z,η) be Q-topological spaces and θ be a Q-topology on $[(Y,\sigma),(Q,\langle id_Q\rangle)]$. Let $\psi \in \theta$ and $\beta \in \eta$. Define a map $\theta(\psi,\beta) : [(Y,\sigma),(Z,\eta)] \to Q$ as,

$$\theta(\psi,\beta)(f) = \psi(\beta \circ f), \text{ for every } f \in [(Y,\sigma),(Z,\eta)].$$

Here we point out that by Lemma 2.4.6, $\beta \circ f \in [(Y,\sigma),(Q,\langle id_Q\rangle)]$. Now let $\theta_{(Z,\eta)} = \langle\{\theta(\psi,\beta) \mid \psi \in \theta,\ \beta \in \eta\}\rangle$. Then $\theta_{(Z,\eta)}$ is a Q-topology on $[(Y,\sigma),(Z,\eta)]$, called as the Q-topology induced by θ.

The following Propositions 2.4.7, 2.4.8 and Theorem 2.4.9 are concerned with the extensions of the results of Lemma 3.1 in [13], for Q-topological spaces.

Proposition 2.4.7. Let (Y,σ) be a Q-topological space and θ be a Q-topology on $[(Y,\sigma),(Q,\langle id_Q\rangle)]$. Then θ is splitting if and only if the Q-topology $\theta_{(Z,\eta)}$ on $[(Y,\sigma),(Z,\eta)]$ induced by θ is splitting for every Q-topological space (Z,η).

Proof. Suppose first that θ is splitting and let $g : (X,\tau) \times (Y,\sigma) \to (Z,\eta)$ be Q-continuous. We have to show that $\overline{g} : (X,\tau) \to ([(Y,\sigma),(Z,\eta)],\theta_{(Z,\eta)})$ is Q-continuous. Let $\psi \in \theta$ and $\beta \in \eta$. Then,

$$(\theta(\psi,\beta) \circ \overline{g})(x) = \theta(\psi,\beta)(\overline{g}(x)) = \psi(\beta \circ \overline{g}(x)), \text{ for every } x \in X.$$

Now consider the map $\beta \circ g : (X,\tau) \times (Y,\sigma) \to (Q,\langle id_Q\rangle)$. Since $g : (X,\tau) \times (Y,\sigma) \to (Z,\eta)$ is Q-continuous and $\beta \in \eta$, $\beta \circ g$ belongs to the product Q-topology of the Q-topological spaces (X,τ) and (Y,σ). Hence the map $\beta \circ g : (X,\tau) \times (Y,\sigma) \to (Q,\langle id_Q\rangle)$ is Q-continuous. Now since θ is splitting, $\overline{\beta \circ g} : (X,\tau) \to ([(Y,\sigma),(Q,\langle id_Q\rangle)],\theta)$ is Q-continuous. Also we have,

$$\overline{\beta \circ g}(x)(y) = (\beta \circ g)(x,y) = \beta(g(x,y)) = \beta(\overline{g}(x)(y)) = (\beta \circ \overline{g}(x))(y), \text{ for every}$$
$$y \in Y.$$

Thus $\overline{\beta \circ g}(x) = \beta \circ \overline{g}(x)$. Now,

$$(\psi \circ \overline{\beta \circ g})(x) = \psi(\overline{\beta \circ g}(x)) = \psi(\beta \circ \overline{g}(x)) = (\theta(\psi, \beta) \circ \overline{g})(x), \text{ for every } x \in X.$$

This implies that $\psi \circ \overline{\beta \circ g} = \theta(\psi, \beta) \circ \overline{g}$. Now since $\overline{\beta \circ g} : (X, \tau) \to ([(Y, \sigma), (Q, \langle id_Q \rangle)], \theta)$ is Q-continuous and $\psi \in \theta$, $\psi \circ \overline{\beta \circ g} = \theta(\psi, \beta) \circ \overline{g} \in \tau$. Note that $\theta_{(Z,\eta)} = \langle \{\theta(\psi, \beta) \mid \psi \in \theta, \ \beta \in \eta\} \rangle$ and hence by Proposition 1.2.40, $\overline{g} : (X, \tau) \to ([(Y, \sigma), (Z, \eta)], \theta_{(Z,\eta)})$ is Q-continuous. Therefore $\theta_{(Z,\eta)}$ is splitting.

Conversely, assume that the Q-topology $\theta_{(Z,\eta)}$ on $[(Y, \sigma), (Z, \eta)]$ induced by θ is splitting for every Q-topological space (Z, η). Then the Q-topology $\theta_{(Q,\langle id_Q \rangle)}$ on $[(Y, \sigma), (Q, \langle id_Q \rangle)]$ induced by θ is splitting. Let $\psi \in \theta$ and $f \in [(Y, \sigma), (Q, \langle id_Q \rangle)]$. Consider $\theta(\psi, id_Q)(f) = \psi(id_Q \circ f) = \psi(f)$. Thus $\theta(\psi, id_Q) = \psi$ and since $\theta(\psi, id_Q) \in \theta_{(Q,\langle id_Q \rangle)}$, $\psi \in \theta_{(Q,\langle id_Q \rangle)}$. This implies that $\theta \subseteq \theta_{(Q,\langle id_Q \rangle)}$. Now since $\theta_{(Q,\langle id_Q \rangle)}$ is splitting and $\theta \subseteq \theta_{(Q,\langle id_Q \rangle)}$, by Proposition 2.3.9, θ is splitting. □

Proposition 2.4.8. Let (Y, σ) be a Q topological space and θ be a Q-topology on $[(Y, \sigma), (Q, \langle id_Q \rangle)]$. If θ is conjoining, then the Q-topology $\theta_{(Z,\eta)}$ on $[(Y, \sigma), (Z, \eta)]$ induced by θ is conjoining for every Q-topological space (Z, η).

Proof. Suppose that θ is conjoining and let $\overline{g} : (X, \tau) \to ([(Y, \sigma), (Z, \eta)], \theta_{(Z,\eta)})$ be Q-continuous. We have to show that $g : (X, \tau) \times (Y, \sigma) \to (Z, \eta)$ is Q-continuous. Let $\beta \in \eta$. Note that $\overline{\beta \circ g}(x) = \beta \circ \overline{g}(x)$. Now, since $\overline{g}(x) \in [(Y, \sigma), (Z, \eta)]$ and $\beta \in \eta$, by Lemma 2.4.6, $\beta \circ \overline{g}(x) = \overline{\beta \circ g}(x) \in [(Y, \sigma), (Q, \langle id_Q \rangle)]$. So $\overline{\beta \circ g}$ is a mapping from X to $[(Y, \sigma), (Q, \langle id_Q \rangle)]$. Next, let $\psi \in \theta$, then,

$$(\theta(\psi, \beta) \circ \overline{g})(x) = \theta(\psi, \beta)(\overline{g}(x)) = \psi(\beta \circ \overline{g}(x)) = \psi(\overline{\beta \circ g}(x)) = (\psi \circ \overline{\beta \circ g})(x), \text{ for every } x \in X.$$

This implies that $\theta(\psi, \beta) \circ \overline{g} = \psi \circ \overline{\beta \circ g}$. Since $\overline{g} : (X, \tau) \to ([(Y, \sigma), (Z, \eta)], \theta_{(Z,\eta)})$ is Q-continuous and $\theta(\psi, \beta) \in \theta_{(Z,\eta)}$, $\theta(\psi, \beta) \circ \overline{g} = \psi \circ \overline{\beta \circ g} \in \tau$. Thus $\overline{\beta \circ g} : (X, \tau) \to ([(Y, \sigma), (Q, \langle id_Q \rangle)], \theta)$ is Q-continuous. Now since θ is conjoining, $\beta \circ g : (X, \tau) \times (Y, \sigma) \to (Q, \langle id_Q \rangle)$ is Q-continuous. Thus $\beta \circ g$ belongs to the product Q-topology of the Q-topological spaces (X, τ) and (Y, σ). Hence the map $g : (X, \tau) \times (Y, \sigma) \to (Z, \eta)$ is Q-continuous. Therefore $\theta_{(Z,\eta)}$ is conjoining. □

Theorem 2.4.9. Let (Y, σ) be a Q topological space and θ be a Q-topology on $[(Y, \sigma), (Q, \langle id_Q \rangle)]$. If θ is splitting-conjoining, then the Q-topology $\theta_{(Z,\eta)}$ on $[(Y, \sigma), (Z, \eta)]$ induced by θ is splitting-conjoining for every Q-topological space (Z, η).

Further, if the Q-topology $\theta_{(Z,\eta)}$ on $[(Y,\sigma),(Z,\eta)]$ induced by θ is splitting-conjoining for every Q-topological space (Z,η), then θ is splitting-conjoining if and only if $\theta = \theta_{(Q,\langle id_Q \rangle)}$.

Proof. The proof of the first part of the Theorem follows from Propositions 2.4.7 and 2.4.8.

Now we will prove the second part. We are given that $\theta_{(Z,\eta)}$ is splitting-conjoining for every Q-topological space (Z,η). Let θ be splitting-conjoining, then $\theta_{(Z,\eta)}$ is splitting-conjoining (in view of Propositions 2.4.7 and 2.4.8). In particular, $\theta_{(Q,\langle id_Q \rangle)}$ is splitting-conjoining. Now using Proposition 2.3.9(4), $\theta = \theta_{(Q,\langle id_Q \rangle)}$. Conversely, if $\theta = \theta_{(Q,\langle id_Q \rangle)}$, then θ is splitting-conjoining (it follows from the given condition itself).

□

Thus we have obtained the following characterization of exponential objects in the category Q-**TOP**.

Theorem 2.4.10. A Q-topological space (Y,σ) is exponential if and only if there exists an splitting-conjoining Q-topology on $[(Y,\sigma),(Q,\langle id_Q \rangle)]$. In this case, for a given Q-topological space (Z,η), the splitting-conjoining Q-topology on $[(Y,\sigma),(Z,\eta)]$ is the Q-topology induced by the splitting-conjoining Q-topology on $[(Y,\sigma),(Q,\langle id_Q \rangle)]$.

2.5 Conclusion

It is known that a topological space $(Y,\mathcal{T})$ is exponential in the category **TOP** of topological spaces if and only if there exists an splitting-conjoining topology on $C((Y,\mathcal{T}),\mathbb{S})$, where $\mathbb{S}$ is the Sierpinski topological space with two points 1 and 0 such that $\{1\}$ is open but $\{0\}$ is not (cf. [13]). This chapter extends this characterization to the category Q-**TOP** of Q-topological spaces introduced by Solovyov [36]. As mentioned in the introduction, in the proofs, our approach is not category theoretic, only some basic concepts of Q-topological spaces are required. Our study is motivated by Escardo and Heckmann [13], who obtained the above characterization of exponential topological spaces in the category **TOP** without using categorical concepts. In the results, presented in this chapter, the

Q-Sierpinski space plays a key role. It is well known that a topological space is exponential in the category **TOP** if and only if it is core compact (cf. [13]). The problem of characterizing exponential Q-topological spaces in terms of core compactness is still open. To tackle this problem, first it is required to extend the concept of core compactness for Q-topological spaces and then to study the characterization of exponential Q-topological spaces in terms of core compactness.