Chapter 1

Introduction and Preliminaries

1.1 Introduction

Fuzzy sets were introduced by Zadeh [43] in 1965. Mathematically, a fuzzy set in a set X is a function from X to the unit interval $[0,1]$. This concept clearly generalizes the concept of a subset of a set if we identify a subset by its characteristic function. Zadeh also extended the notion of inclusion, union, intersection, complement etc., as for sets, to fuzzy sets and gave various properties related to these notions in the context of fuzzy sets.

Chang [10] in 1968, introduced a fuzzy topology on a set X as a collection of fuzzy sets in X which is closed under arbitrary union, finite intersection and contains X and $\emptyset$. He also introduced many basic concepts in fuzzy topology, e.g., continuity, compactness etc. In 1967, Goguen [15] generalized the notion of a fuzzy set and introduced an L-fuzzy set, by replacing $[0,1]$ in the definition of a fuzzy set by a suitable lattice L. In 1973, Goguen [16] introduced an L-topological space as a generalization of a fuzzy topological space in an obvious way.

Lowen [22], in 1976, observed the absence of a basic feature in the Chang's concept of a fuzzy topology (when compared with topology), viz., constant maps between such spaces need not be continuous. He believed that this feature in general, ought to be present in fuzzy topology also. So he modified Chang's definition of a fuzzy topology by redefining a fuzzy topology on a set X as a collection of fuzzy sets in X which is closed under arbitrary union, finite intersection and contains all constant fuzzy sets. Lowen's fuzzy topological spaces satisfy several other desirable properties, which were not satisfied by Chang's fuzzy topological spaces (cf. Lowen and Wuyts [23]).

We mention that there have also been other approaches to fuzzy topology, e.g., as given by Höhle [20], Hutton [21], Šostak [37], Rodabaugh [28].

If L is a lattice, we observe that it has both, an order structure as well as an algebraic structure through its 'join' and 'meet' operations. In 2008, Solovyov [36] generalized the Goguen's L-valued set (i.e. L-fuzzy set) by introducing algebra-valued Q-set for some fixed Ω-algebra Q. Mathematically, if Q is a fixed Ω-algebra, then a Q-set on a set X is a map from X to Q. Continuing further, Solovyov introduced Q-topological spaces as follows. Given a set X and a fixed Ω-algebra Q, a subset τ of Q^X (where Q^X denotes the set of all maps from X to Q), was called a Q-topology on X by Solovyov, if τ is a subalgebra of the Ω-algebra Q^X (more details will appear in the Preliminaries section). The pair (X, τ) is called a Q-topological space. Solovyov also introduced and studied the category Q-**TOP** of Q-topological spaces. It must be noted that many results in L-topological spaces, use the order structure of L. However, in passing from a lattice to an algebra Q, the order structure is lost. But even then, analogues of many concepts and results in L-topology, were shown to exist in Q-topology, by Solovyov. In particular, Solovyov introduced the Q-Sierpinski space, T_0-Q-topological space, stratified Q-topological space, sober Q-topological space etc. and studied many properties related to these notions, including those of categorical nature.

Before continuing further with Q-topological spaces, we wish to point out that Q-topological spaces are a special case of 'affine sets' of Giuli [14], which we briefly describe next.

An affine set over a given set A is a pair $(X, A(X))$, where X is a set and $A(X)$ is a subset of the set A^X of all functions from X to A. If $(X, A(X))$ and $(Y, A(Y))$ are two affine sets, both over A, then a map $f : X \to Y$ is called an affine map if for each $\beta \in A(Y)$, $\beta \circ f \in A(X)$. Many well known structures, e.g., topological spaces, closure spaces, fuzzy topological spaces etc. are examples of affine sets. Giuli [14] initiated the study of the category **ASet** of all affine sets over A and affine maps between them, in 2005. Giuli also noted that similar objects have been studied earlier by Diers [11] for the special case when the 'base-set' A is an Ω-algebra (in the sense of universal algebra), in this case the resulting category is denoted by $\mathbf{ASet}(\Omega)$.

The study of Q-topological spaces was continued further by Singh and Srivastava [30–33] and Noor et al. [27] etc.

In particular, Singh and Srivastava [30] gave a characterization of Q-**TOP** in terms of the Q-Sierpinski space.

Further, they also identified the category Q-$\mathbf{TOP}_0$ of T_0-Q-topological spaces and the category Q-**Sob** of sober Q-topological spaces as the epireflective hull of the Q-Sierpinski space in the category Q-**TOP** (cf. [32]) and Q-$\mathbf{TOP}_0$ (cf. [33]) respectively.

They also studied connectedness and disconnectedness for Q-topological spaces with respect to a class of Q-topological spaces (cf. [31]).

Noor et al. [27] considered the category Q-**BTOP** of Q-bitopological spaces and determined two Sierpinski objects in Q-**BTOP**. They also showed that the epireflective hulls of both these Sierpinski objects in the category Q-**BTOP** is the category Q-$\mathbf{BTOP}_0$ of T_0-Q-bitopological spaces.

A large part of the present thesis is concerned with a further study of Q-topological spaces. In particular, we have studied exponential Q-topological spaces, injective objects and existence of injective hulls in the comma category Q-$\mathbf{TOP}/(Y,\sigma)$, some coreflective hulls in the category **Str**-Q-**TOP** of stratified Q-topological spaces. In a somewhat different direction, we have also given in this thesis, a characterization of the category **FCS** of fuzzy closure spaces, as considered in Srivastava et al. [40], in terms of the Sierpinski fuzzy closure space.

The thesis is divided into five chapters.

Chapter 1 is introductory, which contains a brief introduction of the subject, basic definitions and results which are used in the thesis.

In chapter 2, motivated by Escardo and Heckmann [13], we have obtained a characterization of exponential objects in the category Q-**TOP** of Q-topological spaces. We have not used categorical concepts in our proofs, only some basic concepts of Q-topological spaces are required.

In chapter 3, motivated by Cagliari and Mantovani [8], we have obtained a characterization of injective objects (with respect to the class of embeddings in the category Q-**TOP** of Q-topological spaces) in the comma category Q-$\mathbf{TOP}/(Y,\sigma)$, when (Y,σ) is a stratified Q-topological space, with the help of their T_0-reflection.

Further, we have proved that for any Q-topological space (Y, σ), the existence of an injective hull of $((X, \tau), f)$ in the comma category Q-**TOP**$/(Y, \sigma)$ is equivalent to the existence of an injective hull of its T_0-reflection $((\tilde{X}, \tilde{\tau}), \tilde{f})$ in the comma category Q-**TOP**$/(\tilde{Y}, \tilde{\sigma})$ (and in the comma category Q-**TOP**$_0/(\tilde{Y}, \tilde{\sigma})$, where Q-**TOP**$_0$ denotes the category of T_0-Q-topological spaces).

In chapter 4, motivated by Singh [34], we have determined the coreflective hull of $(Q, \langle\{id_Q\} \cup \{\underline{q} \mid q \in Q\}\rangle)$ in the category **Str**-Q-**TOP** of stratified Q-topological spaces. We have also determined the coreflective hulls of the categories **Str-Dis**-Q-**TOP** of discrete Q-topological spaces and **Str-Ind**-Q-**TOP** of stratified indiscrete Q-topological spaces in the category **Str**-Q-**TOP**, motivated by the works of Hoffmann [19] and Singh and Srivastava [35].

In chapter 5, we have introduced the Sierpinski fuzzy closure space and given a characterization of the category **FCS** of fuzzy closure spaces, with the help of the Sierpinski fuzzy closure space, which is similar to the characterization of the category **TOP** given by Manes [24].

In the last, we have given conclusion and future scope of the work presented in the thesis.

1.2 Preliminaries

In this section we mention the definitions, notations and results which will be used throughout the thesis.

Definition 1.2.1. [43] Let X be a non empty set. Then a **fuzzy set** in X is a function from X to $[0, 1]$.

The set of all fuzzy sets in X, is denoted by $[0, 1]^X$.

A **constant fuzzy set** $\underline{c}$ in X is a fuzzy set $f \in [0, 1]^X$ such that $f(x) = c$, for each x, for some $c \in [0, 1]$.

Chang [10] gave the following definition of a fuzzy topology.

Definition 1.2.2. [10] Let X be a non-empty set. A **fuzzy topology** on X is a subset τ of $[0, 1]^X$ such that τ is closed under arbitrary union, finite intersection and contains $\underline{1}$ and $\underline{0}$. (X, τ) is called a **fuzzy topological space**.

In 1973, Lowen [22] modified the Chang's definition of a fuzzy topology and proposed the following definition of a fuzzy topology.

Definition 1.2.3. [22] Let X be a set. A **fuzzy topology** on X is a subset τ of $[0,1]^X$ which is closed under arbitrary union, finite intersection and contains all constant fuzzy sets. The pair (X,τ) is called a **fuzzy topological space**.

The fuzzy topological space in the sense of Lowen, will be called as a stratified fuzzy topological space.

Let L be a frame (i.e., a complete lattice satisfying 'first infinite distributive law') with 0 and 1 as the least and greatest elements respectively.

Definition 1.2.4. [15] If X is a set, then a map $f : X \to L$ is called an L-**fuzzy set** (or an L-**set**).

Now L^X is also a frame with respect to the order on it, induced by L and it has the least element $\underline{0}$ and greatest element $\underline{1}$, where $\underline{0}$ and $\underline{1}$ denote the constant L-fuzzy sets taking values 0 and 1 respectively.

Definition 1.2.5. [16] Let X be a set and L be a frame with 0 and 1 as the least and greatest elements respectively. Then an L-**topology** on X is a subframe of L^X.

All category theoretic notions, used here, but not defined or explained, are fairly standard (and can be found in [1]). Accordingly, we assume familiarity with some of the most basic notions in the category theory, viz., categories, subcategories, functors, epimorphisms, monomorphisms, isomorphisms, regular monomorphisms, extremal epimorphisms, retractions, sources, sinks, products, coproducts.

We now recall some more notions and results from category theory.

Definition 1.2.6. [1] A full subcategory $\mathbf{W}$ of a category $\mathbf{C}$ is called **isomorphism-closed** if every $\mathbf{W}$-object that is isomorphic to some $\mathbf{C}$-object, is itself a $\mathbf{W}$-object.

Definition 1.2.7. [1] Let M be a class of monomorphisms in a category $\mathbf{C}$. An M-**subobject** of an object B is a pair (A, g), where $g : A \to B$ belongs to M.

Definition 1.2.8. [1] Let (A_1, g_1) and (A_2, g_2) be two subobjects of an object B in a category $\mathbf{C}$. We say that (A_1, g_1) and (A_2, g_2) are **isomorphic** if there exists an isomorphism $g : A_1 \to A_2$ such that $g_1 = g_2 \circ g$.

Definition 1.2.9. [1] Let M be the class of monomorphisms in a category $\mathbf{C}$. $\mathbf{C}$ is called **wellpowered** provided that no $\mathbf{C}$-object has a proper class of pairwise non-isomorphic M-subobjects.

Definition 1.2.10. [1] Let E be a class of epimorphisms in a category $\mathbf{C}$. An E-**quotient object** of an object A is a pair (h, B), where $h : A \to B$ belongs to E. In case E consists of all extremal epimorphisms, E-quotient objects are called **extremal quotient objects**.

Definition 1.2.11. [1] Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F : \mathbf{C} \to \mathbf{D}$ be a functor. We say that F is **faithful** if all the hom-set restrictions

$$F : hom_{\mathbf{C}}(A, B) \to hom_{\mathbf{D}}(FA, FB)$$

are injective.

Definition 1.2.12. [1] Let $\mathbf{K}$ be a category. A **concrete category** over $\mathbf{K}$ is a pair $(\mathbf{C}, F)$, where $\mathbf{C}$ is a category and $F : \mathbf{C} \to \mathbf{K}$ is a faithful functor. Sometimes F is called the **forgetful** (or **underlying**) **functor** of the concrete category and $\mathbf{K}$ is called the **base category** for $(\mathbf{C}, F)$.

Definition 1.2.13. [1] A concrete category over **Set** is called a **construct**, where **Set** is the category of sets and maps.

Remark 1.2.14.

1. We will denote a concrete category $(\mathbf{C}, F)$ over $\mathbf{K}$ by $\mathbf{C}$ alone and denote the underlying functor F by $|\ |$.

2. Let $\mathbf{C}$ be a concrete category over $\mathbf{K}$. The expression "$|A| \xrightarrow{g} |B|$ is a $\mathbf{C}$-morphism" means that for the $\mathbf{K}$-morphism $|A| \xrightarrow{g} |B|$ there exists a (necessarily unique) $\mathbf{C}$-morphism $A \to B$, which will also be denoted by g, with $|A \to B| = |A| \xrightarrow{g} |B|$.

Definition 1.2.15. [1] Let $\mathbf{C}$ be a category.

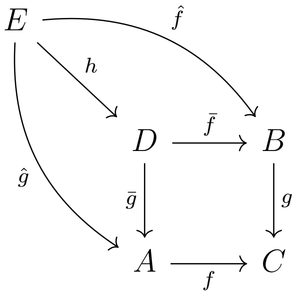
1. A square

$$\begin{array}{ccc} D & \xrightarrow{\bar{f}} & B \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \tag{1.2.1}$$

in **C** is called a **pullback square** if it commutes and for any commutative square of the form

$$\begin{array}{ccc} E & \xrightarrow{\hat{f}} & B \\ {\scriptstyle \hat{g}}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \tag{1.2.2}$$

there exists a unique morphism $h : E \to D$ such that the following diagram commutes



(1.2.3)

2. If the diagram 1.2.1 is a pullback square, then the 2-source $A \xleftarrow{\bar{g}} D \xrightarrow{\bar{f}} B$ is called a pullback of the 2-sink $A \xrightarrow{f} C \xleftarrow{g} B$ and $\bar{f}$ is called a **pullback of** f **along** g.

Definition 1.2.16. [1] A class M of morphisms in a category **C** is called **pullback stable** (or **closed under the formation of pullbacks**) provided that for each pullback square

$$\begin{array}{ccc} D & \xrightarrow{\bar{f}} & B \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array} \tag{1.2.4}$$

with $f \in M$, it follows that $\bar{f} \in M$.

Definition 1.2.17. [1] Let **C** be a concrete category over **K**.

1. A **C**-morphism $f : A \to B$ is called **initial** provided that for any **C**-object D, a **K**-morphism $g : |D| \to |A|$ is a **C**-morphism whenever $f \circ g : |D| \to |B|$ is a **C**-morphism.

2. An initial morphism $f : A \to B$ that has a monomorphic underlying **K**-morphism $f : |A| \to |B|$ is called an **embedding**.

Proposition 1.2.18. ([1], Proposition 8.9) Let **C** be a concrete category over **K**.

1. Let $g : A \to B$ and $h : B \to C$ be initial morphisms (resp. embeddings) in the category **C**. Then $h \circ g : A \to C$ is an initial morphism (resp. an embedding) in **C**.
2. Let $g : A \to B$ and $h : B \to C$ be **C**-morphisms. If $h \circ g : A \to C$ is an initial morphism (resp. an embedding) in **C**, then $g : A \to B$ is initial (resp. an embedding) in **C**.

Definition 1.2.19. [1] In a concrete category an embedding $e : A \to B$ is called an **essential embedding** if every morphism $g : B \to C$ is an embedding, whenever $g \circ e : A \to C$ is an embedding.

Proposition 1.2.20. ([1], Proposition 9.14) Let **C** be a concrete category over **K**. Then,

1. Every isomorphism in **C** is an essential embedding.
2. Composition of essential embeddings is an essential embedding.

Definition 1.2.21. [1] Let **C** be a concrete category over **K**.

1. A **C**-morphism $f : A \to B$ is called **final** provided that for any **C**-object D, a **K**-morphism $g : |B| \to |D|$ is a **C**-morphism whenever $g \circ f : |A| \to |D|$ is a **C**-morphism.
2. A final morphism $f : A \to B$ that has an epimorphic underlying **K**-morphism $f : |A| \to |B|$ is called a **quotient morphism**.

Proposition 1.2.22. ([1], Proposition 8.13) Let **C** be a concrete category over **K**.

1. Let $g : A \to B$ and $h : B \to C$ be final morphisms (resp. quotient morphisms) in the category **C**. Then $h \circ g : A \to C$ is a final morphism (resp. a quotient morphism) in **C**.

2. Let $g : A \to B$ and $h : B \to C$ be **C**-morphisms. If $h \circ g : A \to C$ is a final morphism (resp. a quotient morphism) in **C** then, $h : B \to C$ is a final (resp. a quotient) morphism in **C**.

We mention here that monomorphisms (resp. epimorphisms, isomorphisms) in the category **Set** are precisely injective (resp. surjective, bijective) functions (cf. [1]). Thus by Proposition 8.14 in [1], we have the following result:

Proposition 1.2.23. Let **C** be a concrete category over **Set** and let $f : A \to B$ be a **C**-morphism. Then the following statements are equivalent:

1. $f : A \to B$ is a **C**-isomorphism.
2. $f : A \to B$ is an initial morphism and the underlying function $f : |A| \to |B|$ in **Set** is bijective.
3. $f : A \to B$ is a final morphism and the underlying function $f : |A| \to |B|$ in **Set** is bijective.

Definition 1.2.24. [1] Let **C** be a concrete category over **K**. A source $\{g_j : A \to A_j \mid j \in J\}$ in **C** is called **initial** provided that a **K**-morphism $g : |B| \to |A|$ is a **C**-morphism whenever each composite $g_j \circ g : |B| \to |A_j|$ is a **C**-morphism.

Definition 1.2.25. [1] Let **C** be a concrete category over **K**. A class $\mathcal{X}$ of **C**-objects is called **initially dense** in **C** if for every **C**-object A, there exists an initial source $\{g_j : A \to A_j \mid j \in J\}$ in **C** with $A_j \in \mathcal{X}$, for every $j \in J$.

Definition 1.2.26. ([26], [24]) Let **C** be a concrete category over **K**. A **C**-object S is called a **Sierpinski object** if for every object B of **C**, the source $\{f \mid f : B \to S$ is a **C**-morphism $\}$ in **C** is initial .

Remark 1.2.27. The categories **TOP** of topological spaces and **FTOP** of fuzzy topological spaces (in the sense of Chang [10]) are concrete categories of over **Set** and the usual two-point Sierpinski space and the fuzzy Sierpinski space (of [39]) are Sierpinski objects in **TOP** and **FTOP** respectively.

Definition 1.2.28. [1] Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. A source $\mathcal{S} = \{f_j : A \to A_j \mid j \in J\}$ in **C** is called F-**initial** if for each source $\mathcal{T} = \{g_j : B \to A_j \mid j \in J\}$ in **C** with the same codomain as $\mathcal{S}$ and each **D**-morphism $h : FB \to FA$ with

$Fg_j = Ff_j \circ h$, for every $j \in J$, there exists a unique **C**-morphism $\hat{h} : B \to A$ such that $g_j = f_j \circ \hat{h}$, for every $j \in J$ and $h = F\hat{h}$.

$$\begin{array}{ccc} B & & \\ {\scriptstyle \hat{h}}\downarrow & \searrow^{g_j} & \\ A & \xrightarrow[f_j]{} & A_j \end{array} \tag{1.2.5}$$

$$\begin{array}{ccc} FB & & \\ {\scriptstyle F\hat{h}=h}\downarrow & \searrow^{Fg_j} & \\ FA & \xrightarrow[Ff_j]{} & FA_j \end{array} \tag{1.2.6}$$

Remark 1.2.29. [1] If $(\mathbf{C}, F)$ is a concrete category over $\mathbf{K}$, then F-initial sources are precisely the initial sources in $(\mathbf{C}, F)$.

Definition 1.2.30. [1] Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and let $\{f_j : B \to FA_j \mid j \in J\}$ be a F-structured source. A source $\{\hat{f}_j : A \to A_j \mid j \in J\}$ in $\mathbf{C}$ is called a F-**initial lift** of $\{f_j : B \to FA_j \mid j \in J\}$ if

1. the source $\{\hat{f}_j : A \to A_j \mid j \in J\}$ is F-initial,
2. $F(A \xrightarrow{\hat{f}_j} A_j) = B \xrightarrow{f_j} FA_j$, for every $j \in J$.

Definition 1.2.31. [1] A functor $F : \mathbf{C} \to \mathbf{D}$ is called **topological** if every F-structured source $\{f_j : B \to FA_j \mid j \in J\}$ has a unique F-initial lift $\{\hat{f}_j : A \to A_j \mid j \in J\}$.

Definition 1.2.32. [1] Let $(\mathbf{C}, F)$ be a concrete category over $\mathbf{K}$, then $(\mathbf{C}, F)$ is called **topological** if the functor $F : \mathbf{C} \to \mathbf{K}$ is topological.

Proposition 1.2.33. ([1], Proposition 21.13) In topological construct, the following hold:

1. embeddings = regular monomorphisms.
2. quotient morphisms = extremal epimorphisms.

We now recall the notions of Ω-algebras and their homomorphisms.

Let $\Omega = (n_\lambda)_{\lambda \in K}$ be a class of cardinal numbers.

Definition 1.2.34. [36] A pair $(Z,(\omega_\lambda^Z)_{\lambda\in K})$, where Z is a set and $(\omega_\lambda^Z)_{\lambda\in K}$ is a family of maps $\omega_\lambda^Z : Z^{n_\lambda} \to Z$, is called an Ω-**algebra**. A subset M of Z is called a **subalgebra** of the Ω-algebra $(Z,(\omega_\lambda^Z)_{\lambda\in K})$ if $\omega_\lambda^Z((m_j)_{j\in n_\lambda}) \in M$, for every $\lambda \in K$ and for every $(m_j)_{j\in n_\lambda} \in M^{n_\lambda}$.

Definition 1.2.35. [36] Let $(Z,(\omega_\lambda^Z)_{\lambda\in K})$ and $(S,(\omega_\lambda^S)_{\lambda\in K})$ be Ω-algebras. A map $g : Z \to S$ is said to be an Ω-**homomorphism** if the diagram

$$\begin{array}{ccc} Z^{n_\lambda} & \xrightarrow{g^{n_\lambda}} & S^{n_\lambda} \\ {\scriptstyle \omega_\lambda^Z}\downarrow & & \downarrow{\scriptstyle \omega_\lambda^S} \\ Z & \xrightarrow[g]{} & S \end{array} \tag{1.2.7}$$

commutes for every $\lambda \in K$.

Alg(Ω) will denote the category of Ω-algebras and Ω-homomorphisms.

Definition 1.2.36. [36] Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A **variety of** Ω-**algebras** is a full subcategory of **Alg**(Ω), which is closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras), and $\mathcal{E}$-quotients (homomorphic images).

From now onwards $(Q,(\omega_\lambda^Q)_{\lambda\in K})$ will denote a fixed member of a fixed variety of Ω-algebras.

([36]) Let R be a subset of Q. Then it is easy to check that the intersection of all subalgebras of $(Q,(\omega_\lambda^Q)_{\lambda\in K})$ containing R is a subalgebra of $(Q,(\omega_\lambda^Q)_{\lambda\in K})$. We will denote it by $\langle R\rangle$.

([36]) Let Z be a set and Q^Z be the set of all functions from Z to Q. All operations on Q lift point-wise to Q^Z as:

$$(\omega_\lambda^{Q^Z}(\langle p_j\rangle_{j\in n_\lambda}))(z) = \omega_\lambda^Q(\langle p_j(z)\rangle_{j\in n_\lambda}), \text{ for every } \langle p_j\rangle_{j\in n_\lambda} \in (Q^Z)^{n_\lambda} \text{ and every } z \in Z\ .$$

In particular $(Q^Z,(\omega_\lambda^{Q^Z})_{\lambda\in K})$ is an Ω-algebra.

From now onwards, the Ω-algebra $(Q,(\omega_\lambda^Q)_{\lambda\in K})$ and its underlying set, both will be denoted by Q.

Now we recall the definition a of Q-topological space.

Definition 1.2.37. [36] Let Z be a set. A subset η of Q^Z is called a Q-**topology** on Z if η is a subalgebra of the Ω-algebra $(Q^Z, (\omega_\lambda^{Q^Z})_{\lambda\in K})$. A pair (Z,η), where Z is a set and η is a Q-topology on Z, is called a Q-**topological space**. Let (Z,η) and (X,τ) be Q-topological spaces and $h : Z \to X$ be a function. Then we say that $h : (Z,\eta) \to (X,\tau)$ is Q-**continuous** if $\alpha \circ h \in \eta$, for every $\alpha \in \tau$.

Q-**TOP** will denote the category of Q-topological spaces and Q-continuous maps. Q-**TOP** is a construct via the obvious forgetful functor $|-|$: Q-**TOP**$\to$ **Set**.

Definition 1.2.38. [36] A Q-topological space (X,τ) is called T_0 if for every $x_1, x_2 \in X$ such that $x_1 \neq x_2$, there exists $\alpha \in \tau$ such that $\alpha(x_1) \neq \alpha(x_2)$.

Q-**TOP**$_0$ will denote the full subcategory of Q-**TOP** consisting of T_0-Q-topological spaces. It can be easily seen that Q-**TOP**$_0$ is an isomorphism closed subcategory of Q-**TOP**.

Definition 1.2.39. [36] A Q-topological space (X,τ) is said to be **stratified** if $\underline{q} \in \tau$, for every $q \in Q$, where $\underline{q} : X \to Q$ is defined as $\underline{q}(x) = q$, for every $x \in X$.

Str-Q-**TOP** will denote the category of stratified Q-topological spaces and Q-continuous maps.

Proposition 1.2.40. [36] Let (Y,σ) and (Z,η) be Q-topological spaces and let $\eta = \langle R\rangle$, where $R \subseteq Q^Z$. Then a map $g : (Y,\sigma) \to (Z,\eta)$ is Q-continuous if and only if $\alpha \circ g \in \sigma$, for every $\alpha \in R$.

Proposition 1.2.41. [30] Let $\{g_k : Y \to |(Y_k,\sigma_k)|| \; k \in K\}$ be a $|-|$-structured source (where Y is a set and (Y_k,σ_k) is a Q-topological space for each k). Then the initial lift of the source $\{g_k : Y \to |(Y_k,\sigma_k)|| \; k \in K\}$ in Q-**TOP** is $\{g_k : (Y,\sigma) \to (Y_k,\sigma_k) \mid k \in K\}$, where $\sigma = \langle\{\alpha_k \circ g_k \mid \alpha_k \in \sigma_k, k \in K\}\rangle$.

Theorem 1.2.42. Q-**TOP** is a topological category over **Set**.

Proof. It immediately follows from the Proposition 1.2.41. □

Definition 1.2.43. [36] The Q-topological space $(Q, \langle id_Q\rangle)$ is called the Q-**Sierpinski space**.

Theorem 1.2.44. [30] The Q-Sierpinski space $(Q, \langle id_Q\rangle)$ is a Sierpinski object in the category Q-**TOP**.

Proposition 1.2.45. [30] Let $\{(Y_k, \sigma_k) \mid k \in K\}$ be a family of Q-topological spaces. Let $\{p_k : (\prod Y_k, \eta) \to (Y_k, \sigma_k) \mid k \in K\}$ be the initial lift of the family of all projections $\{p_k : \prod Y_k \to |(Y_k, \sigma_k)|\mid k \in K\}$ in Q-**TOP**. Then $(\prod Y_k, \eta)$ is the product of the family $\{(Y_k, \sigma_k) \mid k \in K\}$ in Q-**TOP**.

The following Proposition can be verified on similar lines as in the category **TOP** of topological spaces (cf. [1]).

Proposition 1.2.46. Let (X, τ) and (Y, σ) be Q-topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ be a Q-continuous map. Then,

1. $f : (X, \tau) \to (Y, \sigma)$ is initial in Q-**TOP** if and only if $\tau = \{\beta \circ f \mid \beta \in \sigma\}$,
2. $f : (X, \tau) \to (Y, \sigma)$ is an embedding in Q-**TOP** if and only if it is initial in Q-**TOP** and f is one-one,
3. $f : (X, \tau) \to (Y, \sigma)$ is final in Q-**TOP** if and only if $\sigma = \{v \in Q^Y \mid v \circ f \in \tau\}$,
4. $f : (X, \tau) \to (Y, \sigma)$ is a quotient morphism in Q-**TOP** if and only if it is final in Q-**TOP** and f is onto.