Numerical study of fractional diffusion equations and its applications

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by

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Chapter 6

Conclusion and scope of future work

Diffusion phenomenon is most occurring phenomenon in nature which can be seen in many fields. It is governed by a PDE known as diffusion equation. Diffusion equation has many applications in science and engineering. So, the present work emphasizes on applications of diffusion equation. Here the fractional order diffusion equation is considered with different types of fractional derivative like as Caputo, Caputo-Fabrizio and Mittag-Leffler kernel fractional derivative in constant and variable forms. Applications are shown in one and two dimensional groundwater problems, tumor diagnosis and other types of diffusion phenomena. The spectral method along with operational matrix has been used by using different polynomials and wavelets.

In chapter 2, a model of non-linear fractional diffusion with reaction process is studied with Dirichlet boundary conditions. This model contains both space and time derivatives in fractional form. Here, it is considered that the concentration of contaminant first increases in spatial direction up to half distance and becomes zero on boundary by gradually decreasing. At any time, the concentration is zero at boundary. The study shows that overshoots of sub diffusion decrease with the increase in the order of time fractional order. In other words, pollute diffuses with slow rate as order of system changes from fractional order to standard order. In case of spatial fractional order, sub-diffusion occurs with faster rate as the order of spatial derivative tends to one from fractional order and this nature becomes reverse after mid of space length.

The variation of the reaction coefficient λ shows that sub-diffusion of contaminant is lower in case of sink term.

In the chapter 3, a two-dimensional version of reaction-diffusion model with both space-time fractional derivatives has been considered. The Neumann boundary conditions have been taken for this model. The operational matrix method based on Genocchi polynomial has been used for numerical study. It is seen that the proposed method is applicable and has higher accuracy for such type of two-dimensional non-linear fractional order PDE.

Here, overshoot of sub-diffusion decreases with the increasing time-fractional order. Similar nature of sub-diffusion can be found for space fractional order. The effect of reaction process on sub-diffusion is similar to the one-dimensional process.

In chapter 4, the study of reaction-diffusion has been extended to variable order with addition of Galilei invariant advection-diffusion model. The error bound of any function with Gegenbauer wavelet expansion has been derived. These models have been solved with the help of operational matrix. The validation of method has been shown by solving different types linear and non-linear examples. The value of concentration is more in case of source term as compare to sink term. The variations of solution profile by oscillating time-fractional order with a small positive and negative increments depict the dynamic behaviour of the model.

In chapter 5, the application of diffusion equation is shown in analysis of dynamic behavior of tumor cells in presence and absence of chemotherapeutic treatment. A system consisting of fractional order diffusion equation with exponential kernel has been studied with the help of Chebyshev spectral method. This study shows that growth of tumor cells is slower in presence of chemotherapy as compared to no treatment. Sub-diffusion of tumor cells decreases when system approaches from standard order to fractional order in both direction time and space. Same nature is found for immune cells and concentration of chemotherapeutic drug. This study also shows that growth of tumor cells is much slower in a person having strong immune system and chemotherapeutic treatment works efficiently on that patient.

Scope of future work

The method presented in chapter 2 and chapter 3 can be applied for study of complex and higher dimensional mathematical models. The method based on ultra-spherical wavelet can be extended for other non-singular fractional order derivative. Other biological models like breast cancer, Zika virus, population model and COVID-19 model can be studied in fractional order system following the study given in chapter 5.

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