Chapter 3

Numerical solution of two dimensional reaction-diffusion equation using operational matrix method based on Genocchi polynomial

3.1 Introduction

Many analytical and numerical techniques on two dimensional diffusion equation have been developed [63, 64]. For example, Finite Element method [65], Legendre Collocation Method [66]. Chen [67] have developed method for finding exact analytical solutions for two-dimensional advection-diffusion equation in the cylindrical coordinates. The Genocchi polynomial [13, 14, 68] collocation method is used to solve non-linear fractional reaction-diffusion equation. After finding the operational matrix of fractional differentiation, the given non-linear fractional equation model and boundary conditions are collocated. By collocating a non-linear system of algebraic equations are obtained which are solved by using Newton method.

3.2 Preliminaries

In this section, Kronecker product of two matrices has been given which will be used throughout the chapter.

3.2.1 Kronecker product of two matrix

Suppose F is a field like as R and C. If $A \in F^{m\times n}$ and $B \in F^{p\times q}$ are any matrices then their Kronecker product denoted as $A \otimes B$ is defined as [69]

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix},$$

Some properties of Kronecker product are given as follows where $A \in F^{m\times n}, C \in F^{n\times p}, B \in F^{q\times r}, D \in F^{r\times s}$

- $(\alpha A) \otimes B= A \otimes (\alpha B)=\alpha(A \otimes B)$
- $(A + B) \otimes C = A \otimes C + B \otimes C$
- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

- $A \otimes B = (A \otimes I_p)(I_n \otimes B) = (I_m \otimes B)(A \otimes I_q)$
- $(A \otimes B)(C \otimes D) = AC \otimes BD$
- $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

3.2.2 Approximation of an arbitrary function

Let us Suppose $\{G_1(t), G_2(t), ..., G_M(t)\} \subset L^2[0,1]$ is the set of Genocchi polynomials. A function $u(t)$ which belongs to $L^2[0,1]$ can be expressed as

$$u(t) = \sum_{l=1}^{M} c_l G_l(t) = C^T G(t), \tag{3.1}$$

where $c_l = (u(t), G_l(t))$ and (.) denotes the inner product. C and G(t) are column vectors. Similarly, an arbitrary function $u(x,y,t)$ belongs to $L^2[0,1] \times L^2[0,1] \times L^2[0,1]$ of three variables can be expressed in terms of Genocchi polynomials as

$$u(x,y,t) = \sum_{l=1}^{M}\sum_{m=1}^{M}\sum_{n=1}^{M} c_{lmn} G_m(t) G_l(x) G_n(x) = \psi^T(t).V.(\psi(x) \otimes \psi(y)), \tag{3.2}$$

where $V = [c_{lmn}]_{M \times M^2}$ and $\otimes$ denotes Kronecker product.

3.2.3 Genocchi operational matrix of fractional derivative

The operational matrix of fractional derivative has been given in section 2.2.3 of chapter 2.

3.3 Error bound and stability analysis

In this section, the upper bound is found out by means of Genocchi polynomial on the error which is being expecting in the approximation. Consider the following space

$$\prod_{M} = Span\{G_1(x), G_2(x), ...G_M(x), G_1(y), G_2(y), ...G_M(y), G_1(t), G_2(t), ...G_M(t)\}. \tag{3.3}$$

Let $\tilde{u}(x, y, t) \in \prod_M$ be the best approximation of $u(x, y, t)$. Then using the definition of best approximation, the following is obtained

$$\|u(x, y, t) - \tilde{u}(x, y, t)|_\infty \leq \|u(x, y, t) - w(x, y, t)|_\infty, \forall w(x, y, t) \in \prod_M. \tag{3.4}$$

The above inequality will remain true if $w(x, y, t)$ represents the interpolating polynomial for $u(x, y, t)$ at points (x_i, y_j, t_k) where x_i, y_j, t_k for $0 \leq i, j, k \leq M$ are respective roots of $G_{M+1}(x), G_{M+1}(y)$ and $G_{M+1}(t)$. Then we get

$$\begin{aligned} u(x, y, t) - w(x, y, t) =& \frac{1}{(M+1)!}\frac{\partial^{M+1}u(\mu, y, t)}{\partial x^{M+1}}\prod_{i=0}^{M}(x - x_i) \\ &+ \frac{1}{(M+1)!}\frac{\partial^{M+1}u(x, \zeta, t)}{\partial y^{M+1}}\prod_{i=0}^{M}(y - y_j) \\ &+ \frac{1}{(M+1)!}\frac{\partial^{M+1}u(x, y, \eta)}{\partial t^{M+1}}\prod_{i=0}^{M}(t - t_k) - \\ &\frac{\partial^{3M+3}u(\mu', \zeta', \eta')}{\partial x^{M+1}\partial y^{M+1}\partial t^{M+1}} \times \frac{\prod_{i=0}^{M}(x - x_i)\prod_{j=0}^{M}(y - y_j)\prod_{k=0}^{M}(t - t_k)}{(M+1)!(M+1)!(M+1)!} \end{aligned} \tag{3.5}$$

where $\mu', \zeta', \eta', \mu, \zeta, \eta \in [0,1]$. Now by the properties of norm

$$\begin{aligned}
\|u(x,y,t) - w(x,y,t)\| \leq & \frac{1}{(M+1)!} \max_{x,y,t\in[0,1]} \left| \frac{\partial^{M+1} u(\mu,y,t)}{\partial x^{M+1}} \right| \left\| \prod_{i=0}^{M} (x - x_i) \right\| \\
& + \frac{1}{(M+1)!} \max_{x,y,t\in[0,1]} \left| \frac{\partial^{M+1} u(x,\zeta,t)}{\partial y^{M+1}} \right| \left\| \prod_{i=0}^{M} (y - y_j) \right\| \\
& + \frac{1}{(M+1)!} \max_{x,y,t\in[0,1]} \left| \frac{\partial^{M+1} u(x,y,\eta)}{\partial t^{M+1}} \right| \left\| \prod_{i=0}^{M} (t - t_k) \right\| \\
& + \max_{x,y,t\in[0,1]} \left| \frac{\partial^{3M+3} u(\mu',\zeta',\eta')}{\partial x^{M+1} \partial y^{M+1} \partial t^{M+1}} \right| \\
& \times \frac{\left\| \prod_{i=0}^{M} (x - x_i) \right\| \left\| \prod_{j=0}^{M} (y - y_j) \right\| \left\| \prod_{k=0}^{M} (t - t_k) \right\|}{(M+1)!(M+1)!(M+1)!}.
\end{aligned} \tag{3.6}$$

Since $u(x,y,t)$ is a continuous differential function on the interval $[0,1]$ then there exist constants A_1, A_2, A_3 and A_4 such that

$$\max_{x,y,t\in[0,1]} \left| \frac{\partial^{M+1} u(\mu,y,t)}{\partial x^{M+1}} \right| \leq A_1, \tag{3.7}$$

$$\max_{x,y,t\in[0,1]} \left| \frac{\partial^{M+1} u(x,\zeta,t)}{\partial y^{M+1}} \right| \leq A_2, \tag{3.8}$$

$$\max_{x,y,t\in[0,1]} \left| \frac{\partial^{M+1} u(x,y,\eta)}{\partial t^{M+1}} \right| \leq A_3, \tag{3.9}$$

$$\max_{x,y,t\in[0,1]} \left| \frac{\partial^{3M+3} u(\mu',\zeta',\eta')}{\partial x^{M+1} \partial y^{M+1} \partial t^{M+1}} \right| \leq A_4. \tag{3.10}$$

Now to minimize the factor, let us apply the following procedure

$$\min_{x_i\in[0,1]} \max_{x\in[0,1]} \left| \prod_{i=0}^{M} (x - x_i) \right| = \min_{x_i\in[0,1]} \max_{x\in[0,1]} \left| \frac{G_{M+1}(x)}{M+1} \right| \tag{3.11}$$

where $(M+1)$ is the leading coefficient of Genocchi polynomial of order $(M+1)$ and Genocchi polynomial satisfy

$$\max_{x\in[0,1]}\Big|G_{M+1}(x)\Big| \leq \frac{4e^{\pi}\pi^{-M-1}(-2^{M+1}\Gamma(2+M,\pi)+e^{\pi}\Gamma(2+M,2\pi)}{-2+2^{M+1}}. \tag{3.12}$$

From above inequalities, it can be find out

$$\|u(x,y,t)-\tilde{u}(x,y,t)\| \leq \frac{4e^{\pi}\pi^{-M-1}(-2^{M+1}\Gamma(2+M,\pi)+e^{\pi}\Gamma(2+M,2\pi)}{-2+2^{M+1}} \times \Big(\frac{A_1+A_2+A_3}{(M+1)!}+\frac{A_4}{((M+1)!)^3}\Big). \tag{3.13}$$

So, an upper bound has been derived for approximate solution of the absolute errors, which shows that approximation $\tilde{u}(x,y,t)$ converges to the exact solution $u(x,y,t)$ and validates the stability of proposed scheme.

3.4 Solution of the problem

In this section, an operational matrix method based is applied with the help of Genocchi polynomials of fractional derivative to obtain the solution of the following two dimensional non-linear time-space fractional order PDE given by

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \kappa_1(x,y)\frac{\partial^{\beta}u(x,y,t)}{\partial x^{\beta}}+\kappa_2(x,y)\frac{\partial^{\gamma}u(x,y,t)}{\partial y^{\gamma}}+\kappa_3 u(1-u)+f(x,y,t), \tag{3.14}$$

where $0<\alpha\leq 1, 1\leq\beta\leq 2$, $1\leq\gamma\leq 2$, $\kappa_1(x,y)$ is the longitudinal diffusion coefficient, and $\kappa_2(x,y)$ is the transverse diffusion coefficient, $\kappa_3(x,y)$ is the reaction coefficient and $f(x,y,t)$ is the force term. The initial and boundary conditions are considered as

$$u(x,y,0) = f_1(x,y), \tag{3.15}$$

$$u(0,y,t) = f_2(y,t), \tag{3.16}$$

$$u(x,0,t) = f_3(x,t), \tag{3.17}$$

$$\frac{\partial u(1,y,t)}{\partial x} = f_4(x,y,t), \tag{3.18}$$

$$\frac{\partial u(x,1,t)}{\partial y} = f_5(x,y,t), \tag{3.19}$$

where $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq t \leq 1$. If $\kappa_3 = 0$, then system is known as conservative system and $\kappa_3 \neq 0$ gives the non-conservative. $\kappa_3 < 0$ is known as sink term and $\kappa_3 > 0$ is known as source term. An unknown function $u(x,y,t)$ is approximated by Genocchi polynomial as

$$u(x,y,t) = \sum_{l=1}^{M} \sum_{m=1}^{M} \sum_{n=1}^{M} c_{lmn} G_m(t) G_l(x) G_n(x), \tag{3.20}$$

where c_{lmn} are unknown coefficients for $l = 1, 2, \cdots M$, $m = 1, 2, \cdots M$ and $n = 1, 2, \cdots M$.

Now rewrite the equation (3.20) in matrix form as

$$u(x,y,t) = \psi^T(t).C.(\psi(x) \otimes \psi(y)), \tag{3.21}$$

where $C = [c_{lmn}]_{M\times M^2}$ is an $M\times M^2$ matrix of unknowns and $\psi(t) = (G_1(t).....G_M(t))^T$ is a column vector. If $\psi(x,t)$ is column vector formed by Genocchi polynomial then

$$\begin{aligned}\frac{\partial^\beta}{\partial x^\beta}(\psi(x,t)) &= \frac{\partial^\beta}{\partial x^\beta}(\psi(x)\otimes\psi(t),\\ &= \frac{\partial^\beta\psi(x)}{\partial x^\beta}\otimes\psi(t),\\ &= P^\beta\psi(x)\otimes I\psi(t),\\ &= (P^\beta\otimes I)(\psi(x)\otimes\psi(t)). \end{aligned} \tag{3.22}$$

Similarly,

$$\begin{aligned}\frac{\partial^\alpha}{\partial t^\alpha}(\psi(x,t)) &= \frac{\partial^\alpha}{\partial t^\alpha}(\psi(x)\otimes\psi(t),\\ &= \psi(x)\otimes\frac{\partial^\alpha\psi(t)}{\partial t^\alpha},\\ &= I\psi(x)\otimes P^\alpha\psi(t),\\ &= (I\otimes P^\alpha)(\psi(x)\otimes\psi(t)). \end{aligned} \tag{3.23}$$

Now operating fractional derivative of order α on (3.21) w.r.to x and using theorem 1, we get

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = (P^\alpha\phi(t))^T.C.(\psi(x)\otimes\psi(y)). \tag{3.24}$$

Similarly,

$$\frac{\partial^\beta u(x,y,t)}{\partial x^\beta} = \phi^T(t).C.(P^\beta\otimes I)(\psi(x)\otimes\psi(y)), \tag{3.25}$$

$$\frac{\partial^\gamma u(x,y,t)}{\partial y^\gamma} = \phi^T(t).C.(I\otimes P^\gamma)(\psi(x)\otimes\psi(y)). \tag{3.26}$$

Now equations (3.15) − (3.19) with the aid of the equation (3.21) give rise to

$$\phi^T(0).C.(\psi(x)\otimes\psi(y)) = f_1(x,y), \tag{3.27}$$

$$\phi^T(t).C.(\psi(0)\otimes\psi(y)) = f_2(y,t), \tag{3.28}$$

$$\phi^T(t).C.(\psi(x)\otimes\psi(0)) = f_3(x,t), \tag{3.29}$$

$$\phi^T(t).C.(P^1\otimes I)(\psi(1)\otimes\psi(y)) = f_4(x,y,t), \tag{3.30}$$

$$\phi^T(t).C.(I\otimes P^1)(\psi(x)\otimes\psi(1)) = f_5(x,y,t). \tag{3.31}$$

After putting the values of $u(x,y,t)$, $\frac{\partial^\alpha u}{\partial t^\alpha}$, $\frac{\partial^\beta u}{\partial x^\beta}$ and $\frac{\partial^\beta u}{\partial y^\gamma}$ from equations (3.21)−(3.28), the following residue is obtained as

$$\begin{aligned}\Xi(x,y,t) = {} & (P^\alpha\phi(t))^T.C.(\psi(x)\otimes\psi(y)) - \kappa_1(x,y)\phi^T(t).C.(P^\beta\otimes I)(\psi(x)\otimes\psi(y)) \\ & - \kappa_2(x,y)\phi^T(t).C.(I\otimes P^\gamma)(\psi(x)\otimes\psi(y)) - \\ & \kappa_3\psi^T(t).C.(\psi(x)\otimes\psi(y))(1-\psi^T(t).C.(\psi(x)\otimes\psi(y))) - f(x,y,t)\end{aligned} \tag{3.32}$$

Let us collocate equations (3.32), (3.27) − (3.31) at points $x_i = \frac{i}{M}, y_j = \frac{j}{M}$ and $t_k = \frac{k}{M}$ for $i,j,k = 1,2,\cdots,M$. After collocating, the non linear system of algebraic equations is obtained. By Solving that system of equations and finding C, the numerical solution of proposed model (3.14) can be found.

3.5 Results and discussion

In this section, a drive has been taken to validate the effectiveness of the proposed method first through applying it on two spatial fractional order problems ($\alpha = 1$) and compare these obtained results with the existing analytical results for different particular cases.

TABLE 3.1: variations of absolute error for different time and space for first case taking $M = 4$.

$x \downarrow$ t $\rightarrow$	0.2	0.4	0.6	0.8	1
0.2	1×10^{-5}	2.6×10^{-5}	4.5×10^{-5}	5.9×10^{-5}	6.5×10^{-5}
0.4	2.2×10^{-5}	5.4×10^{-5}	8.6×10^{-5}	1.1×10^{-4}	1.2×10^{-4}
0.6	3.4×10^{-5}	7.7×10^{-5}	1.1×10^{-4}	1.5×10^{-4}	1.6×10^{-4}
0.8	4.3×10^{-5}	9.2×10^{-5}	1.8×10^{-4}	1.8×10^{-4}	1.8×10^{-4}
1	4.5×10^{-5}	9.4×10^{-5}	1.3×10^{-4}	1.7×10^{-4}	1.8×10^{-4}

Considering $\beta = \gamma = 1.5, \kappa_3 = 0$, the proposed model is reduced to

$$\frac{\partial u(x,t)}{\partial t} = \kappa_1(x,y)\frac{\partial^{1.5}u(x,t)}{\partial x^{1.5}} + \kappa_2(x,y)\frac{\partial^{1.5}u(x,t)}{\partial y^{1.5}} + f(x,y,t) \tag{3.33}$$

where, $\kappa_1(x,y) = \dfrac{(3-2x)\Gamma(3-\alpha)}{2}$, $\kappa_2(x,y) = \dfrac{(4-y)\Gamma(3-\beta)}{6}$ and the forcing function as $f(x,y,t) = e^{-t}(x^2(-y^{3/2}+y-4)y^{3/2} + \sqrt{x}(2x-3)y^3)$.

For the initial and boundary conditions

$$u(x,y,0) = x^2y^3, \tag{3.34}$$

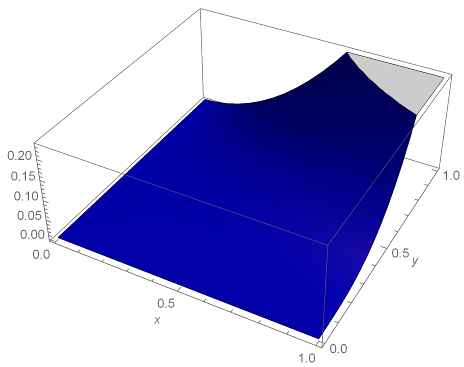
$$u(0,y,t) = 0, \tag{3.35}$$

$$u(x,0,t) = 0, \tag{3.36}$$

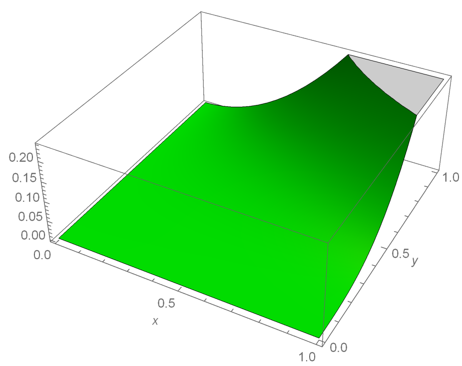
$$\frac{\partial u(1,y,t)}{\partial x} = 2e^{-t}y^3, \tag{3.37}$$

$$\frac{\partial u(x,1,t)}{\partial y} = 2e^{-t}x^2, \tag{3.38}$$

the exact solution of above the problem is $u(x,y,t) = e^{-t}x^2y^3$. The absolute error is calculated between the exact solution and the approximate solution using the proposed method for $M = 4$, which one is displayed through Table 3.1. The results clearly predict that the numerical results are in complete agreement with the existing results. The similarity nature for both the solutions can also be found from Fig.3.1. Considering another particular case of the proposed model (3.14) as $\beta = 1.8, \gamma = 1.6$, and $\kappa_3 = 0$ so that the model is reduced to a non-linear two dimensional fractional



(a) Numerical solution



(b) Exact solution

FIGURE 3.1: Plots of $u(x, y, t)$ vs. x and y for $M = 4$ in case of numerical and exact solution for $t = 0.5$.

differential equation as

$$\frac{\partial u(x,t)}{\partial t} = \kappa_1(x,y)\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}} + \kappa_2(x,y)\frac{\partial^{1.6}u(x,t)}{\partial y^{1.6}} + f(x,y,t), \tag{3.39}$$

where $\kappa_1(x, y) = \dfrac{\Gamma(2.2)x^{2.8}y}{6}$, $\kappa_2(x, y) = \dfrac{2xy^{2.6}}{\Gamma 4.6}$ and the forcing function as

$$f(x, y, t) = e^{-t}(x^2(-y^{3/2} + y - 4)y^{2.6} + \sqrt{x}(2x - 3)y^3). \tag{3.40}$$

For the initial and boundary conditions

$$u(x, y, 0) = x^3y^{3.6}, \tag{3.41}$$

$$u(0, y, t) = 0, \tag{3.42}$$

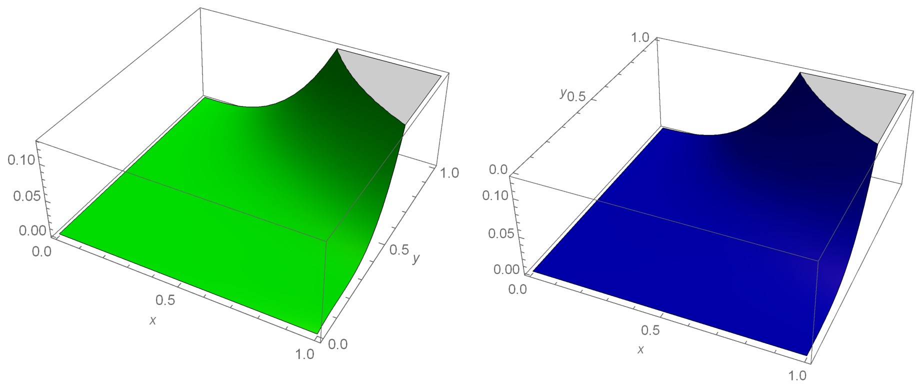
$$u(x, 0, t) = 0, \frac{\partial u(1, y, t)}{\partial x} = 3e^{-t}y^{3.6}, \tag{3.43}$$

$$\frac{\partial u(x, 1, t)}{\partial y} = 3.6e^{-t}x^3, \tag{3.44}$$

the equation (3.39) has the exact solution, which is given by $u(x, y, t) = e^{-t}x^2y^{3.6}$. The absolute error for different values of x and y are shown in the Table 3.1 for $M = 4$. The Fig.3.2 clearly shows the similarity of the results with the exact solution and proposed solution.

TABLE 3.2: variations of absolute error for different time and space for first case taking $M = 4$.

x ↓ t →	0.2	0.4	0.6	0.8	1
0.2	3.9×10^{-5}	3.7×10^{-4}	1.2×10^{-3}	2.8×10^{-3}	5.5×10^{-3}
0.4	3.6×10^{-4}	3×10^{-4}	1×10^{-3}	2.3×10^{-3}	4.5×10^{-3}
0.6	3.9×10^{-5}	2.3×10^{-5}	7.6×10^{-4}	1.8×10^{-3}	3.5×10^{-3}
0.8	4.2×10^{-5}	1.6×10^{-4}	5.3×10^{-4}	1.3×10^{-3}	2.7×10^{-3}
1	4.2×10^{-5}	1.1×10^{-4}	13.8×10^{-4}	1.0×10^{-3}	2.1×10^{-3}



(a) Numerical solution (b) Exact solution

FIGURE 3.2: Plots of $u(x, y, t)$ vs. x and y for $M = 4$ in case of numerical and exact solution for $t = 0.5$.

After the confirmation of accuracy and efficiency of the method, it is applied to find numerical solutions of considered non linear two dimensional space time fractional order non-conservative system (3.14) under the following initial and boundary conditions as

$$u(x, y, 0) = x, \tag{3.45}$$

$$u(0, y, t) = 0, \tag{3.46}$$

$$u(x, 0, t) = xt, \tag{3.47}$$

$$\frac{\partial u(1, y, t)}{\partial x} = t, \tag{3.48}$$

$$\frac{\partial u(x, 1, t)}{\partial y} = 0, \tag{3.49}$$

The numerical results for different particular cases are depicted through Figs.3.3−3.6 at $t = 0.5$. It is seen from Fig 3.3 that for non-linear time fractional reaction diffusion equation ($\beta = \gamma = 2, \kappa_3 = +1$), the sub diffusion phenomena of solute concentration occurs and overshoots of sub diffusion decrease with the increase in α.

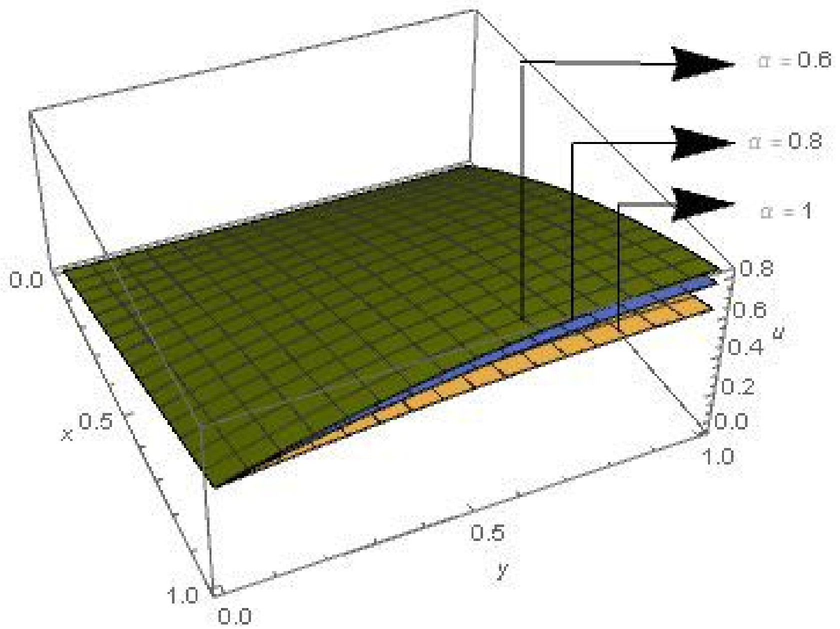


FIGURE 3.3: Plots of $u(x, y, t)$ vs. x and y for various α at t=0.5 when $\kappa_3 = 1$, $\beta = \gamma = 2$.

It is seen from Fig.3.4 that similar sub-diffusions are found for non-linear spatial

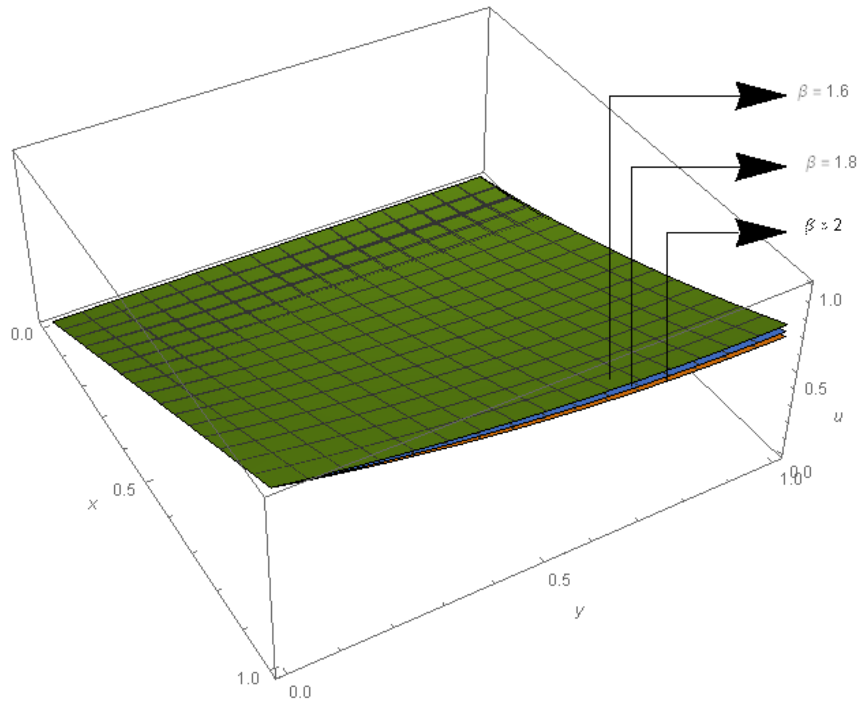


FIGURE 3.4: Plots of $u(x, y, t)$ vs. x and y for various β at t=0.5 when $\kappa_3 = 1$, $\alpha = 1, \gamma = 2$.

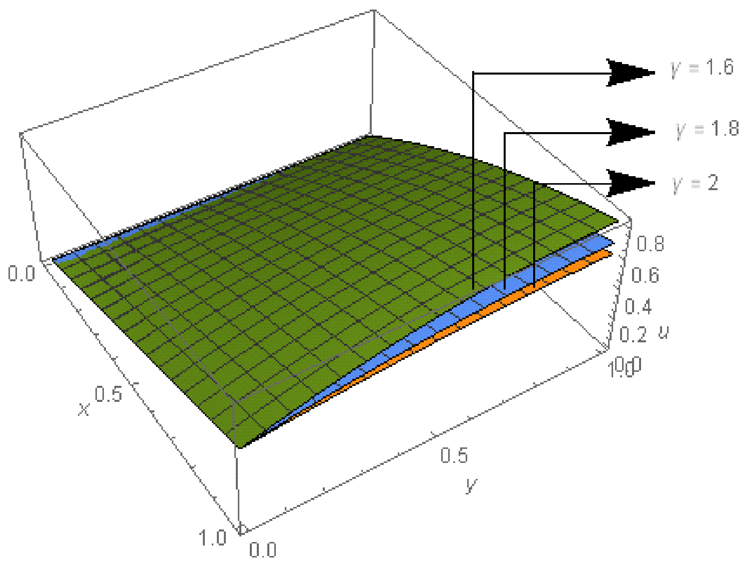


FIGURE 3.5: Plots of $u(x, y, t)$ vs. x and y for various γ at t=0.5 when $\kappa_3 = 1$, $\beta = 2, \alpha = 1$.

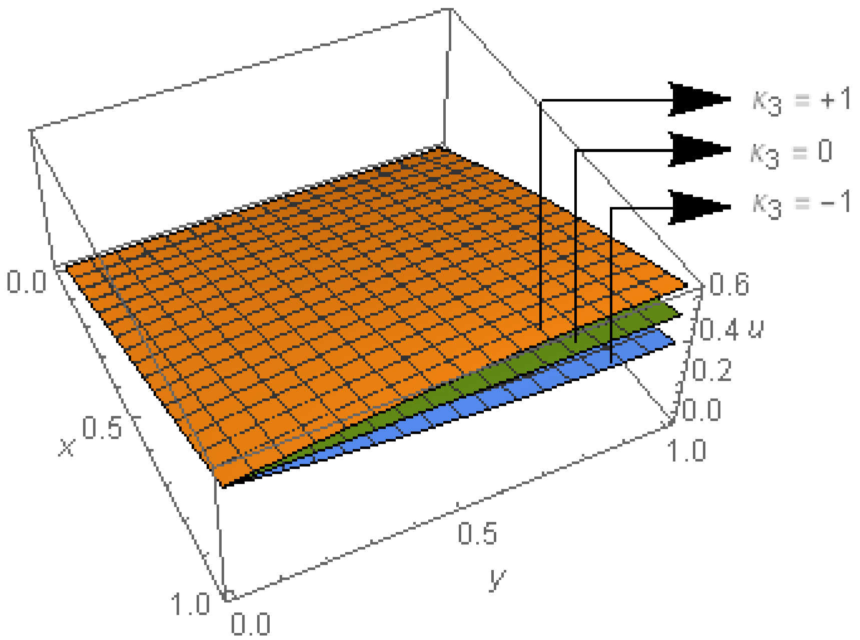


FIGURE 3.6: Plots of $u(x, y, t)$ vs. x and y for various β at t=0.5 when $\beta = 2$, $\alpha = 1, \gamma = 2$.

fractional order reaction-diffusion equation ($\alpha = 1, \gamma = 2, \kappa_3 = 1$). Here initially solute concentration decreases as β increases. It is seen from Fig.3.5 that solute concentration increases as γ increases.

The effect of reaction term on the solution profile for standard order reaction-diffusion equation ($\alpha = 1, \beta = \gamma = 2$) is shown through Fig. 3.6 without the presence of force term. It is clear from the figures that overshoots of the sub-diffusions of solute concentration decrease for the case of sink term ($\kappa_3 = -1$) as compared to source term ($\kappa_3 = +1$).

3.6 Conclusion

In this chapter three important goals have been achieved. First one, the use of collocation method based on Genocchi polynomials to solve the two dimensional non linear reaction-diffusion equation in presence of the forced term. Second one is the pictorial presentations of the nature of overshoots during sub-diffusion as the system approaches from standard order to fractional order. Third one is the exhibition of decrease of solute concentration due to the presence of sink term for standard order as well as fractional order cases.

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