Chapter 2

Operational matrix method for solving nonlinear space-time fractional order reaction-diffusion equation based on Genocchi polynomial

2.1 Introduction

Fractional calculus is an ancient topic of mathematics with history like as ordinary or integer calculus. It is developing progressively now. Theory of fractional calculus is developed by N. H. Abel and J. Liouville. The details can be found in [34, 35]. In last few years fractional calculus has attracted attentions of the researchers of medical physics, chemistry, biology, engineering and mathematics. Fractional calculus and fractional differential equation (FDE) are found in many applications in different fields. Due to increasing applications, the attention is paid to numerical and exact solutions of the FDEs. As there are many difficulties to solve a FDE by analytic method so there is a need of seeking numerical methods. There are many numerical methods available in literature viz., eigen-vector expansion, Adomain decomposition method [36], fractional differential transform method [37], homotopy perturbation method [31, 38, 39], predictor-corrector method [40], multi step homotopy analysis method [41, 42] and generalized block pulse operational matrix method [43] etc. Some numerical methods based upon operational matrices of fractional differentiation and integration with Legendre wavelets [28], Chebyshev wavelets [29, 44, 45], Sine wavelets, Haar wavelets [46] have been developed to find the solutions of fractional order differential and integro-differential equations. The functions which are commonly used include Legendre polynomial [26, 47], Laguerre polynomial [27], Chebyshev polynomial and semi-orthogonal polynomial as Genocchi polynomial [13]. The application of Feng's first integral method applied to the nonlinear mKdV space-time fractional partial differential equation is cited in article [48]. Many different forms of fractional order differential operators had been introduced as the Grunwald-Letnikov, Riemann-Liouville, Hadamard, Caputo, Caputo–Fabrizio, Riesz fractional order and variable order fractional operators. The applications of these fractional derivatives in electrical circuits RC, RL, RLC, power electronics devices and non linear loads are studied in [49].It can be seen that physical problems arose in nature which follow three laws the power law, MittagLeffler law and the exponential decay law. To describe the behavior of these problems a new type of fractional operator is used in [50, 51]. Analytical solutions of a fractional-time wave equation with memory effect and frictional memory kernel of Mittag–Leffler type via the Atangana–Baleanu fractional order derivative is described in article [52]. Applications of variable order fractional differential equations can be found in [53].

A family of generalized fractional Cattaneo's equations for which passive transport in porous media is possible is studied in [54] by using fractional substitutions in integer order rational transform functions. Many applications of fractional derivatives in porous media can be found as gas-flow equation in porous media [55], distributed order Hausdorff derivative diffusion model in porous media [56], modeling of non-Darcian flow and solute transport [57]. Transport phenomena characterize the motion of fluids in porous media structure as heat/mass transfer and chemical reactions, like fluid behaves when flowing through wood or sponge, filtration of water using sand and other porous material. Transport and flow phenomena in porous medium and industrial synthetic porous matrices as well as fractured rock arise in many fields of technology and sciences like soil sciences, agricultural, ceramic, petroleum engineering, construction, chemical and biomedical, food technology etc. Sixty percent part of the original oil is still left behind in a classic oil reservoir after the end of oil recovery process. So there is a need of developing oil recovery methods for the unrecovered oil. However, the recovery processes only establish a small-scale fraction of a extensive amount. Lot of studies on it are available in the literature on porous media. In the oil recovery processes, the another areas like soil science and hydrology are imaginably the most established topics related to porous media. Many study on groundwater flow and the restoration of aquifers which are contaminated by different type of pollutants are relevant areas of research present-days. The ancient research areas of chemical engineers that deal with porous media consist of drying, centrifuging, filtration, multiphase flow in packed columns, flow and transport in micro porous membranes, separation and adsorption, and diffusion and reaction in porous catalysts. Transport of solute in porous media is dependent on solvent and solute properties, velocity field in porous media, size, shape and location of solid part of the medium. Solute transport arises by three processes viz., diffusion, mechanical dispersion and advection in porous media. The process in which solutes are transported by random thermal motion of solute molecules is called diffusion. Rate of solute transport arising in diffusion is given by Fick's Law. The process, in which solutes are transported by bulk motion of flowing groundwater, is called advection. Rate of solute transport arises in advection is given by the product of solute concentration and the components of groundwater velocity. Mechanical dispersion is a spreading process which is caused by a small amount of fluctuations in groundwater velocity along with path of flow within individual pores. Rate of solute transport in mechanical dispersion can be represented by the generalized form of Fick's law of diffusion.

Advective-dispersive theory is used in many physical situations as flow through porous media, mass transfer in fluids, relaxation in polymer systems, tracer dynamics in polymer networks, spread of contaminations in fluids [58, 59]. Contaminations occur on the land of surface and permeate into the surface by pores. Finally, contaminants are transported into groundwater.

The following equation represents solute transport in aquifers:

$$\frac{\partial c(x,t)}{\partial t} = -v\frac{\partial c}{\partial x} + d\frac{\partial^2 c}{\partial x^2}, \tag{2.1}$$

where $c(x,t)$ is solute concentration, $v > 0$ represents average fluid velocity and d represents dispersion coefficient. Equation 2.1 is also called advection-dispersion equation. This equation also describes probability function for location of particles in a continuum. The equation 2.1 is used in groundwater hydrology in which the transport of passive tracers is carried by fluid flow in porous media. Reaction-diffusion process has been investigated since a long time. In the process of reaction-diffusion, reacting molecules are used to move through space due to diffusion. This definition excludes other modes of transports as convection. Drifts of those may arise due to presence of externally imposed fields.

When a reaction occurs within an element of space, molecules can be created or consumed. These events are added to the diffusion equation and lead to reaction-diffusion equation of the form

$$\frac{\partial c}{\partial t} = D\nabla^2 c + R(c,t), \tag{2.2}$$

where $R(c,t)$ denotes reaction term at time t. The extension of the reaction-diffusion equation in fractional order system can be found in the articles [60, 61].

The process of diffusion in porous medium has been studied in many contexts. A crucial equation on the mathematical description of the physical problem is mass conservation equation, which is parabolic type, which induces a non physical behavior. Our purpose is to study the models, such as, contaminant models with diffusion, reaction and convective transport in porous media. In reaction-diffusion equation (RDE), it is assumed that the behavior of various populations described is governed by two processes. First one is local reaction in which the populations interact between themselves. The second one is the diffusion which makes the populations spread out in the space. The concept of population is understood here quite loosely. The RDE constitute a usual description for all complex systems in many areas of physics, ecology, computer sciences, geology and combustion theory. The aim of the authors is to investigate the diffusion phenomenon in a multi-scale porous medium by using collocation method. The medium consists of a connected network made of pores and fractures which are equi-distributed, and diffusion process is modeled by a nonlinear RDE with a non-linear reaction term. More precisely, an attempt has been taken to model a non-linear order RDE and solve it numerically with Dirichlet's type initial and boundary conditions.

Here Genocchi polynomials have been introduced in collocation method to solve non-linear fractional order reaction-diffusion equation. After finding the operational matrix of fractional differentiation, the given non-linear fractional equation model and boundary conditions are collocated. By collocating a non-linear system of algebraic equations are obtained which are solved by using an iteration method called Newton method. The chapter is organized as follows.

In the section 2.2, the definitions, mathematical preliminaries of fractional calculus, Genocchi numbers, Genocchi polynomial and their properties are given. The approximation of a arbitrary function and operational matrix of fractional differentiation by Genocchi polynomial are given in section 2.3 and 2.4, respectively. Section 2.5 contains the error bound and stability analysis of the proposed method. In section 2.6, a drive has been taken to solve proposed model using the operational matrix with Genocchi polynomials. Section 2.7 contains the validation of the method through a comparison of the numerical results with analytical results for two particular cases and also illustrations of numerical results of proposed model through graphical presentations are given in section 2.7. The conclusion of over all work is presented in section 2.8.

2.2 Preliminaries

Here, few definitions and important properties of fractional calculus have been introduced [34]. It is well known that the Riemann-Liouville definition has disadvantages when it comes to modeling real world problems. But definition of fractional differentiation given by M. Caputo [35] is more reliable for application point of view. Basic properties of Caputo fractional derivative are as follows.

$$D_c^{\vartheta} M = 0, \tag{2.3}$$

where M is a constant.

$$D_c^{\vartheta} t^{\sigma} = \begin{cases} 0, & \sigma \in N \cup 0, and\sigma < \lceil \vartheta \rceil, \\ \dfrac{\Gamma\,(1+\sigma)}{\Gamma(1-\vartheta+\sigma)} t^{-\vartheta+\sigma}, & \sigma \in N \cup 0 and\sigma \geq \lceil \vartheta \rceil\, or\sigma \notin N and\sigma > \lfloor \vartheta \rfloor, \end{cases} \tag{2.4}$$

where $\lfloor \vartheta \rfloor$ is floor function.

The operator D_c^{ϑ} is linear, since

$$D_c^{\vartheta}(Af(t) + BG(t)) = AD_c^{\vartheta} f(t) + BD_c^{\vartheta} g(t), \tag{2.5}$$

where A and B are constants.

Caputo operator and Riemann-Liouville operator have a relation given by

$$(I^{\vartheta} D_c^{\vartheta} g)(t) = g(t) - \sum_{k=0}^{l-1} g^k(0^+) \frac{t^k}{k!}, l-1 < \vartheta \leq l. \tag{2.6}$$

2.2.1 Genocchi polynomial and its properties [13, 14]

Genocchi polynomials and numbers have been investigated by many Mathematicians and Physicists. Genocchi numbers G_n and Genocchi polynomials $G_n(x)$ can be derived respectively by following exponential generating functions.

$$\frac{2x}{e^x + 1} = \sum_{n=0}^{\infty} G_n \frac{x^n}{n!}, (|x| < \pi), \tag{2.7}$$

$$\frac{2xe^{tx}}{e^x + 1} = \sum_{n=0}^{\infty} G_n(t) \frac{x^n}{n!}, (|x| < \pi), \tag{2.8}$$

where $G_n(t)$is Genocchi polynomial whose degree is n. Genocchi polynomial is given by

$$G_n(t) = \sum_{k=0}^{l} \binom{l}{k} G_{l-k} t^k. \tag{2.9}$$

Here, G_{l-k} is the Genocchhi numbers. Few examples of some Genocchi polynomials are

$$
\begin{aligned}
&G_0(t) = 0,\\
&G_1(t) = 1,\\
&G_2(t) = 2t - 1,\\
&G_3(t) = 3t^2 - 3t,\\
&G_4(t) = 4t^3 - 6t^2 + 1.
\end{aligned}
$$

Some properties of Genocchi polynomial are given below:

$$\int_0^1 G_n(t)G_m(t)dt = \frac{2(-1)^l m!l!}{(m+l)!}G_{m+l}, l, m \geq 1, \tag{2.10}$$

$$G_n(1) + G_n(0) = 0, n > 1, \tag{2.11}$$

$$\frac{dG_l(t)}{dt} = lG_{l-1}(t), l \geq 1. \tag{2.12}$$

Genocchi polynomials have many advantages over the classical orthogonal polynomials during approximation of a function as it has lesser terms and smaller coefficients of individual terms. In certain cases, these polynomials provide better accuracy than other polynomials. It has a lot of applications in many branches of physics and mathematics viz; differential structures on spheres in differential topology, p-adic analytic number theory, theory of modular forms in Eisenstein series and quantum groups in quantum physics.

2.3 Approximation of a arbitrary function

Let us Suppose $\{G_1(t), G_2(t), \cdots, G_M(t)\} \subset L^2[0,1]$ is the set of Genocchi polynomials. A function $u(t)$ which belongs to $L^2[0,1]$ can be expressed as

$$u(t) = \sum_{l=1}^{M} c_l G_l(t) = C^T G(t), \tag{2.13}$$

where $c_l = (u(t), G_l(t))$ and (,) denotes the inner product. C and G(t) are column vectors.

Similarly, an arbitrary function $u(x,t)$ belongs to $L^2[0,1] \times L^2[0,1]$ of two variables can be expressed in terms of Genocchi polynomials as

$$u(x,t) = \sum_{l=1}^{M} \sum_{m=1}^{M} u_{lm} G_m(t) G_l(x), \tag{2.14}$$

where $V = [u_{lm}]$ and $u_{lm} = (G_l(x), (u(x,t), G_m(t)))$.

2.4 Genocchi operational matrix of fractional derivative [15]

2.4.1 Lemma

. **Statement:** Let us consider $G_j(x)$ be the Genocchi polynomial then $D^{\vartheta} G_j(x) = 0$ for $j = 1, \cdots, \lceil \vartheta \rceil$, $\vartheta > 0$.

Theorem 1: Let, $\psi(y) = (G_1(y), G_2(y), \cdots, G_N(y))^T$ is the Genoochi vector and $\vartheta > 0$. Then

$$D^{\vartheta} \psi(y) = Q^{\vartheta} \psi(y), \tag{2.15}$$

where Q^{ϑ} is a $M \times M$ operational matrix of fractional derivative of order ϑ defined by:

$$Q^{\vartheta} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\vartheta\rceil}^{\lceil\vartheta\rceil} \varsigma_{\lceil\vartheta\rceil,k,1} & \sum_{k=\lceil\vartheta\rceil}^{\lceil\vartheta\rceil} \varsigma_{\lceil\vartheta\rceil,k,2} & \cdots & \sum_{k=\lceil\vartheta\rceil}^{\lceil\vartheta\rceil} \varsigma_{\lceil\vartheta\rceil,k,M} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\vartheta\rceil}^{i} \varsigma_{i,k,1} & \sum_{k=\lceil\vartheta\rceil}^{i} \varsigma_{i,k,2} & \cdots & \sum_{k=\lceil\vartheta\rceil}^{i} \varsigma_{i,k,M} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\vartheta\rceil}^{M} \varsigma_{M,k,1} & \sum_{k=\lceil\vartheta\rceil}^{M} \varsigma_{M,k,2} & \cdots & \sum_{k=\lceil\vartheta\rceil}^{M} \varsigma_{M,k,M} \end{bmatrix},$$

where $\varsigma_{i,k,j}$ can be written as

$$\varsigma_{i,k,j} = \frac{G_{i-k}\, i!}{\Gamma(1-\vartheta+k)(i-k)!} h_j. \tag{2.16}$$

Here G_{i-k} is Genocchi numbers and h_j can be obtained by

$$h_j = \frac{Gram_j(G_1(y), G_2(y), \cdots, G_N(y))}{Gram(G_1(y), G_2(y), \cdots, G_N(y))}, \tag{2.17}$$

where $Gram(G_1(y), G_2(y), \cdots, G_M(y))$ is equal to

$$\begin{vmatrix} \langle\, G_1(y), G_1(y)\rangle & \langle\, G_1(y), G_2(y)\rangle & \cdots \langle\, G_1(y), G_M(y)\rangle \\ \langle\, G_2(y), G_1(y)\rangle & \langle\, G_2(y), G_2(y)\rangle & \cdots \langle\, G_2(y), G_M(y)\rangle \\ \vdots & & \\ \langle\, G_N(y), G_1(y)\rangle & \langle\, G_N(y), G_2(y)\rangle & \cdots \langle\, G_M(y), G_M(y)\rangle \end{vmatrix}. \tag{2.18}$$

Here, $Gram_j(G_1(y), G_2(y), \cdots, G_M(y))$ can be obtained by replacing the j^{th} column of

$$Gram(G_1(y), G_2(y), \cdots, G_M(y)) \tag{2.19}$$

by column whose elements are

$$\langle\, G_1(t), f(y)\rangle \langle\, G_2(y), f(y)\rangle, \cdots, \langle\, G_M(y), f(y)\rangle. \tag{2.20}$$

2.5 Error bound and stability analysis

In this section, the upper bound is found by means of Genocchi polynomial on the error which is being expecting in the approximation. Considering the following space

$$\prod_M = Span\{G_1(x), G_2(x), \cdots, G_M(x), G_1(t), G_2(t), \cdots, G_M(t)\}. \tag{2.21}$$

Let $\tilde{u}(x,t) \in \prod_M$ be the best approximation of $u(x,t)$. Then using the definition of best approximation, the following is obtained

$$\|u(x,t) - \tilde{u}(x,t)|_\infty \leq \|u(x,t) - w(x,t)|_\infty, \forall w(x,t) \in \prod_M. \tag{2.22}$$

The above inequality will remains true if $w(x,t)$ represents the interpolating polynomial for $u(x,t)$ at points (x_i, t_k) where x_i, t_k for $0 \leq i, k \leq M$ are respectively roots of $G_{M+1}(x)$ and $G_{M+1}(t)$. Then by the procedure given in article [62]

$$\begin{aligned} u(x,t) - w(x,t) = &\frac{1}{(M+1)!}\frac{\partial^{M+1}u(\mu,t)}{\partial x^{M+1}}\prod_{i=0}^{M}(x-x_i) + \frac{1}{(M+1)!}\frac{\partial^{M+1}u(x,\eta)}{\partial t^{M+1}}\prod_{i=0}^{M}(t-t_k) \\ &- \frac{\partial^{2M+2}u(\mu',\eta')}{\partial x^{M+1}\partial t^{M+1}} \times \frac{\prod_{i=0}^{M}(x-x_i)\prod_{k=0}^{M}(t-t_k)}{(M+1)!(M+1)!}, \end{aligned} \tag{2.23}$$

where $\mu', \eta', \mu, \eta \in [0,1]$. Now by the properties of the norm

$$\begin{aligned}\|u(x,t)-w(x,t)\| \leq & \frac{1}{(M+1)!}\max_{x,t\in[0,1]}\left|\frac{\partial^{M+1}u(\mu,t)}{\partial x^{M+1}}\right|\Big\|\prod_{i=0}^{M}(x-x_i)\Big\| \\ & + \frac{1}{(M+1)!}\max_{x,t\in[0,1]}\left|\frac{\partial^{M+1}u(x,\eta)}{\partial t^{M+1}}\right|\Big\|\prod_{k=0}^{M}(t-t_k)\Big\| \\ & + \max_{x,t\in[0,1]}\left|\frac{\partial^{2M+2}u(\mu',\eta')}{\partial x^{M+1}\partial t^{M+1}}\right| \times \frac{\Big\|\prod_{i=0}^{M}(x-x_i)\Big\|\Big\|\prod_{k=0}^{M}(t-t_k)\Big\|}{(M+1)!(M+1)!}.\end{aligned} \tag{2.24}$$

Since $u(x,t)$ is a continuous differential function on the interval $[0,1]$ then there exist constants A_1, A_2 and A_3 such that

$$\max_{x,t\in[0,1]}\left|\frac{\partial^{M+1}u(\mu,t)}{\partial x^{M+1}}\right| \leq A_1, \tag{2.25}$$

$$\max_{x,t\in[0,1]}\left|\frac{\partial^{M+1}u(x,\eta)}{\partial t^{M+1}}\right| \leq A_2, \tag{2.26}$$

$$\max_{x,t\in[0,1]}\left|\frac{\partial^{2M+2}u(\mu',\eta')}{\partial x^{M+1}\partial t^{M+1}}\right| \leq A_3. \tag{2.27}$$

Now to minimize the factor, the following procedure is obtained

$$\min_{x_i\in[0,1]}\max_{x\in[0,1]}\left|\prod_{i=0}^{M}(x-x_i)\right| = \min_{x_i\in[0,1]}\max_{x\in[0,1]}\left|\frac{G_{M+1}(x)}{M+1}\right|, \tag{2.28}$$

where $(M+1)$ is the leading coefficient of Genocchi polynomial of order $(M+1)$ and Genocchi polynomial satisfies

$$\max_{x\in[0,1]}\Big|G_{M+1}(x)\Big| \leq \frac{4e^{\pi}\pi^{-M-1}(-2^{M+1}\Gamma(2+M,\pi)+e^{\pi}\Gamma(2+M,2\pi))}{-2+2^{M+1}}. \tag{2.29}$$

From the above inequality, we get

$$\|u(x,t)-\tilde{u}(x,t)\| \leq \frac{4e^{\pi}\pi^{-M-1}(-2^{M+1}\Gamma(2+M,\pi)+e^{\pi}\Gamma(2+M,2\pi)}{-2+2^{M+1}} \times \Big(\frac{A_1+A_2}{(M+1)!}+\frac{A_3}{((M+1)!)^2}\Big). \tag{2.30}$$

So, an upper bound is obtained for approximate solution of the absolute errors, which shows that approximation $\tilde{u}(x,t)$ converges to the exact solution $u(x,t)$ and validate the stability of the proposed scheme.

2.6 Solution of the problem

In this section, operational matrix method is applied based on Genocchi polynomials of fractional derivatives to obtain the solution of following non-linear fractional order diffusion equation of the type

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{\beta}}{\partial x^{\beta}}\Big(u^m\frac{\partial u}{\partial x}\Big)+\lambda u(1-u)+\psi(x,t), 0<\alpha\leq 1, 0\leq\beta\leq 1, \tag{2.31}$$

with initial and boundary conditions as

$$u(x,0) = f(x), \tag{2.32}$$

$$u(1,t) = g(t), \tag{2.33}$$

$$u(0,t) = h(t), \tag{2.34}$$

where $0 \leq x \leq 1$, $0 \leq t \leq 1$. If $\lambda = 0$, then system is known as conservative system and $\lambda \neq 0$ implies the non-conservative. $\lambda < 0$ is known as sink term and $\lambda > 0$ as the source term. $\psi(x,t)$ is the forced term. The function $u(x,t)$ is approximated with the help of Genocchi polynomial as

$$u(x,t) = \sum_{i=1}^{M}\sum_{j=1}^{M} c_{ij} G_i(x) G_j(t), \tag{2.35}$$

where c_{ij} are unknown coefficients for $i = 1, 2, \cdots$; and $j = 1, 2, \cdots$; This equation can be rewritten in matrix form as

$$u(x,t) = \phi^T(x).C.\phi(t), \tag{2.36}$$

where $C = [c_{ij}]_{M\times M}$ is an $M\times M$ matrix of unknowns and $\phi(t) = (G_1(t), \cdots, G_M(t))^T$ is a column vector. Now operating fractional derivative of order α on the expression (2.36) with respect to x and using theorem 1, the following expression is obtained

$$\frac{\partial^\alpha u}{\partial t^\alpha} = Q^\alpha u(x,t) = \phi^T(x).C.Q^\alpha \phi(t), \tag{2.37}$$

Similarly,

$$\frac{\partial^\beta u}{\partial x^\beta} = Q^\beta u(x,t) = Q^\beta \phi^T(x).C.\phi(t), \tag{2.38}$$

Now equations (2.32) – (2.34) with the aid of the equation (2.36) give

$$\phi^T(x).C.\phi(0) = f(x), \tag{2.39}$$

$$\phi^T(1).C.\phi(t) = g(t), \tag{2.40}$$

$$\phi^T(0).C.\phi(t) = h(t). \tag{2.41}$$

After putting the values of $u(x,t)$, $\frac{\partial^\alpha u}{\partial x^\alpha}$ and $\frac{\partial^\beta u}{\partial t^\beta}$ from equations (2.36), (2.37) and (2.38) in equation (2.31), the following residue is obtained

$$\phi^T(x).C.Q^\alpha\phi(t) - \big((Q^\beta.\phi(x))^T.C.\phi(t)\big)^m \times Q^1.\phi^T(x).C.\phi(t) - \big(\phi^T(x).C.\phi(t)\big)^m \\ \times Q^{\beta+1}.\phi^T(x).C.\phi(t) - \lambda\phi^T(x).C.\phi(t)\big(1-\phi^T(x).C.\phi(t)\big) - \psi(x,t) = 0 \qquad (2.42)$$

Now collocating the equations (2.39) − (2.41) and (2.42) at points $x_i = \frac{i}{M}$ for $i = 1, 2, \cdots, M$ and $t_i = \frac{i}{M}$ for i=$1, 2, \cdots, M$, a non linear system of algebraic equations ia obtained. By Solving that system of equations and finding C, the numerical solution of the proposed model (2.31) can be found.

2.7 Results and discussion

In this section, a drive has been taken to validate the effectiveness of the proposed method through applying it in three standard order problems ($\alpha = 1$) and compare the obtained results with the existing analytical results for different particular cases.

Example 1:

Considering $\beta = 0, m = 1, \lambda = 0, \psi(x,t) = x - xt^2$, so that model is reduced to

$$\frac{\partial u(x,t)}{\partial t} = u(x,t)\frac{\partial u(x,t)}{\partial x} + x - xt^2. \qquad (2.43)$$

For the initial and boundary conditions

$$u(x,0) = 0, u(0,t) = 0, u(1,t) = t, \qquad (2.44)$$

TABLE 2.1: variations of L_∞ and L_2 for different time for first case taking $M = 3$.

t	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1$
L_∞	2.77×10^{-17}	2×10^{-17}	1.1×10^{-17}	5.5×10^{-17}	5×10^{-17}
L_2	3.72×10^{-17}	0	6.4×10^{-17}	0	1×10^{-16}

the exact solutions of above problem is $u(x,t) = xt$. Root mean square error (RMSE) and maximum absolute error (MAE) denoted by L_2 and L_∞ respectively have been commonly used to show the accuracy of the numerical solution over the existing analytical result of a considered model. RMSE is used to measure model performance in air quality, climate research studies and in meteorology. RMSE is more appropriate to represent model performance compared to MAE when error distribution is expected to be Gaussian type. Another advantage of RMSE over MAE is it avoids the use of absolute value which is highly undesirable in many mathematical calculations, whereas it requires less calculation. The MAE L_∞ and RMSE L_2 are defined by

$$L_\infty = Max|u(x_i,t_i) - \overline{u}(x_i,t_i)|, \tag{2.45}$$

$$L_2 = \sqrt{\frac{1}{M}\sum |u(x_i,t_i) - \overline{u}(x_i,t_i)|^2}, \tag{2.46}$$

which are calculated between the existing exact solution and the numerical solution using proposed method for $M = 3$. The obtained results are displayed through Table 2.1, which clearly depict that obtained numerical results are in complete agreement with the existing results. The similarity nature for both the solutions can also be found from Fig.2.1 and Fig.2.2.

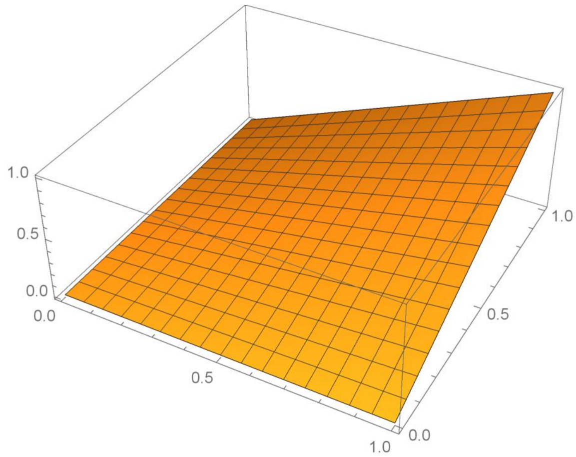

FIGURE 2.1: Plots of $u(x,t)$ vs. x and t for $M=3$ in case of approximate solution.

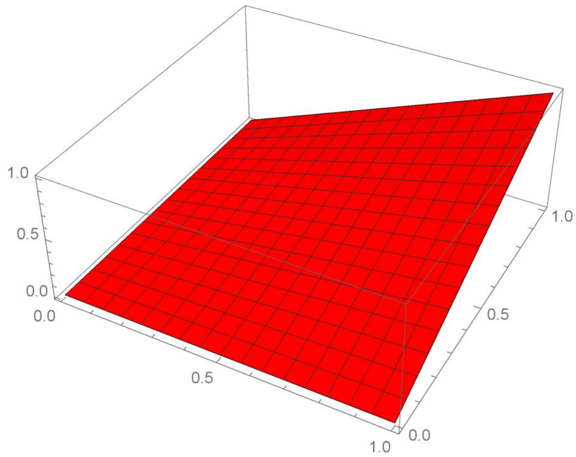

FIGURE 2.2: Plots of $u(x,t)$ vs. x and t for $M=3$ in case of exact solution.

Example 2:

Considering another particular case for the proposed method as $\beta = 1, m = 1, \lambda = 1, \alpha = 1$, so that the model is reduced to a standard order non-linear reaction diffusion equation as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}(u(x,t)\frac{\partial u(x,t)}{\partial x}) + u(x,t)(1-u(x,t)), \tag{2.47}$$

TABLE 2.2: variations of L_∞ and L_2 for different time for second case taking $M = 3$.

t	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1$
L_∞	1.1×10^{-3}	9.9×10^{-4}	4.4×10^{-4}	9×10^{-4}	5.8×10^{-4}
L_2	8.8×10^{-4}	6.4×10^{-4}	3.1×10^{-4}	7.6×10^{-4}	3.3×10^{-4}

TABLE 2.3: variations of L_∞ and L_2 for different time for second case taking $M = 4$.

t	t=0.2	t=0.4	t=0.6	t=0.8	t=1
L_∞	9.7×10^{-5}	3×10^{-5}	2×10^{-5}	4×10^{-5}	2.2×10^{-5}
L_2	6.4×10^{-5}	2.1×10^{-5}	1.3×10^{-5}	2.7×10^{-5}	1.2×10^{-5}

whose exact solution under the initial and boundary conditions

$$u(x,0) = \frac{1}{4}(1 - tanh(\frac{x}{2\sqrt{6}}))^2, \tag{2.48}$$

$$u(1,t) = \frac{1}{4}(1 - tanh(\frac{1}{2\sqrt{6}}(1 - \frac{5t}{\sqrt{6}}))^2, \tag{2.49}$$

$$u(0,t) = \frac{1}{4}(1 - tanh(\frac{1}{2\sqrt{6}}(-\frac{5t}{\sqrt{6}}))^2, \tag{2.50}$$

is given by $u(x,t) = \dfrac{1}{4}(1 - tanh(\frac{1}{2\sqrt{6}}(x - \frac{5t}{\sqrt{6}}))^2$. The maximum absolute error L_∞ and the root mean square error L_2 for different values of t are shown in Table 2.2 and Table 2.3, respectively for $M = 3$ and $M = 4$.

Example 3:

Consider the Burgers-Fisher equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u\frac{\partial u(x,t)}{\partial x} + u(1-u), \quad 0 < x < 1, \quad t > 0, \tag{2.51}$$

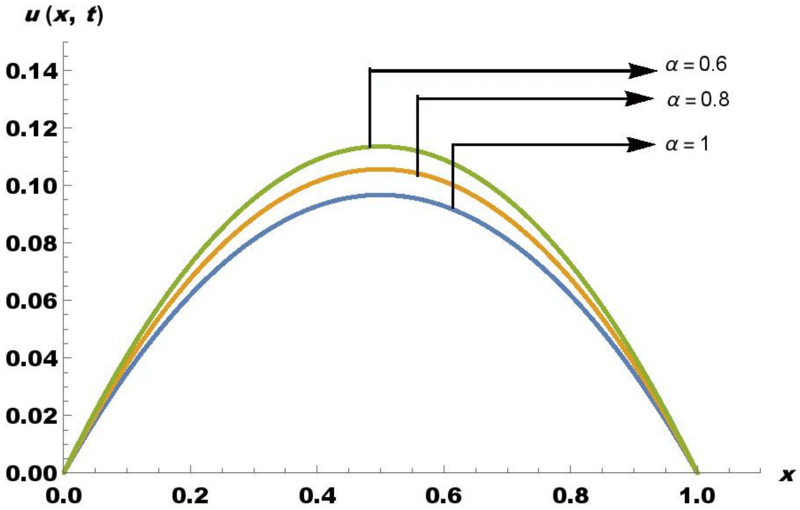

FIGURE 2.3: Plots of $u(x,t)$ vs. x for various α at t=1 when $\lambda = -1$, $\beta = 1$.

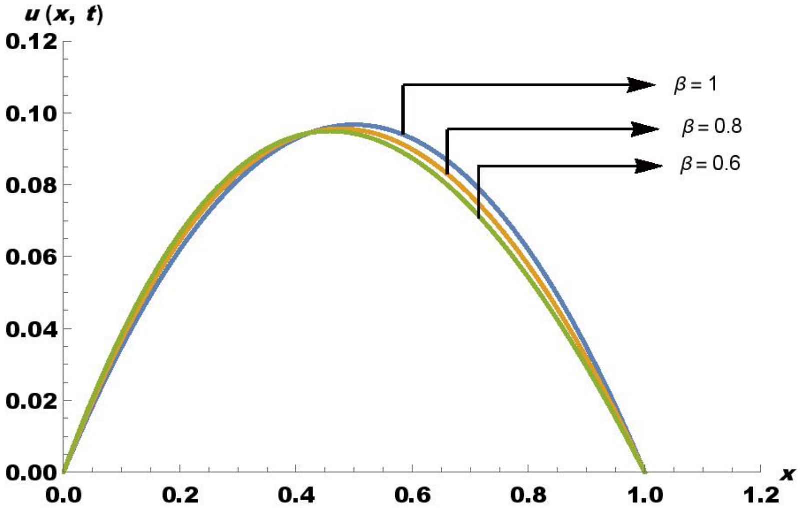

FIGURE 2.4: Plots of of $u(x,t)$ vs. x for various β at t=1 when $\lambda = -1$, $\alpha = 1$.

with following initial and boundary conditions as

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{x}{4}), \tag{2.52}$$

$$u(0,t) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{5t}{8}), \tag{2.53}$$

$$u(1,t) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{4}(1 + \frac{5t}{2})). \tag{2.54}$$

TABLE 2.4: Comparison of absolute errors for proposed method and the method given in [17].

$(x,t)\downarrow$	method given in literature	proposed method
(0.1,1)	1.4×10^{-4}	2.51×10^{-7}
(0.2,1)	1.9×10^{-4}	1.07×10^{-6}
(0.3,1)	8.8×10^{-5}	2.13×10^{-6}
(0.4,1)	2.1×10^{-5}	3.21×10^{-6}
(0.5,1)	1.21×10^{-5}	4.13×10^{-6}
(0.6,1)	5.7×10^{-4}	4.74×10^{-6}
(0.7,1)	7.12×10^{-4}	4.89×10^{-6}
(0.8,1)	8.1×10^{-4}	4.37×10^{-6}
(0.9,1)	5.15×10^{-5}	2.8×10^{-6}

where exact solution of above problem is $\frac{1}{2}+\frac{1}{2}\tanh(\frac{1}{4}(x+\frac{5t}{2}))$. A comparison of the absolute error obtained using method used in this chapter and the exact solution with the absolute error obtained by [17], is shown in Table 2.4. It is seen from the table that proposed method in this chapter is much superior as compared to the existing numerical method when maximum absolute error is computed for the given example. After the confirmation of accuracy and efficiency of the method, proposed numerical method is applied to find numerical solution of the considered non linear space time fractional order non-conservative system (2.31) under the following initial and boundary conditions as

$$u(x,0)=x(1-x), u(1,t)=0, u(0,t)=0. \tag{2.55}$$

The numerical results of the solute concentration vs. x at time $t=1$ are calculated numerically for various values of α and β for conservative and non-conservative systems. All these effects on pollute concentration in ground water are depicted through Figs.2.3 − 2.6. It is seen from Fig.2.3 that for non-linear time fractional reaction diffusion equation the sub diffusion phenomena of solute concentration occurs for the case when the order of non-linearity is $m=2$. It is also found that overshoots of sub diffusion decrease with the increase in α. i.e, pollute diffuses with slower rate in groundwater as system approaches fractional order to standard order $\alpha = 1$. It is seen from Fig.2.4 that similar sub-diffusions are found for non-linear spatial fractional reaction-diffusion equation ($\alpha = 1, m = 2, \lambda = -1$). Here pollute diffuses with faster rate as β approaches to standard order ($\beta = 1$) from fractional order. This means initially solute concentration decreases as β increases and after a small duration it becomes opposite in nature. The effects of reaction term on the solution profile for standard order reaction-diffusion equation ($\alpha = 1, \beta = 1$) are shown through Fig.2.5 and Fig.2.6, respectively without and with the presence of forced term. It is clear from the figures that overshoots of the sub-diffusions of solute concentration decrease for the case of sink term($\lambda = -1$) as compared to source term($\lambda = +1$). Thus the rate of diffusion of the pollute will be more due to the effect of source term and effect of forced term causes the increment in diffusion rate of pollute concentration. Fig.2.5 justifies the fact that the overshoots of the solute concentration will be higher in presence of force term (Fig.2.6) as compared to the case during its absence (Fig.2.5.)

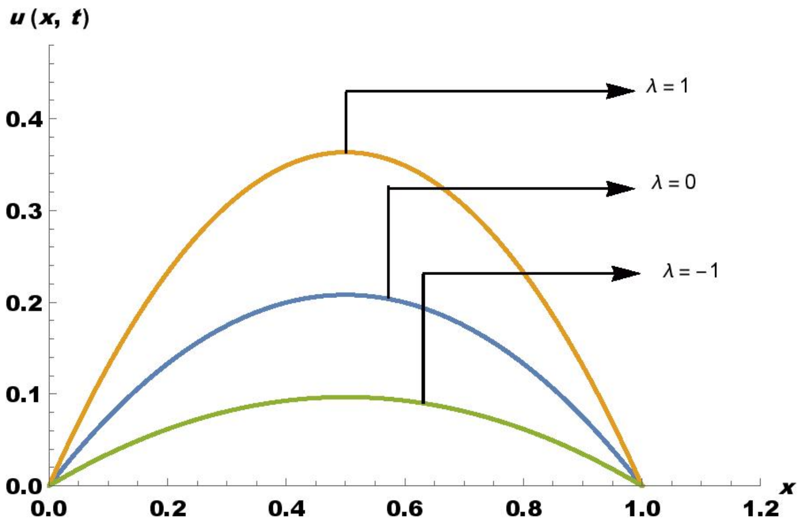

FIGURE 2.5: Plots of $u(x,t)$ vs. x for different values of λ at t=1 when $\alpha = 1, \beta = 1, \psi(x,t) = 0$.

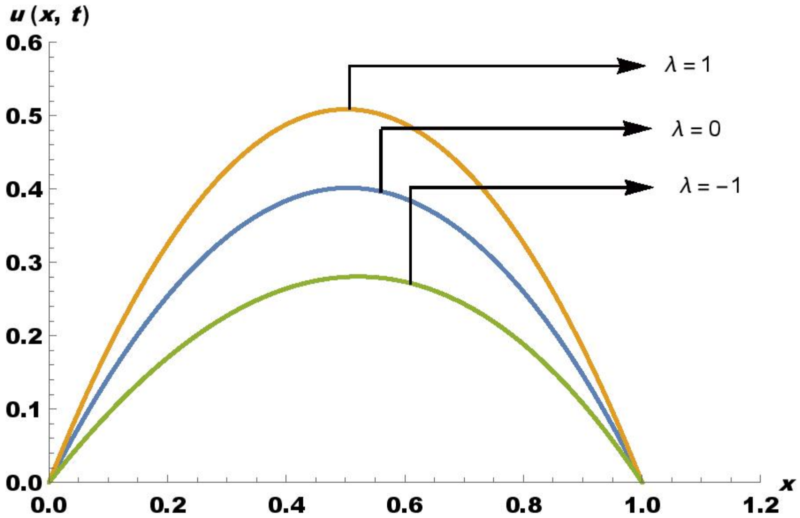

FIGURE 2.6: Plots of $u(x,t)$ vs. x for different values of λ at t=1 when $\alpha = 1, \beta = 1, \psi(x,t) = x - xt^2$.

2.8 Conclusion

The present scientific contribution has achieved four important goals. First one is the use of collocation method based on Genocchi polynomials to solve the considered non linear time-space reaction-diffusion equation in presence of the forced term. The efficiency and effectiveness of the proposed method are validated by comparing obtained numerical results with existing analytical results of Burgers and Burgers-Fisher equations. Second one is the pictorial presentations of the nature of overshoots during sub-diffusion as the system approaches from standard order to fractional order and the effect of fractional order time and spatial derivatives on pollute concentration in ground water is shown through graphs. The important feature of the present study on proposed mathematical model is the graphical exhibitions of the rate of pollute diffusion for spatial fractional case as compared to time fractional case. Third one is the exhibition of decrease of solute concentration due to the presence of sink term for standard order as well as fractional order systems and also increase in solution profile due to the effect of forced term. The last one is finding the error bound and stability analysis of the proposed model. In future study, the model can be extended to diffusion equation to variable order and riesz fractional order diffusion equation. The aim will be to develop some numerical methods to solve those models in two dimensional cases.

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