Chapter 6

Finite difference with collocation method to solve multi term variable-order fractional reaction-advection-diffusion equation in heterogeneous medium

6.1 Introduction

Partial differential equations (PDEs) with fractional derivative have gained considerable popularity and importance due to its capability of modeling many anomalous phenomena and complex system in natural science and engineering fields [210, 211, 212, 213, 214, 215]. In modeling the real process and phenomenon, fractional derivatives gives a more accurate model than integer order derivatives [216, 217, 218, 219]. In particular, modeling of anomalous diffusion in a specific type of porous media is one of the most significant applications of fractional derivatives [43, 220]. In this chapter, a attention has been made to solve a class of fractional advection-diffusion equation (FADE) in a heterogeneous medium. Many models have been developed in heterogeneous medium by various researchers viz. [221, 222, 223, 224]. Diffusion in the heterogeneous medium does not obey the laws of standard diffusion in fundamental ways generally. In recent years, variable-order fractional diffusion equations(VOFDE) becomes a useful tool to describe mathematical models in various fields but finding an analytical solution to VOFDE is quite difficult. Generally, the numerical method with sufficient accuracy is acceptable in finding the highly accurate solution of VOFDE. There are few numerical methods for solving VOFDEs which can be found [225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241].

In this chapter, the following multi-term variable fractional order reaction advection diffusion (MTV-FRADE) in heterogeneous medium has been solved.

$$
\begin{aligned}
P_{\alpha_1,\alpha_2,\ldots\alpha_p}({}_0^C D_t)(u(x,t)) = A\left(\frac{1}{2}+\frac{q}{2}\right){}_aD_x^{\beta(x,t)}u(x,t) + A\left(\frac{1}{2}-\frac{q}{2}\right){}_xD_b^{\beta(x,t)}u(x,t) \\
- v\frac{\partial u(x,t)}{\partial x} + ku(x,t) + f(x,t),
\end{aligned}
\tag{6.1}
$$

with initial and boundary conditions

$$
\begin{aligned}
u(x,0) &= \psi_0(x), \qquad && a \le x \le b, \\
u(0,t) &= \psi_1(t), \qquad && 0 \le t \le T, \\
u(t,t) &= \psi_2(t), \qquad && 0 \le t \le T,
\end{aligned}
\tag{6.2}
$$

where

$$
P_{\alpha_1,\alpha_2,\ldots\alpha_p}({}_0^c D_t)(u(x,t)) = \sum_{i=1}^{p} d_i \, {}_0^C D_t^{\alpha_i(x,t)} u(x,t),
\tag{6.3}
$$

$0 \le \alpha_i(x,t) \le 1$, for $i = 1,2,..,p, p \in N$, ${}_0^C D_t^{\alpha_i(x,t)}$ is left Caputo derivative. $1 < \beta(x,t) \le 2$, ${}_aD_x^{\beta(x,t)}$ and ${}_xD_b^{\beta(x,t)}$ left and right Riemann-Liouville(RL) variable-order fractional derivatives, respectively. $A > 0$ is fractional dispersion coefficient which generally controls the rate of spreading of solute, $-1 \le q \le 1$ is the skewness parameter. If $q = 1$, then the considered MTV-FRADE will be positively skewed while for $q = -1$ the solution will be negatively skewed. If $q = 0$, the solution will be symmetrical.

This chapter consists of the MTV-FRADE which allows for modeling the transient dispersion. An accurate numerical method has been developed to find solutions of this type of models. This technique is based on Fibonacci collocation method in conjunction with the standard finite difference method. Firstly, the solution is approximated with a series of Fibonacci polynomials in space with unknown coefficients in time. Using the introduced technique the MTV-FARDE is converted into the system of variable-order differential equations which can easily be translated in the system of algebraic equations. This system of algebraic equations can easily be solved with the help of any iteration methods.

This chapter is arranged as follows. In section 6.2, contains necessary definitions of the variable-order derivatives in Caputo and left and right RL senses. The certain properties of Fibonacci polynomial are also discussed here. Section 6.3 consists of certain theorems which help to develop the finite difference scheme. This section also contains the development of the proposed numerical technique. In section 6.4, the accuracy of the developed numerical technique is verified through comparing the solutions of the particular cases of considered MTV-FRADE equation (6.1) with the previously existing results which clearly predict that the proposed numerical method is more accurate than the existing methods. In section 6.5, the considered model (6.1) has been solved with the proposed method and discussed the effect on solute profile due to various parameters through graphical presentations. Overall work of the chapter is concluded in the section 6.6.

6.2 Preliminaries

This section includes some definitions of various types of variable-order derivatives which are necessary for the subsequent sections.

6.2.1 Caputo variable order fractional derivative

For the function $u(x,t)$ in the interval $[a,b]$, the expression

$$ {}_0^C D_t^{\alpha(x,t)} u(x,t) = \frac{1}{\Gamma(n-\alpha(x,t))} \int_0^t (t-\eta)^{n-1-\alpha(x,t)} \frac{\partial^n u(x,\eta)}{\partial \eta^n} d\eta, \quad n-1 < \alpha(x,t) \leq n, \tag{6.4} $$

is called the Caputo time-variable fractional derivative of order $\alpha(x,t)$, where $\Gamma(.)$ denotes the Gamma function.

6.2.2 Riemann-Liouville left and right variable fractional derivatives

The left and right variable fractional derivatives of order $m-1 < \beta(x,t) \leq m$ of function $u(x,t)$ with respect to x in the domain $[a,b]$ of Riemann-Liouville type are defined as

$$ {}_a D_x^{\beta(x,t)} = \frac{1}{\Gamma(m-\beta(x,t))} \frac{\partial^m}{\partial x^m} \int_a^x (x-\eta)^{m-1-\beta(x,t)} u(\eta,t) d\eta, $$
$$ {}_x D_b^{\beta(x,t)} = \frac{(-1)^m}{\Gamma(m-\beta(x,t))} \frac{\partial^m}{\partial x^m} \int_x^b (\eta-x)^{m-1-\beta(x,t)} u(\eta,t) d\eta, $$

From the definition of variable fractional order Riemann-Liouville derivative, it is easily concluded that [212]

$$\begin{aligned} {}_aD_x^{\beta(x,t)}(x-a)^{\gamma} &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta(x,t))}(x-a)^{\gamma-\beta(x,t)}, \ \ \gamma > -1, \\ {}_xD_b^{\beta(x,t)}(x-b)^{\gamma} &= \frac{(-1)^{\gamma}\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta(x,t))}(b-x)^{\gamma-\beta(x,t)}, \ \ \gamma > -1. \end{aligned} \tag{6.5}$$

The Fibonacci polynomial (1.15) can be rewritten around $(x-a)$ with the help of Taylor series expansion as

$$F_n(x) = \sum_{i=0}^{n} \frac{1}{i!}\Bigg\{ \sum_{\substack{j=i \\ (j+n)=odd}}^{n} \frac{(\frac{n+j-1}{2})!}{(\frac{n-j-1}{2})!(j-i)!} a^{j-i} \Bigg\}(x-a)^i. \tag{6.6}$$

A square integrable function $f(x)$ in [0,1] can be expressed in terms of Fibonacci polynomial as the equation (5.4). In general the series can be approximated by finite sum of *(n+1)*-terms of Fibonacci polynomial as given in equation (5.5).

6.3 Construction of finite difference scheme

In this section, some necessary theorems are provided to develop the numerical technique that will convert the considered model (6.1) in the set of algebraic equations which can be solved with the help of appropriate numerical technique.

Theorem 2.1. Suppose $f_n(x)$ is approximated in terms of Fibonacci polynomial. Then for any integer $l > 0$,

$$\frac{d^l}{dx^l} f_n(x) = \sum_{k=l+1}^{n+1} c_k \sum_{\substack{i=l \\ (i+k)=odd}}^{k} \frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!(i-l)!} x^{i-l}. \tag{6.7}$$

Proof. As the differential operator is linear, therefore

$$\frac{d^l}{dx^l} f_n(x) = \sum_{k=0}^{n+1} c_k \frac{d^l}{dx^l} F_k(x). \tag{6.8}$$

Now from equation (1.18), we have

$$\begin{aligned}\frac{d^l}{dx^l}F_k(x) &= \sum_{\substack{i=0\\(i+k)=odd}}^{k} \frac{(\frac{k+i-1}{2})!}{i!(\frac{k-i-1}{2})!}\frac{d^l x^i}{dx^l},\\ &= \sum_{\substack{i=0\\(i+k)=odd}}^{k} \frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!(i-l)!}x^{i-l}.\end{aligned} \tag{6.9}$$

With the help of above equation (6.9), one can easily conclude that

$$\begin{cases} \frac{d^l}{dx^l}F_k(x) = 0, & \text{for } k = 1, 2, 3, ..., l, \\ \frac{d^l}{dx^l}F_k(x) = \sum_{\substack{i=0\\(i+k)=odd}}^{k} \frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!(i-l)!}x^{i-l}, & \text{for } k \geq l+1. \end{cases} \tag{6.10}$$

Now using equation (6.10) in equation (6.8), we have

$$\frac{d^l}{dx^l}f_n(x) = \sum_{k=l+1}^{n+1} c_k \sum_{\substack{i=l\\(i+k)=odd}}^{k} \frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!(i-l)!}x^{i-l}.$$

Theorem 2.2. Suppose $f_n(x)$ is approximated in terms of Fibonacci polynomial, then Riemann-Liouville left and right derivatives of f_n of order $\beta(x,t)$ will be

$${}_aD_x^{\beta(x,t)}f_n(x) = \sum_{k=m+1}^{n+1} c_k \sum_{i=m}^{k} \left\{ \sum_{\substack{j=1\\(j+k)=odd}}^{k} \frac{(\frac{k+j-1}{2})!a^{j-i}}{(\frac{k-j-1}{2})!(j-i)!} \right\} \frac{(x-a)^{i-\beta(x,t)}}{\Gamma(i+1-\beta(x,t))} \tag{6.11}$$

and

$${}_xD_b^{\beta(x,t)}f_n(x) = \sum_{k=m+1}^{n+1} c_k \sum_{i=m}^{k} \left\{ \sum_{\substack{j=1\\(j+k)=odd}}^{k} \frac{(\frac{k+j-1}{2})!a^{j-i}}{(\frac{k-j-1}{2})!(j-i)!} \right\} \frac{(-1)^i(b-x)^{i-\beta(x,t)}}{\Gamma(i+1-\beta(x,t))}, \tag{6.12}$$

where $m-1 < \beta(x,t) \leq m$.

Proof. On taking left R-L variable-order derivative of the function $f_n(x)$, we have

$${}_aD_x^{\beta(x,t)}f_n(x) = \sum_{k=1}^{n+1} c_k F_k(x). \tag{6.13}$$

Now from equation (6.6) by using equation (6.5), we have

$$\begin{aligned} {}_aD_x^{\beta(x,t)}F_k(x) &= \sum_{i=0}^{k}\frac{1}{i!}\Bigg\{\sum_{\substack{j=i\\(j+k)=odd}}^{k}\frac{(\frac{k+j-1}{2})!}{(\frac{k-j-1}{2})!(j-i)!}a^{j-i}\Bigg\}{}_aD_x^{\beta(x,t)}(x-a)^i \\ &= \sum_{i=m}^{k}\Bigg\{\sum_{\substack{j=i\\(j+k)=odd}}^{k}\frac{(\frac{k+j-1}{2})!}{(\frac{k-j-1}{2})!(j-i)!}a^{j-i}\Bigg\}\frac{1}{\Gamma(i+1-\beta(x,t))}(x-a)^{i-\beta(x,t)}. \end{aligned} \tag{6.14}$$

From equation (6.14), one can easily say that

$$\begin{cases} {}_aD_x^{\beta(x,t)}F_k(x)=0, & \text{for } k=0,1,2,...,m. \\ {}_aD_x^{\beta(x,t)}F_k(x)=\sum_{i=m}^{k}\Bigg\{\sum_{\substack{j=i\\(j+k)=odd}}^{k}\frac{(\frac{k+j-1}{2})!}{(\frac{k-j-1}{2})!(j-i)!}a^{j-i}\Bigg\}\frac{1}{\Gamma(i+1-\beta(x,t))}(x-a)^{i-\beta(x,t)}, & \\ & \text{for } \beta(x,t)\geq m+1. \end{cases} \tag{6.15}$$

Now using equation (6.15) in equation (6.13), we get the desired result as

$${}_aD_x^{\beta(x,t)}f_n(x)=\sum_{k=m+1}^{n+1}c_k\sum_{i=m}^{k}\Bigg\{\sum_{\substack{j=1\\(j+k)=odd}}^{k}\frac{(\frac{k+j-1}{2})!a^{j-i}}{(\frac{k-j-1}{2})!(j-i)!}\Bigg\}\frac{(x-a)^{i-\beta(x,t)}}{\Gamma(i+1-\beta(x,t))}.$$

Similarly one can easily derive for right-hand derivative

$${}_xD_b^{\beta(x,t)}f_n(x)=\sum_{k=m+1}^{n+1}c_k\sum_{i=m}^{k}\Bigg\{\sum_{\substack{j=1\\(j+k)=odd}}^{k}\frac{(\frac{k+j-1}{2})!a^{j-i}}{(\frac{k-j-1}{2})!(j-i)!}\Bigg\}\frac{(-1)^i(b-x)^{i-\beta(x,t)}}{\Gamma(i+1-\beta(x,t))}.$$

Now approximating the numerical solution of considered model (6.1) with initial and boundary conditions (6.2) in terms of the series Fibonacci polynomials in space and unknown constants in time as

$$u(x,t)=\sum_{k=1}^{n+1}c_k(t)F_k(x), \tag{6.16}$$

where $F_k(x)$ is k-th order Fibonacci polynomial and $c_k(t)$'s are the unknowns which are to be determined at time t. Now the proposed model (6.1) at the time level $(m+1)$ using the above approximation will be

$$\begin{aligned}\sum_{k=1}^{n+1} P_{\alpha_1,\alpha_2,\ldots\alpha_p}({}_0^C D_t)c_k(t)F_k(x) = {} & A\left(\frac{1}{2}+\frac{q}{2}\right){}_aD_x^{\beta(x,t)}\sum_{k=1}^{n+1} c_k(t)F_k(x)\\ & + A\left(\frac{1}{2}-\frac{q}{2}\right){}_xD_b^{\beta(x,t)}\sum_{k=1}^{n+1} c_k(t)F_k(x)\\ & - v\frac{\partial}{\partial x}\sum_{k=1}^{n+1} c_k(t)F_k(x) + k\sum_{k=1}^{n+1} c_k(t)F_k(x) + f(x,t).\end{aligned} \tag{6.17}$$

The equation (6.17) with the aid of the equations (6.3) becomes

$$\begin{aligned}\sum_{k=1}^{n+1}\left(\sum_{i=1}^{p} d_i {}_0^C D_t^{\alpha_i(x,t)} c_k(t)\right)F_k(x) = {} & A\left(\frac{1}{2}+\frac{q}{2}\right){}_aD_x^{\beta(x,t)}\sum_{k=1}^{n+1} c_k(t)F_k(x)\\ & + A\left(\frac{1}{2}-\frac{q}{2}\right){}_xD_b^{\beta(x,t)}\sum_{k=1}^{n+1} c_k(t)F_k(x)\\ & - v\frac{\partial}{\partial x}\sum_{k=1}^{n+1} c_k(t)F_k(x) + k\sum_{k=1}^{n+1} c_k(t)F_k(x) + f(x,t).\end{aligned} \tag{6.18}$$

Now on dividing the time domain $[0,T]$ in M equal parts with equal interval of $h = T/M$ and $t_m = mh$ for $0 \le m \le M$, we obtain from equation (6.4) as

$$\begin{aligned}{}_0^C D_t^{\alpha(x,t)} c_k(t) &= \frac{1}{\Gamma(1-\alpha(x,t))}\int_0^t (t-\eta)^{\alpha(x,t)}\frac{dc_k(\eta)}{d\eta}d\eta,\\ &= \frac{1}{\Gamma(1-\alpha(x,t))}\int_0^t \frac{1}{\eta^{\alpha(x,t)}}\frac{dc_k(t-\eta)}{d\eta}d\eta,\\ {}_0^C D_t^{\alpha(x,t)} c_k(t_{m+1}) &= \frac{1}{\Gamma(1-\alpha(x,t))}\sum_{j=0}^{m}\int_{jh}^{(j+1)h}\frac{1}{\eta^{\alpha(x,t)}}\frac{dc_k(t_{m+1}-\eta)}{d\eta}d\eta,\\ &\approx \frac{1}{\Gamma(1-\alpha(x,t))}\sum_{j=0}^{m}\frac{c_k(t_{m+1}-jh)-c_k(t-(j+1)h)}{h}\int_{jh}^{(j+1)h}\frac{1}{\eta^{\alpha(x,t)}}d\eta,\\ &\approx \frac{h^{-\alpha(x,t)}}{\Gamma(2-\alpha(x,t))}\sum_{j=0}^{m}\{(j+1)^{1-\alpha(x,t)} - j^{1-\alpha(x,t)}\}(c_k^{m-j+1} - c_k^{m-j}).\end{aligned} \tag{6.19}$$

Using **Theorem 2.1** and **Theorem 2.2** and the equation (6.19), the equation (6.18) at time level $(m+1)$ gives rise to

$$\sum_{k=1}^{n+1}\Bigg(\sum_{i=1}^{p} d_i\frac{h^{-\alpha_i(x,t_{m+1})}}{\Gamma(2-\alpha_i(x,t_{m+1}))}\sum_{j=0}^{m}\{(j+1)^{1-\alpha_i(x,t_{m+1})}-j^{1-\alpha_i(x,t_{m+1})}\}(c_k^{m-j+1}-c_k^{m-j})\Bigg)F_k(x)$$
$$=A\Big(\frac{1}{2}+\frac{q}{2}\Big)\sum_{k=3}^{n+1}c_k\sum_{i=2}^{k}\Bigg\{\sum_{\substack{j=1\\(j+k)=odd}}^{k}\frac{(\frac{k+j-1}{2})!a^{j-i}}{(\frac{k-j-1}{2})!(j-i)!}\Bigg\}\frac{(x-a)^{i-\beta(x,t_{m+1})}}{\Gamma(i+1-\beta(x,t_{m+1}))}$$
$$+A\Big(\frac{1}{2}-\frac{q}{2}\Big)\sum_{k=3}^{n+1}c_k\sum_{i=2}^{k}\Bigg\{\sum_{\substack{j=1\\(j+k)=odd}}^{k}\frac{(\frac{k+j-1}{2})!a^{j-i}}{(\frac{k-j-1}{2})!(j-i)!}\Bigg\}\frac{(-1)^i(b-x)^{i-\beta(x,t_{m+1})}}{\Gamma(i+1-\beta(x,t_{m+1}))}$$
$$-v\sum_{k=2}^{n+1}c_k^{m+1}\sum_{\substack{i=1\\(i+k)=odd}}^{k}\frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!(i-1)!}x^{i-1}+k\sum_{k=1}^{n+1}c_k(t)F_k(x)+f(x,t_{m+1}).\tag{6.20}$$

Equation (6.20) can be collocated at $(n-1)$ points x_l as

$$\sum_{k=1}^{n+1}\Bigg(\sum_{i=1}^{p} d_i\frac{h^{-\alpha_i(x_l,t_{m+1})}}{\Gamma(2-\alpha_i(x_l,t_{m+1}))}\sum_{j=0}^{m}\{(j+1)^{1-\alpha_i(x_l,t_{m+1})}-j^{1-\alpha_i(x_l,t_{m+1})}\}(c_k^{m-j+1}-c_k^{m-j})\Bigg)F_k(x_l)$$
$$=A\Big(\frac{1}{2}+\frac{q}{2}\Big)\sum_{k=3}^{n+1}c_k\sum_{i=2}^{k}\Bigg\{\sum_{\substack{j=1\\(j+k)=odd}}^{k}\frac{(\frac{k+j-1}{2})!a^{j-i}}{(\frac{k-j-1}{2})!(j-i)!}\Bigg\}\frac{(x_l-a)^{i-\beta(x_l,t_{m+1})}}{\Gamma(i+1-\beta(x_l,t_{m+1}))}$$
$$+A\Big(\frac{1}{2}-\frac{q}{2}\Big)\sum_{k=3}^{n+1}c_k\sum_{i=2}^{k}\Bigg\{\sum_{\substack{j=1\\(j+k)=odd}}^{k}\frac{(\frac{k+j-1}{2})!a^{j-i}}{(\frac{k-j-1}{2})!(j-i)!}\Bigg\}\frac{(-1)^i(b-x_l)^{i-\beta(x_l,t_{m+1})}}{\Gamma(i+1-\beta(x_l,t_{m+1}))}$$
$$-v\sum_{k=2}^{n+1}c_k^{m+1}\sum_{\substack{i=1\\(i+k)=odd}}^{k}\frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!(i-1)!}x_l^{i-1}+k\sum_{k=1}^{n+1}c_k(t)F_k(x_l)+f(x_l,t_{m+1}),\tag{6.21}$$

where the collocation points $x_l=\frac{l}{n+1}$, $i=1,2,...,n-1$. To obtain the unknown coefficients at initial state $t=0$ we substitute the equation (6.16) in the initial condition of equation (6.2). Moreover, to obtain two more equations we will use the boundary conditions. Now from initial condition we have

$$\sum_{k=1}^{n+1}c_k^0F_k(x_l)=\psi_0(x_l).\tag{6.22}$$

and from the boundary conditions we have

$$\begin{aligned} \sum_{k=1}^{n+1} c_k^{m+1} F_k(0) = \psi_1(t_{m+1}), \\ \sum_{k=1}^{n+1} c_k^{m+1} F_k(1) = \psi_2(t_{m+1}). \end{aligned} \tag{6.23}$$

Here, equation (6.21) together with two equations of the boundary conditions (6.23) generate $(n+1)$ system of ordinary differential equations which can be translated in to a system of algebraic equations. This system of algebraic equations can be solved numerically using any iteration method to gain the unknowns coefficients $c_i (i = 1, 2, ..., n+1)$ at each time level. Hence, the desired approximated solution of the MTV-FRADE in heterogeneous medium can be obtained.

6.4 Error Analysis

In this section, the proposed method has been applied on certain special cases of considered model (6.1) having exact solutions and compared the numerical results with the existing methods. Moreover, the order of convergence and CPU time to calculate the solution have been calculated with the help of with the help of Mathematica software.

$$\text{Maximum Absolute Error at time } t \text{ is defined by } E_n(t) = \underset{0\leq x\leq 1}{Max} |U(x,t) - u_n(x,t)|, \tag{6.24}$$

$$\text{and order of convergence } = \frac{log(\frac{E_{n_1}(t)}{E_{n_2}(t)})}{log(\frac{n_2}{n_1})}. \tag{6.25}$$

where $U(x,t)$ is exact solution and $u_n(x,t)$ is numerical solution of the problem with n as degree of approximation.

Example 1. Consider the following variable-order fractional diffusion equation which can be obtained by choosing appropriate values of $d_i, \alpha_i(x,t), D, q, v$ and k as

$$\begin{aligned} d_0\, {}_0^C D_t^{\alpha}(u(x,t)) + d_1\, {}_0^C D_t^{\alpha_1}(u(x,t)) + d_2\, {}_0^C D_t^{\alpha_2}(u(x,t)) =& \frac{1}{2}{}_0 D_x^{\beta(x,t)}(u(x,t)) + \frac{1}{2}{}_x D_1^{\beta(x,t)}(u(x,t)) \\ &- \frac{\partial u(x,t)}{\partial x} + f(x,t), \end{aligned} \tag{6.26}$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) &= 0, && 0 \le x \le 1, \\ u(0,t) &= 0, && 0 \le t \le T, \\ u(1,t) &= 0, && 0 \le t \le T, \end{aligned} \tag{6.27}$$

where $0 < \alpha_1 < \alpha_2 < \alpha = 1$, $1 < \beta(x,t) \le 2$,. Taking $\beta(x,t) = 1.5 + 0.4sin(0.5\pi x)$ and $f(x,t) = f_1(x,t) + f_2(x,t) + f_3(x,t) + f_4(x,t)$, where

$$f_1(x,t) = 2x^2(1-x)^2\left\{d_0 t + \frac{d_1 t^{2-\alpha_1}}{\Gamma(3-\alpha_1)} + \frac{d_1 t^{2-\alpha_2}}{\Gamma(3-\alpha_2)}\right\},$$

$$f_2(x,t) = 2t^2 x(1-x)(1-2x),$$

$$f_3(x,t) = -\frac{t^2 x^{2-\beta(x,t)}}{\Gamma(5-\beta(x,t))}\left\{12x^2 - 6(4-\beta(x,t))x + (3-\beta(x,t))(4-\beta(x,t))\right\},$$

$$f_4(x,t) = -\frac{t^2 (1-x)^{2-\beta(x,t)}}{\Gamma(5-\beta(x,t))}\left\{12(1-x)^2 - 6(4-\beta(x,t))(1-x) + (3-\beta(x,t))(4-\beta(x,t))\right\}.$$

The exact solution of the problem given as $u(x,t) = t^2x^2(1-x)^2$ [242].

TABLE 6.1: Comparison of maximum absolute error of previous method with our proposed method when $\alpha_1 = 0.15, \alpha_2 = 0.95, d_0 = d_1 = d_3 = 1$ at $T = 1.5$.

M	Maximum error with our method	Maximum error with [242]	Convergence rate	CPU time(Sec)
100	1.5635e-4	2.3260e-3	-	2.984
200	7.6960e-5	1.1817e-3	1.02	7.391
400	3.7904e-5	5.9541e-4	1.02	23.03
800	1.8678e-5	2.9940e-4	1.02	77.61

TABLE 6.2: Comparison of maximum absolute error of previous method with our proposed method when $\alpha_1 = 0.2, \alpha_2 = 0.8, d_0 = 0, d_1 = d_3 = 1$ at $T = 1.5$.

M	Maximum error with our method	Maximum error with [242]	Convergence rate	CPU time(Sec)
100	2.9581e-5	1.0975e-3	-	2.953
200	1.2798e-5	4.8724e-4	1.02	7.422
400	5.5471e-6	2.1448e-4	1.02	23.06
800	2.4073e-6	9.3598e-4	1.02	78.68

TABLE 6.3: Comparison of maximum absolute error of previous method with our proposed
method when $\alpha_1 = 0.2, \alpha_2 = 0.5, d_0 = d_1 = d_3 = 1$ at $T = 1.5$.

M	Maximum error with our method	Maximum error with [242]	Convergence rate	CPU time(Sec)
100	9.0585e-5	1.3992e-2	-	2.938
200	4.4488e-5	7.2402e-3	1.02	7.407
400	2.1962e-5	3.6576e-3	1.01	23.07
800	1.0882e-5	1.8352e-4	1.01	79.79

Tables 6.1, 6.2 and 6.3 are formed on solving the considered example with the proposed method for different particular cases for $n = 5$ degrees of approximation to validate the accuracy, efficiency and effectiveness of the proposed method. It is clear from these tables that the proposed method is much more accurate and takes less time to compute the solution even for less degree of approximation.

6.5 Numerical results and discussions

After validation of the accuracy and effectiveness of the proposed method, an endeavour has been made to solve the special form of the considered mathematical model (6.1) by using the proposed method. In this section, the effects on solute concentration due to the variations of different parameters have been discussed for a particular case of the model (6.1).

$$\frac{\partial u(x,t)}{\partial t} + d_1\, {}_0D_t^{0.2t}u(x,t) + d_2\, {}_0D_t^{0.8t}u(x,t) = d\left(\frac{1}{2}+\frac{q}{2}\right){}_0^cD_x^{\beta(x,t)}u(x,t) + d\left(\frac{1}{2}-\frac{q}{2}\right){}_x^cD_1^{\beta(x,t)}u(x,t) - v\frac{\partial u(x,t)}{\partial x} + ku(x,t) + f(x,t), \tag{6.28}$$

under the prescribed initial and boundary conditions as

$$\begin{aligned} u(x,0) &= 0, \qquad 0 \le x \le 1, \\ u(0,t) &= 0, \qquad 0 \le t \le 1, \\ u(t,t) &= 0, \qquad 0 \le t \le 1, \end{aligned} \tag{6.29}$$

where v is average velocity of the plume, d is fractional dispersion coefficient, q is skewness parameter, k is source term and $f(x,t) = 5x^2(1-x)^2$ is forced term.

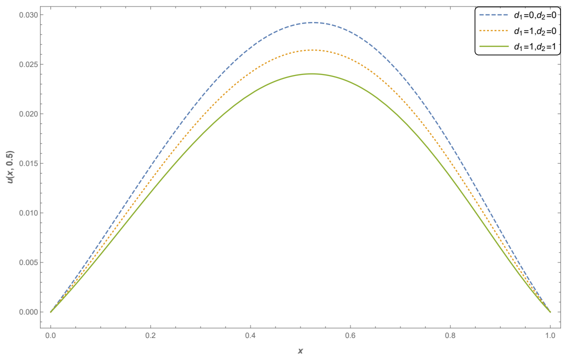

FIGURE 6.1: Plots of solute profile for $q = 0$, $d = v = k = 1$ and $\beta(x,t) = 2$ at $t = 0.5$.

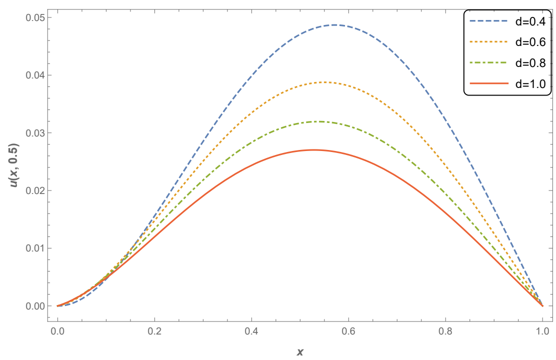

FIGURE 6.2: Plots of solute profile for $d_1 = d_2 = v = k = 1$, $q = 0$ and $\beta(x,t) = 1.5 + 0.5sin(0.5\pi x)$ at $t = 0.5$.

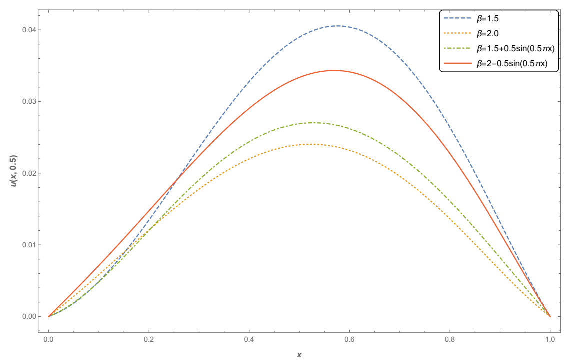

FIGURE 6.3: Plots of solute profile for $d_1 = d_2 = d = v = k = 1$, $q = 0$ at $t = 0.5$.

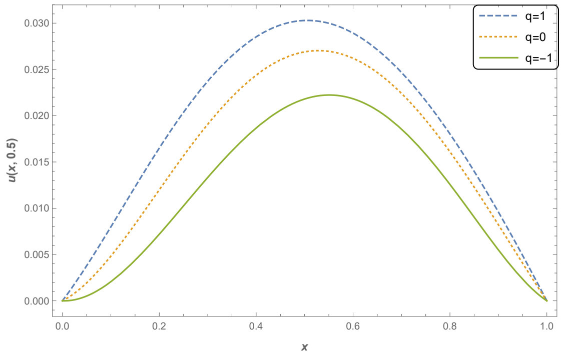

FIGURE 6.4: Plots of solute profile for $d_1 = d_2 = d = v = k = 1$, and $\beta(x,t) = 1.5 + 0.5\ sin(0.5\pi x)$ at $t = 0.5$.

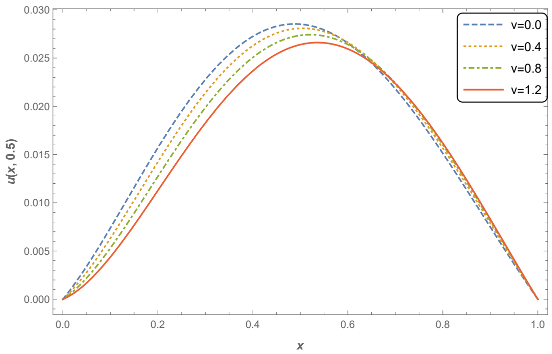

FIGURE 6.5: Plots of solute profile for $d_1 = d_2 = d = k = 1$, $q = 0$ and $\beta(x,t) = 1.5 + 0.5\ sin(0.5\pi x)$ at $t = 0.5$.

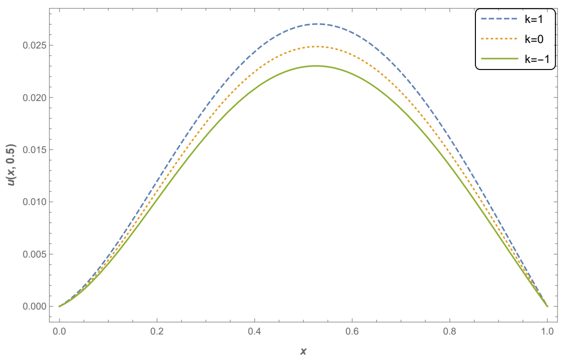

FIGURE 6.6: Plots of solute profile for $d_1 = d_2 = d = v = 1$, $q = 0$ and $\beta(x,t) = 1.5 + 0.5\ sin(0.5\pi x)$ at $t = 0.5$.

Figures 6.1-6.6 are drawn after solving the model (6.28) with the initial and boundary conditions given in equation (6.29) by using the proposed numerical method. Fig. 6.1 is drawn to show the effect of increasing the multi terms on solute concentration. From the figure one can easily observe that on increasing the number of multi-terms, it decelerates the diffusion in a heterogeneous medium. Fig. 6.2 is drawn to observe the effect on diffusion in heterogeneous medium on increasing the value of dispersion coefficient, as one can easily observe that on increasing in the value of dispersion coefficient, it decelerates the diffusion. Fig. 6.3 is drawn to observe the effect on diffusion due to variable-order spatial derivative. It is observed that spatial order $\beta(x,t) = 1.5$ and $\beta(x,t) = 1.5+0.5\ sin(0.5\Pi x)$ have the similar shape and $\beta(x,t) = 1.5 + 0.5\ sin(0.5\Pi x)$ decelerates the diffusion. It is also observed that for $\beta(x,t) = 2.0$ and $\beta(x,t) = 2 - 0.5\ sin(0.5\pi x)$ have the similar shape though for $\beta(x,t) = 2 - 0.5\ sin(0.5\pi x)$ the diffusion is accelerated. It is clearly seen from Fig. 6.4 that the nature of the solute profile for some fixed parameters are positively skewed for $q = 1$, symmetric for $q = 0$ and negatively skewed for $q = -1$. Fig. 6.5 is drawn to observe the effect on solute concentration due to change in average plume velocity and it is very much clear that on increase in the average plume velocity decelerates the diffusion. Fig. 6.6 is drawn to observe the effect on diffusion due to reaction term, i.e., for non-conservative ($k \neq 0$) and conservative ($k = 0$) systems. It is seen that sink and source terms respectively decelerate and accelerate the diffusion process as compared to conservative system ($k = 0$).

6.6 Conclusion

In this chapter, a numerical method has been developed to solve the variable-order fractional reaction-advection-diffusion equation in heterogeneous medium and also to show that the developed method is much more efficient than the previously existing methods. The salient feature of this scientific contribution is the discussion of the effect on diffusion in the heterogeneous medium due to the presence of various parameters present in the considered model. The beauty of the contribution is the graphical presentations of deceleration and acceleration of the diffusion process due to the advection and reaction terms.