Chapter 5

Numerical Solution of Nonlinear Diffusion Equation by Using Non-Standard / Standard Finite Difference and Fibonacci Collocation Methods

5.1 Introduction

Nowadays fractional calculus is getting attention due to its fascinating applications in modeling of a variety of scientific and engineering fields, such as plasma turbulence model [171], heat conduction [38], optimal control [172], porous media [173] etc. Due to its great applications in various fields it attracts the attention of many researchers while solving the various fractional order differential models. Ali Akgül and Mahmut Modanli [174] have used Crank-Nicholson difference method to solve third order fractional differential equation. Ali Akgül and Esra Karatas [175] developed a method to solve generalized fractional derivative differential equations. There are several articles found during literature survey on fractional differentiation [176, 177, 178, 179, 180]. The fractional type diffusion in a specific type of the field named as porous media has a lot of applications [181, 182]. Transport of solute through porous medium is of greater interest to the scientists including them who know the importance of ground water flow and variety of tertiary oil recovery process. Generally the Fickian diffusive process [110] is considered as the transport of solute through porous medium. But in the recent years many models have been developed by researches to describe the transport of solute through porous medium. Alamri et al. [183] have given a model to describe Poiseuille flow of nanofluid through porous medium with slip. Rusinque and Brenner [184] have given model to describe the mass transport in porous media at the micro-and nanoscale. All these transports of solute through porous medium cause the diffusion in flow medium. Many researchers have solved various types of diffusion equations with different methods, e.g., Cartalade et al.[185] developed multiple-Relaxation-Time Lattice Boltzmann scheme to solve reaction-diffusion equations. Xiao et al.[186] solved the reaction-diffusion equations on implicit surfaces with lifted local Galerkin method. Yuste et al. [187] used the finite difference method with non-uniform time steps to solve the diffusion equation, Atangana [188, 189] gave the analytical solution of the groundwater flow equation and gave the stability and convergence of time-fractional variable order telegraph equation. Pandey et al.[190] have given the approximate solution of the coupled Burger fractional diffusion equation. During modeling of any physical phenomenon, non-linearity of the system plays an important role, e.g., Fitzhugh-Nagumo(F-N) equation, Burger equation, Fisher equation, etc. In this chapter, the following fractional-order extended Burger-Fisher- Fitzhugh-Nagumo has been considered.

$$\frac{\partial u(x,t)}{\partial t} = r(x)\left({}_0^C D_x^{\beta} u(x,t)\right) - \lambda u^q \frac{\partial u(x,t)}{\partial x} + ku(x,t)\{1 - u^q(x,t)\}\{au(x,t) - \rho\} + f(x,t), \tag{5.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad 0 < x < L, \tag{5.2}$$

and the boundary conditions

$$u(0,t) = g_1(t), \;\; u(L,t) = g_2(t), \;\; 0 < t \le T, \tag{5.3}$$

where $r(x)$ is the diffusion coefficient, λu^q is advection coefficient representing the velocity of the flow medium, k is reaction term and $f(x,t)$ represents the forced term which accelerates / decelerates the diffusion process. If $k = 0$, the system is called conservative system and if $k \neq 0$, the system is called non-conservative. If $k < 0$ is called the sink term while $k > 0$ is source term.

Taking $a = 0$ and $\rho = -1$, the equation (5.1) becomes the Burger-Fisher equation, which has been solved by many researchers for the case of integer order system. Javidi et al.

[191] have solved the Burger-Fisher equation with spectral collocation method. Zhu et el.[192] have solved the equation with the help of cubic B-spline. Rashidi et al. [193] have solved with homotopy perturbation. Burger-Fisher equation arises in field of financial mathematics, gas dynamics and traffic flow. This equation shows the interaction between reaction mechanics, convection effect and diffusion transport. Taking $a = 1, \lambda, k = 1$ and $q = 1$, the resultant equation is reduced to the Fitzhugh-Nagumo(F-N) equation. This equation has its physical importance like, it is employed to describe the transmission of nerve impulses [194, 195]. Many researches have solved the F-N equation and showed its applications in various fields. For example, Li et al. [196] gave the exact solution of the F-N equation. Shih et al. [197] investigated and showed its application in population and circuit theory, Abbasbandy et al. [198] have solved the F-N equation with homotopy method.

In this chapter, the non-standard finite difference(NSFD) and collocation method are used to solve the mathematical model (5.1). Mickens [199] has shown that any ordinary differential equation with its exact solution can be matched exactly by difference equation with sampled solution. He usually modifies the denominators according to the expected form of solution. NSFD improves the standard finite difference scheme with higher accuracy and efficiency. Many models have been solved with NSFD scheme viz., Mickens and Ronald [200] have developed NSFD scheme for Lotka-Volterra system. Buckmire and Ron [201] have used Mickens's finite difference scheme to find the numerical solution of the cylindrical Bratu-Gelfand problem. Pandey [202] has used NSFD scheme to solve linear Fredholm integro-differential equations of second order.

The organization of the chapter is given as follows. Section 5.2 consists of a brief introduction about no-standard finite difference method. In section 5.3, the formula has been derived to obtain the fractional-order derivative of any function in terms of the Fibonacci polynomial. To check the accuracy of the formula, it has been applied to obtain a fractional-order derivative of x^2 and compared with the existing results. Furthermore, the non-standard finite difference scheme has been developed and discussed the method to solve one-dimensional fraction order advection reaction-advection-diffusion equation. In section 5.4, the stability of the method is shown by applying it on the linear and integer order of the considered model. In section 5.5, the proposed method is applied to three existing examples having exact solutions to obtain the accuracy of the method. The section also contains the endeavor of showing the higher accuracy of the proposed method compared to other existing methods through maximum error analysis. Solutions of the considered model under prescribed initial and boundary conditions and its discussions are presented in section 5.6. The overall conclusion of the work is given in section 5.7.

5.2 Preliminaries

5.2.1 NSFD method

A NSFD scheme is used through discretization of the discrete model of differential equation which can be constructed by many rules [203, 199, 204].

In general NSFD scheme of first-order derivative takes the form

$$\frac{du}{dt} = \frac{u_{k+1} - \psi(h)u_k}{\phi(h)},$$

where $\psi(h)$ and $\phi(h)$ depend on $h = \delta t$ with

$$\psi(h) = 1 + o(h) \;\;\&\;\; \phi(h) = h + o(h^2).$$

The functions $\psi(h)$ and $\phi(h)$ may depend on various parameters appeared in differential equations. Moreover $\phi(h)$ is a continuous function satisfying $0 < \phi(h) < 1$, $h \to 0$. There are certain examples of the $\phi(h)$ satisfying these conditions which are given as [205]

$$\phi(h) = h, \;\; \phi(h) = sinh(h), \;\; \phi(h) = exp(h) - 1, \;\; \text{etc.}$$

For the best choices of $\phi(h)$ and $\psi(h)$ and to know more about nonstandard finite difference method see [203, 199, 204].

A square integrable function $f(x)$ in [0,1] can be expressed in terms of Fibonacci polynomial as [45]

$$f(x) = \textstyle\sum_{k=1}^{\infty} c_k F_k \;,$$

where

$$c_k = \sum_{j=0}^{\infty} \frac{k(-1)^j f^{(2j+k-1)}(0)}{j!(j+k)!}. \tag{5.4}$$

In general above series can be approximated by finite sum of *(n+1)*-terms of Fibonacci Polynomial as

$$f_n(x) = \sum_{k=1}^{n+1} c_k F_k, \tag{5.5}$$

where c_k's are the set of unknowns which are to be determined.

5.3 NSFD Scheme Construction

Theorem 1. Suppose $f_n(x)$ is approximated in terms of Fibonacci polynomial as in equation (5.5). Assume $\beta > 0$, then the fractional order derivative of the function $f_n(x)$ is defined as

$${}_0^C D_x^{\beta}[f_n(x)] = \sum_{k=\lceil\beta\rceil+1}^{n+1} \sum_{\substack{i=\lceil\beta\rceil \\ (i+k)=odd}}^{k} c_k \frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!\Gamma(i+1-\beta)} x^{i-\beta}. \tag{5.6}$$

Proof. As the Caputo fractional operator is linear given in 9^{th} property of subsection 1.3.4, therefore

$${}_0^C D_x^{\beta}[f_n(x)] = \sum_{k=1}^{n+1} c_k \left({}_0^C D_x^{\beta}[F_k(x)]\right). \tag{5.7}$$

Now from equation (1.15), we have

$${}_0^C D_x^{\beta} F_k(x) = \sum_{\substack{i=1 \\ (i+k)=odd}}^{k} \frac{(\frac{k+i-1}{2})!}{i!(\frac{k-i-1}{2})!} \left({}_0^C D_x^{\beta} x^i\right).$$

using sixth property of sub-section 1.3.4 of Caputo fractional operator, we get

$${}_0^C D_x^{\beta}[F_k(x)] = 0, \quad i = 1, 2, 3..., \lceil\beta\rceil, \quad \lceil\beta\rceil > 0, \tag{5.8}$$

$${}_0^C D_x^{\beta}[F_k(x)] = \sum_{\substack{i=\lceil\beta\rceil \\ (i+k)=odd}}^{k} \frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!\Gamma(i+1-\beta)} x^{i-\beta}. \tag{5.9}$$

Now using equations (5.8) and (5.9) in equation (5.7), we get

$${}_0^C D_x^{\beta}[f_n(x)] = \sum_{k=\lceil\beta\rceil+1}^{n+1} \sum_{\substack{i=\lceil\beta\rceil \\ (i+k)=odd}}^{k} c_k \frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!\Gamma(i+1-\beta)} x^{i-\beta}.$$

Let the field variable of the considered nonlinear fractional order reaction diffusion equation (5.1) with the prescribed initial and boundary conditions (5.2) and (5.3) are approximated as

$$u_n(x,t) = \sum_{k=1}^{n+1} c_k(t) F_k(x).$$

On discretizing above equation at time level m, we have

$$u_n(x,t_m) = \sum_{k=1}^{n+1} c_k^m F_k(x). \tag{5.10}$$

Now the equation (5.1) with the initial and boundary conditions (5.2) and (5.3) at the time level m are expressed as

$$\begin{aligned}\frac{\partial u(x,t_m)}{\partial t} = r(x)\big({}_0^C D_x^{\beta} u(x,t_m)\big) - \lambda u^q(x,t_m)\frac{\partial u(x,t_m)}{\partial x}\\ + ku(x,t_m)\{1-u^q(x,t_m)\}\{au(x,t_m)-\rho\} + f(x,t).\end{aligned} \tag{5.11}$$

Since it is quite difficult to solve a system of nonlinear equations, the nonlinear terms of the equation (5.11) are linearised in the following forms

$$\begin{aligned}u^q(x,t_m)u_x(x,t_m) = qu^{q-1}(x,t_{m-1})u_x(x,t_{m-1})u(x,t_m) + u^q(x,t_{m-1})u_x(x,t_m)\\ - qu_x(x,t_{m-1})u^q(x,t_{m-1}),\end{aligned} \tag{5.12}$$

$$\begin{aligned}u(x,t_m)[1-u^q(x,t_m)][au(x,t_m)-\rho] &= u(x,t_m)[1-u^q(x,t_{m-1})][au(x,t_{m-1})-\rho]\\ &- qu(x,t_{m-1})u^{q-1}(x,t_{m-1})[au(x,t_{m-1})-\rho]u(x,t_m)\\ &+ au(x,t_{m-1})[1-u^q(x,t_{m-1})]u(x,t_m) + qu^{q+1}(x,t_{m-1})[au(x,t_{m-1})-\rho]\\ &- au^2(x,t_{n-1})[1-u^q(x,t_{m-1})].\end{aligned} \tag{5.13}$$

Now using equations (5.12) and (5.13) in equation (5.11), we get

$$\frac{\partial u(x,t_m)}{\partial t} = r(x)\big({}_0^C D_x^{\beta} u(x,t_m)\big) - R^m(x), \tag{5.14}$$

where

$$\begin{aligned}R^m(x) &= \lambda\{qu^{q-1}(x,t_{m-1})u_x(x,t_{m-1})u(x,t_m) \\ &+ u^q(x,t_{m-1})u_x(x,t_m) - qu_x(x,t_{m-1})u^q(x,t_{m-1})\} \\ &+ k\{u(x,t_m)[1-u^q(x,t_{m-1})][au(x,t_{m-1}) \\ &- \rho] - qu(x,t_{m-1})u^{q-1}(x,t_{m-1})[au(x,t_{m-1})-\rho]u(x,t_m) \\ &+ au(x,t_{m-1})[1-u^q(x,t_{m-1})]u(x,t_m) + qu^{q+1}(x,t_{m-1})[au(x,t_{m-1})-\rho] \\ &- au^2(x,t_{n-1})[1-u^q(x,t_{m-1})]\} + f(x,t_n).\end{aligned}$$

Again using equations (5.6) and (5.10) in equation (5.14), we obtain

$$\sum_{k=1}^{n+1}\frac{dc_k^m}{dt}F_k(x) = r(x)\sum_{k=\lceil\beta\rceil}^{n+1}\sum_{\substack{i=\lceil\beta\rceil \\ (i+k)=odd}}^{k} c_k^m\frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!\Gamma(i+1-\alpha)}x^{i-\beta} - R^m(x),$$

$$\sum_{k=1}^{n=1}\frac{c_k^m - c_k^{m-1}}{\phi(h)}F_k(x) = r(x)\sum_{k=\lceil\beta\rceil}^{n+1}\sum_{\substack{i=\lceil\beta\rceil \\ (i+k)=odd}}^{k} c_k^m\frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!\Gamma(i+1-\beta)}x^{i-\beta} - R^m(x). \quad (5.15)$$

It is necessary to calculate the values $c_k^{\prime}s$ at the initial time level to obtain the values of $c_k^{\prime}s$ at further time levels. To calculate these values we will use the initial and boundary conditions as

$$\sum_{k=1}^{n+1}c_k^0F_k(x) = u_0(x), \quad (5.16)$$

$$\sum_{k=1}^{n+1}c_k^mF_k(0) = g_1(t_m),\ \sum_{k=1}^{n+1}c_k^mF_k(1) = g_2(t_m). \quad (5.17)$$

From equation (5.15), it is found that at each time level, $(n+1)$ unknowns have to be obtained. In order to achieve the unknowns we need $(n+1)$ equations. So the equation (5.15) will be collocated at $(n-1)$ collocation points $\frac{i}{n+1}$, $i = 1,2,3,..,n-1$, together with boundary conditions (5.17). These linear equations can be solved easily and the desired approximate solution of fractional order nonlinear diffusion equation can be computed.

Simple test: Now to test the derivative obtained by Theorem 1, consider $f(x) = x^2$ for $n = 3$ and $\beta = 1.4$. Using 6^{th} property of subsection 1.3.4 of Caputo fractional operator, we obtain

$$D^{1.4}x^2 = 2.23835x^{0.6}.$$

Now derivative obtained by Theorem 1 is

$$D^{1.4}[x^2] = \sum_{k=3}^{4} \sum_{\substack{i=2 \\ (i+k)=odd}}^{k} c_k \frac{(\frac{k+i-1}{2})!}{(\frac{k-i-1}{2})!\Gamma(i-0.4)} x^{i-1.4}.$$

On putting the values of c_3 and c_4 from equation (5.4), we get $D^{1.4}x^2 = 2.23835x^{0.6}$.

5.4 Stability Analysis

For simplicity, it will be shown that the developed scheme (5.14) is unconditionally stable for homogeneous Dirichlet boundary conditions for integer order spatial derivative. Consider the nonlinear terms present in considered model (5.1) are locally constants and as the considered domain of the model is finite so taking maximum value of $r(x)$, we get

$$\frac{\partial u}{\partial t} = k_1 \frac{\partial^2 u}{\partial x^2} - k_2 \frac{\partial u}{\partial x} + k_3 u(x,t) + f(x,t),$$

which can be written in the discretized form for m as

$$\frac{u_n^{m+1} - u_n^m}{h} = k_1 (u_{xx})_n^{m+1} - k_2 (u_x)_n^{m+1} + k_3 u_n^{m+1} + f^{m+1}. \tag{5.18}$$

Now multiplying the above equation by $u(x_j)_n^{m+1}$ and taking summation over each node of x_j, we get

$$\begin{aligned} \sum_{j=0}^{j=N} \{u(x_j)_n^{m+1} - u(x_j)_n^m\} u(x_j)_n^{m+1} = & \sum_{j=0}^{j=N} \{hk_1 (u_{xx}(x_j))_n^{m+1} - hk_2 (u_x(x_j))_n^{m+1} \\ & + hk_3 u(x_j)_n^{m+1} + hf^{m+1}(x_j)\} u(x_j)_n^{m+1} \end{aligned}$$

$$\begin{aligned} \int_0^1 \{u(x_j)_n^{m+1} - u(x_j)_n^m\} u(x_j)_n^{m+1} dx = & \, hk_1 \int_0^1 (u_{xx}(x))_n^{m+1} u(x)_n^{m+1} dx \\ & - hk_2 \int_0^1 (u_x(x))_n^{m+1} u(x)_n^{m+1} dx + hk_3 \int_0^1 u(x)_n^{m+1} u(x)_n^{m+1} dx \\ & + h \int_0^1 f^{m+1}(x) u(x)_n^{m+1} dx. \end{aligned} \tag{5.19}$$

Now

$$\begin{aligned}\int_0^1 (u_{xx}(x))_n^{m+1} u(x)_n^{m+1} dx &= (u_x(x))_n^{m+1} u(x)_n^{m+1}\Big|_{x=0}^{x=1} - \int_0^1 \{(u_x(x)_n^{m+1})\}^2 dx \\ &= -\int_0^1 \{(u_x(x)_n^{m+1})\}^2 dx \leq 0.\end{aligned} \tag{5.20}$$

and

$$\int_0^1 (u_x(x))_n^{m+1} u(x)_n^{m+1} = \frac{1}{2}\{u(x)_n^{m+1}\}^2\Big|_{x=0}^{x=1} = 0. \tag{5.21}$$

Now using equations (5.20) and (5.21) in equation (5.19), we get

$$\begin{aligned}\Big|\langle u_n^{m+1}, u_n^{m+1}\rangle\Big| &= \Big|\langle u_n^{m+1}, u_n^{m}\rangle - h\int_0^1 \{(u_x)_n^{m+1}\}^2 dx + hk_3\langle u_n^{m+1}, u_n^{m+1}\rangle + h\langle f^{m+1}, u_n^{m+1}\rangle\Big|, \\ &\leq \Big|\langle u_n^{m+1}, u_n^{m}\rangle\Big| + hk_3\Big|\langle u_n^{m+1}, u_n^{m+1}\rangle\Big| + h\Big|\langle f^{m+1}, u_n^{m+1}\rangle\Big|,\end{aligned}$$

$$||u_n^{m+1}|| \leq \frac{1}{1-hk_3}\Big\{||u_n^m|| + h||f^{m+1}||\Big\}. \tag{5.22}$$

Now by simple iteration on equation (5.22), one can easily get

$$||u_n^k|| \leq \frac{1}{(1-hk_3)^k}||u_n^0|| + \sum_{j=1}^{j=k}\frac{h}{(1-hk_3)^{k+1-j}}||f^k||. \tag{5.23}$$

Above equation (5.23) clearly shows that the proposed scheme is unconditionally stable.

5.5 Illustrative Examples

In this section, particular cases of the considered mathematical model has been solved by using the proposed nonstandard finite difference Fibonacci collocation method. Next the obtained numerical results have been compared with the exact analytical solutions. The maximum error is defined by

$$L_\infty = |U(x,t_m) - u_n(x,t_m)|_\infty = \underset{0\leq x\leq 1}{Max}|U(x,t_m) - u_n(x,t_m)|,$$

where the $U(x,t)$ is the exact solution and $u_n(x,t)$ is the approximate solution.

Example 1. Consider the space time fractional order diffusion problem

$$\frac{\partial u(x,t)}{\partial t} = r(x)\big({}_0^C D_x^{1.8} u(x,t)\big) + q(x,t), \;\; 0 \leq x \leq 1, \; t \geq 0, \tag{5.24}$$

which can be obtained from the considered model (5.1) by taking $\lambda = 0$ and $k = 0$. For $r(x) = \Gamma(1.2)x^{1.8}$ and $q(x,t) = 3x^2(2x-1)e^{-t}$, the equation (5.24) with initial condition $u(x,0) = x^2(1-x)$ and zero Drichlet boundary conditions has the exact solution $u(x,t) = x^2(1-x)e^{-t}$. Now the problem is solved by using the proposed numerical method, discussed in section 3 for $n = 3$ and $\phi(h) = exp(h) - 1$. Comparison of the errors obtained by discussed method and the existing results obtained in [206, 207] are displayed in Tables 5.1-5.3 for different values of t. Using equation (5.10) in equation (5.24), we have

$$\sum_{k=1}^{4} \frac{dc_k^m}{dt} F_k(x) = r(x)\{2.17825x^{0.2}c_3^m + 5.44562x^{1.2}c_4^m\} + q(x,t). \tag{5.25}$$

with the initial condition

$$\sum_{k=1}^{4} c_k^0 F_k(x) = x^2(1-x) \tag{5.26}$$

and the boundary conditions

$$\begin{aligned} &c_1^m + c_3^m = 0, \\ &c_1^m + c_2^m + c_3^m + 3c_4^m = 0. \end{aligned} \tag{5.27}$$

Equation (5.25) has four unknowns at each time levels, which is to be determined. Now on collocating equations (5.25) and (5.26) at points $x = 1/4, 1/2$ we will have two equations at zero and any time level m. Hence diffusion equation (5.24) will be reduced in to set of differential equations. Now using nonstandard finite difference method and with the help of boundary conditions (5.27), we have the required values of c_k's at each time level. Hence from equation (5.25), we obtain

$$\begin{aligned} &\sum_{k=1}^{4} \frac{c_k^m - c_k^{m-1}}{exp(h)-1} F_k(1/4) = r(1/4)\{1.6508c_3^m + 1.03175c_4^m\} + q(1/4, mh), \\ &\sum_{k=1}^{4} \frac{c_k^m - c_k^{m-1}}{exp(h)-1} F_k(1/2) = r(1/2)\{1.89628c_3^m + 2.37034c_4^m\} + q(1/2, mh), \end{aligned} \tag{5.28}$$

and from initial condition, we get

$$\begin{aligned} &c_1^0 + 0.25c_2^0 + 1.0625c_3^0 + 0.5156c_4^0 = 0.0468, \\ &c_1^0 + 0.5c_2^0 + 1.125c_4^0 = 0.125. \end{aligned} \tag{5.29}$$

It is clear from above equations (5.28) and (5.29) that two equations are there at zero time level and at time level m. On solving equations for c_k's at desired time level and putting it in to the equation (5.10), we get the required solution.

TABLE 5.1: Comparison of absolute errors among our method and the methods given in [206, 207] at $t = 1$ for Example 1.

x	[206] with $n = 3$	[207] with n=3	Our method with $n = 3$
0	0	4.16e-17	0.
0.1	5.46e-6	1.94e-8	1.72085e-15
0.2	8.51e-6	2.69e-8	5.10703e-15
0.3	9.60e-6	2.53e-8	9.35363e-15
0.4	9.18e-6	1.74e-8	1.38223e-14
0.5	7.69e–6	6.18e-9	1.7597e-14
0.6	5.60e-6	6.56e-9	1.99216e-14
0.7	3.33e-6	1.49e-9	2.01783e-14
0.8	1.34e-6	1.90e-8	1.7597e-14
0.9	8.39e-6	1.50e-8	1.11022e-14
1.0	0	0	2.22045e-16

TABLE 5.2: Comparison of absolute errors among our method and the methods given in [206, 207] at $t = 2$ for Example 1.

x	[206] with $n = 3$	[207] with $n = 3$	Our method with $n = 3$
0	0	8.70e-19	0.
0.1	3.336e-6	7.92e-8	2.65066e-15
0.2	5.65e-6	1.23e-8	5.50948e-15
0.3	7.05e-6	1.39e-6	8.27116e-15
0.4	7.64e-6	1.32e-6	1.06581e-14
0.5	7.52e–6	1.11e-6	1.23096e-14
0.6	6.80e-6	8.07e-7	1.29896e-14
0.7	5.59e-6	4.78e-7	1.24206e-14
0.8	3.98e-6	1.90e-7	1.02696e-14
0.9	2.08e-6	9.32e-9	6.27276e-15
1.0	0	8.70e-19	2.77556e-17

TABLE 5.3: Comparison of absolute errors among our method and the methods given in [206, 207] at $t = 10$ for Example 1.

x	[206] for $n = 7$	[207] for $n = 3$	Our method for $n = 3$
0	0	1.05e-21	0.
0.2	2.34e-8	2.64e-10	6.84403e-19
0.4	4.78e-9	2.85e-10	1.83806e-18
0.6	7.30e-9	1.73e-10	2.63935e-18
0.8	2.84e-8	4.08e-11	2.31748e-18
1.0	0	1.05e-21	1.35525e-20

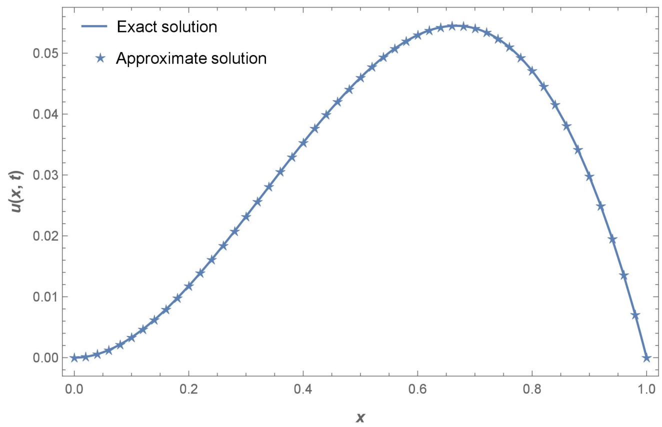

FIGURE 5.1: The behavior of exact solution and approximate solution at $n = 3$ for Example 1.

From Tables 5.1-5.3, it is clear that our proposed method is more accurate and efficient as compared to the existing methods. Fig 5.1 depicts the exactness of the numerical solution of the considered example obtained by using our proposed method.

Example 2. Consider the following fractional diffusion equation as

$$\frac{\partial u(x,t)}{\partial t} = r(x)\left({}_0^C D_x^{1.8} u(x,t)\right) + q(x,t) \;\; 0 \leq x \leq 1, \tag{5.30}$$

which can be obtained from considered model (5.1) by taking $\lambda = 0$ and $k = 0$. For $r(x) = \frac{\Gamma(2.2)}{6} x^{2.8}$ and $q(x,t) = -(1+x)x^3 e^{-t}$ with initial condition $u(x,0) = x^3$ and Drichlet boundary conditions $u(0,t) = 0, \;\; u(1,t) = e^{-t}$, the equation (5.30) has the exact solution $u(x,t) = x^3 e^{-t}$. This problem has been solved for $n = 3$ and $\phi(h) = exp(h) - 1$ with the proposed numerical scheme and compare the numerical results obtained by previous methods [208, 207, 206] through finding maximum absolute error which are given in Table 5.4.

TABLE 5.4: Comparison of maximum errors among different methods and our proposed method for Example 2.

Max error [208]	Max error [208]	Max error [206]	Max error [207]	Max error our method
6.84895e-4	2.82750e-5	8.3830e-10	1.3772e-9	4.37428e-14

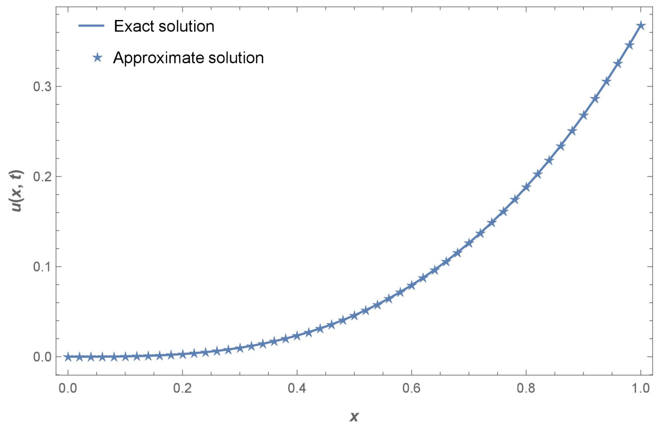

FIGURE 5.2: The behavior of exact solution and approximate solution at $n = 3$ for Example 2.

Table 5.4 shows that the maximum absolute errors obtained between the exact results and approximate result obtained by proposed numerical method is more accurate compared to errors obtained by using the other existing numerical methods [206, 207, 208]. The exactness of the given example for proposed method is displayed through Fig. 5.2.

Example 3. Taking $a = 0, \rho = -1$, the diffusion coefficient $r(x) = 1$ and the forced function $f(x,t) = 0$, the considered model (5.1) is reduced to

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - \alpha u^q(x,t)\frac{\partial u(x,t)}{\partial x} + ku(x,t)\{1 - u^q(x,t)\}. \tag{5.31}$$

The initial condition and drichlet boundary conditions can be obtained through the exact solution of the problem (5.31), which is given bellow [192, 209]

$$u(x,t) = \left(\frac{1}{2} + \frac{1}{2}tanh\left[\frac{-\alpha q}{2(q+1)}\left[x - \left(\frac{\alpha}{q+1} + \frac{k(q+1)}{\alpha}\right)t\right]\right]\right)^{\frac{1}{q}}.$$

The L_∞ norm error for the considered problem (5.31) for $\lambda = 0.1$ and $k = -0.0025$ and comparison of the numerical results obtained by proposed method with the results obtained by the other existing numerical methods [209, 192], are given in Table 5.5.

Table 5.5: Maximum absolute error for $n = 5$, $\phi = h$ and $\delta\ t = 0.0001$.

	Time(t)	Proposed method	[209]	[192]
q=1	0.1	1.33227e-15	1.08646e-12	1.32396e-11
	0.2	1.33227e-15	1.24456e-13	1.78026e-11
	0.3	1.33227e-15	1:24456e-13	1.94258e-11
	0.4	1.33227e-15	1:24456e-13	2.00083e-11
	0.5	1.33227e-15	1:24456e-13	2.02158e-11
q=2	0.1	1.35758e-12	2.17542e-11	2.84700e-10
	0.2	1.91058e-12	3.02457e-11	3.87950e-10
	0.3	2.11819e-12	3.33861e-11	4.24646e-10
	0.4	2.18958e-12	3.45414e-11	4.37589e-10
	0.5	2.21734e-12	3.49593e-11	4.02050e-10
q=4	0.1	1.06114e-11	3.12324 e-11	3.99168 e-10
	0.2	1.48941e-11	4.34227 e-11	5.43802 e-10
	0.3	1.64713e-11	4.79300 e-11	5.95169 e-10
	0.4	1.70429e-11	4.95853 e-11	6.13233 e-10
	0.5	1.72410e-11	5.01746 e-11	6.19407 e-10

It is seen from the Table 5.5 that the results obtained by proposed method are much superior than the existing methods [209, 192] even for less number of spatial and temporal partitions.

5.6 Numerical Solution of Proposed ARDE and discussion

After validating the efficiency and accuracy of the proposed numerical method, an endaviour has been made to find the solution of considered problem (5.1) for different values of parameters as q, $\rho = 1, a = 1, r(x) = 1, \lambda = 1, k = 1,\ f(x,t) = 0$ under the prescribed initial and boundary conditions as

$$u(x,0) = \frac{1}{2} - \frac{1}{2}tanh\left[\frac{1}{4}\left(x\right)\right],$$
$$u(0,t) = \frac{1}{2} + \frac{1}{2}tanh\left[\frac{1}{4}\left(\frac{t}{2}\right)\right],$$
$$u(1,t) = \frac{1}{2} - \frac{1}{2}tanh\left[\frac{1}{4}\left(1-\frac{t}{2}\right)\right].$$

Figs. 5.3-5.6 represent the solute concentration profile of the considered model after solving it using our proposed method. Figs 5.3-5.5 are drawn for solute profile versus x at $t = 0.5$ for various spatial order derivatives at $q = 1, 2$ and 3. The figures show the effects on solute profiles due to increase in order of spatial derivative, i.e., from fractional order to integer order. From these three figures it is found that the behavior of solute profile is same in every case, which justify the correctness of the solution of the model obtained by the proposed method. Fig 5.6 is drawn for different values of q for $\beta = 2$ at time $t = 0.5$. From Fig 5.6, one can easily observe that with the increase in the non-linearity in the model, the curvature of the solute profile decreases.

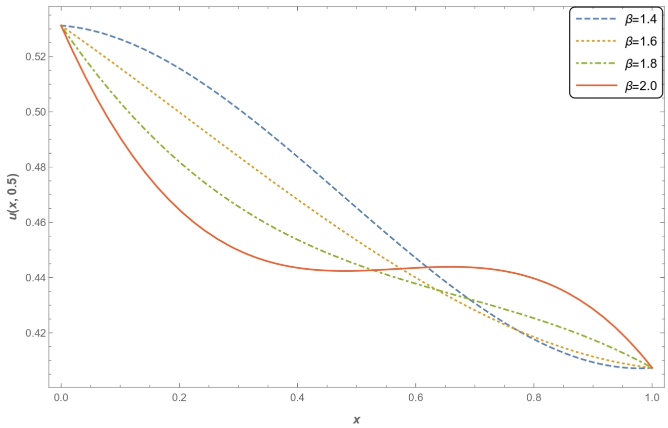

FIGURE 5.3: Plots of solute profile vs. x for various values of β at $q = 1$ at $t = 0.5$.

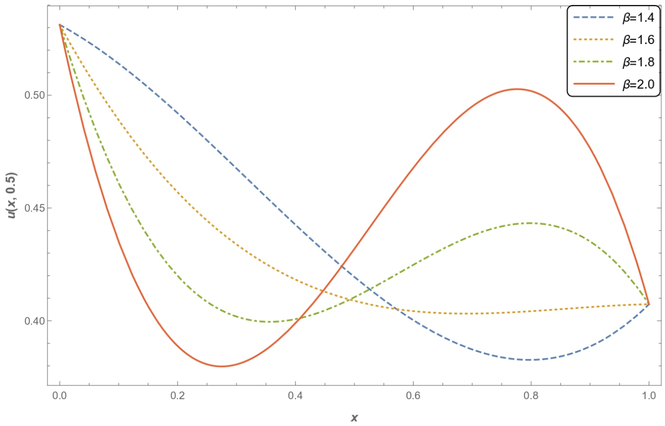

FIGURE 5.4: Plots of solute profile vs. x for various values of β at $q = 2$ at $t = 0.5$.

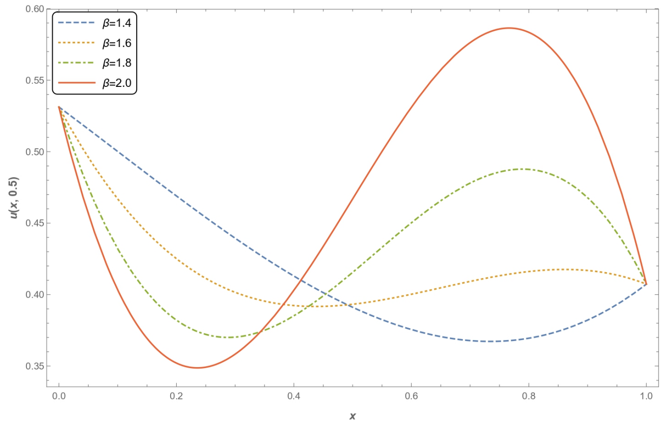

FIGURE 5.5: Plots of solute profile vs. x for various values of β at $q = 4$ at $t = 0.5$.

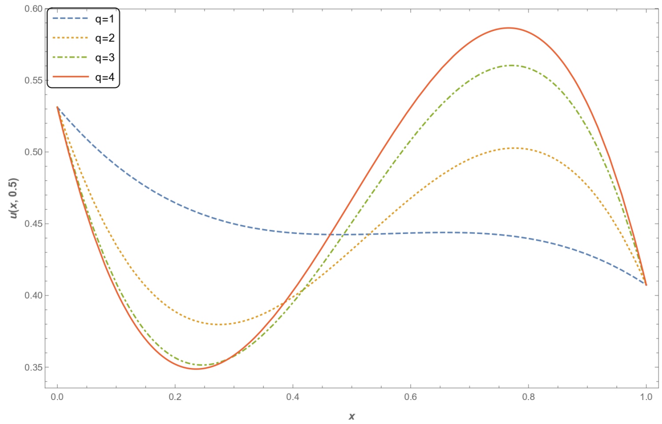

FIGURE 5.6: Variations of solute concentration for different values of q at $\beta = 2$, $t = 0.5$.

5.7 Conclusion

In the present chapter, an effective method has been developed to obtain the approximate solution of fractional order nonlinear reaction-advection-diffusion equation. While developing the method, the model is approximated firstly with Fibonacci polynomial to reduce the considered diffusion equation to the set of ordinary differential equations, which have been solved by using the finite difference scheme. To validate the higher efficiency and accuracy of the proposed algorithm a comparative study through error analysis between the numerical results and analytical results with other existing methods have been done for two linear and one nonlinear cases of the considered model. The most important part of the study is the graphical exhibition of the effect of the solute profile due to increase in the order of the non-linearity in both advection and reaction terms.