Chapter 6

Conclusion and Future Work

This chapter concludes the thesis and presents some future possibilities in the direction of the research work presented in this thesis.

6.1 Conclusion

Numerical simulation of the physical model is necessary to get the approximated solutions when the corresponding differential equation is complex. The complexity of FPDEs increases with the use of newly proposed fractional derivatives. This thesis presented approximations of the Atangana-Baleanu derivative of Caputo type and generalized derivatives containing scale and weight functions. Further, some numerical methods were developed to solve linear and non-linear FPDEs formed by using these derivatives. To maintain simplicity and accuracy, finite difference method was used. For achieving a high ordered order of convergence, Taylor series expansion was used, followed by FDM. The stability and convergence of numerical schemes are derived. Theoretical statements were analyzed through various examples.

Chapter 1 introduced the thesis with basic definitions. The past and recent works on FPDEs were presented. The motivation behind this research work and problem statement of the thesis were explained.

Chapter 2 developed numerical approximations for the Atangana-Baleanu Caputo (ABC) derivative by two methods. These approximations were then used to solve the FADE, whose time derivative is the ABC derivative. The first method used FDM, and in the second, the Taylor expansion series of the function was used, followed by FDM. The stability and convergence of the schemes were established numerically. The convergence orders were estimated as $O(\tau^2 + h^2)$ and $O(\tau^3 + h^2)$ for Method 1 and Method 2, respectively. Numerical examples validated the statements.

Chapter 3 presented a numerical technique to solve a non-linear FPDE, fractional Burgers equation. Here time-derivative was chosen as Atangana-Baleanu fractional derivative. FDM was used to lessen the complexity of calculations. The stability of the scheme was proved, and the order of convergence was estimated numerically as $O(\tau + h^2)$. Some examples were considered to perform numerical simulations.

Chapter 4 approximated the generalized derivative containing scale function and weight functions of Caputo type of order $\alpha \in (0, 1)$. Taylor's expansion was used for a high order of convergence. Further, a numerical scheme was developed for solving the GFADE, which was formed using the generalized Caputo derivative in respect of time. The effects of different parameters were shown on the diffusion process of the equation with the help of some examples.

Chapter 5 discussed another numerical scheme using FDM to solve GTFTE, whose time derivatives are generalized fractional derivatives of Caputo type. This chapter presented the difference in GTFTE's solution when the weight and scale functions in the generalized fractional derivative are changed. The convergence and the stability of the numerical scheme are studied.

6.2 Future Work

Researchers have been working in the field of fractional calculus, especially on its operators, to provide accurate definitions likewise in classical calculus. This thesis presented the approximations of some new operators. So, the method is restricted to finite difference method only for the simplicity of the calculation and accuracy of results. But there are some methods like FEM, non-uniform mess methods, meshless methods, etc. that can provide better results. The approximations using such methods are expected in the near future.

Most of this thesis's work is confined to the interval $(0, 1/2]$ to provide a higher order of convergence. To extend this interval, more studies on the theories and methods are required. This work forms a base for future research with an extended domain. Generalized operators contain scale function and weight function, convergence order of FPDEs containing generalized derivatives, somehow, depends on these functions. The nature of this dependence is still to be found out.

This work also provides some applications for the approximations of real-world problems. Most of the examples were taken from literature surveys with smooth data to provide a comparison. In upcoming research, it is expected to work on some non-smooth data and some more complicated models.