Chapter 5

Finite Difference Scheme for a Generalized Fractional Time-Derivatives Telegraph Equation

This chapter begins with the short introduction in Section 5.1. Section 5.2 describes the finite difference scheme and its computational algorithm for the considered model. The stability and convergence of the proposed numerical scheme are discussed in Section 5.3. Section 5.4 provides some examples to check the proficiency and simplicity of the proposed scheme with different parameters. The changes in the numerical solutions of the fractional telegraph equation concerning the changes in scale and weight functions are discussed in Subsection 5.4.1 and Subsection 5.4.2, respectively. Section 5.5 concludes the chapter.

5.1 Introduction

A finite difference scheme (FDS) is presented for solving the generalized fractional advection-diffusion equation in the previous chapter. This chapter studies generalized time-fractional telegraph equation (GTFTE). An FDS is developed for generalized fractional derivative (GFD) to solve GTFTE numerically. The effects of scale and weight functions on GTFTE are also observed.

Recalling the new definition of GFD proposed by Agarwal [33, 34], which contains a scale function and a weight function. Scale function can be used to change the considered domain. Weight function is used to extend the kernel in operators, which is helpful to make the models more flexible. Hence, by choosing the different types of scale and weight functions, different generalized fractional derivatives and integrals can be obtained, e.g., these operators turn into Riemann-Liouville and Caputo operators for scale function $z(t) = t$ and weight function $w(t) = 1$. The applications of GFD have been studied on fractional diffusion equation, fractional advection-diffusion equations, and fractional Burgers equation [162, 106, 161].

This chapter presents the FDS of the following PDE called fractional telegraph equation (FTE),

$$\frac{{}^*\partial^{2\alpha} u(x,t)}{{}^*\partial t^{2\alpha}} + 2\lambda \frac{{}^*\partial^{\alpha} u(x,t)}{{}^*\partial t^{\alpha}} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \tag{5.1}$$

where

$$\frac{{}^*\partial^{\alpha} u(x,t)}{{}^*\partial t^{\alpha}} = {}^C_a D^{\alpha}_{t,(z,w)} u(x,t),$$

$$\frac{{}^*\partial^{\alpha} u(x,t)}{{}^*\partial t^{\alpha}} = {}^C_a D^{2\alpha}_{t,(z,w)} u(x,t),$$

$\alpha \in (0, 1/2)$, and c, λ are real numbers. The above Eq. (5.1) is confined to a bounded domain ω such that $(x,t) \in \omega = [a,b] \times [0,T]$ and under the initial condition $u(x,0) = \phi(x)$ and boundary conditions $u(a,t) = g_1(t)$ and $u(b,t) = g_2(t)$ for all $t > 0$.

5.2 Finite Difference Scheme

To get the numerical scheme for solving FTE, space-time domain $[a,b] \times [0,T]$ is divided into N and M equal sub-intervals, in a way such that $R_{\Delta x} = \{x_i : 0 \leq i \leq N\}$ is a uniform mesh of the interval $[a,b]$, where $x_i = a + i(\Delta x), i = 0, 1, \ldots, N$ with $\Delta x = (b-a)/N$, and $R_{\Delta t} = \{t_j : 0 \leq j \leq M\}$ is a uniform mesh in time direction, where $t_j = j(\Delta t), j = 0, 1, \ldots, M$, $\Delta t = T/M$ with $t_0 = 0$ is the initial time, and x_0 and x_N denote the boundary points. For simplification, write $u(x_i, t_j) = u_j^i, w(t_j) = w_j$, and $z(t_j) = z_j$. Approximate the first term of Eq. (5.1) as follows

$$\left[\frac{{}^*\partial^{2\alpha} u(x,t)}{{}^*\partial t^{2\alpha}}\right]_{(x_i,t_{j+1})} = \frac{[w(t_{j+1})]^{-1}}{\Gamma(1-2\alpha)} \int_0^{t_{j+1}} \frac{\frac{\partial}{\partial \tau}[w(\tau)u(x_i,\tau)]}{[z(t_{j+1}) - z(\tau)]^{2\alpha}} d\tau.$$

$$\approx \frac{[w(t_{j+1})]^{-1}}{\Gamma(1-2\alpha)} \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} \frac{w(t_{k+1})u(x_i,t_{k+1}) - w(t_k)u(x_i,t_k)}{(t_{k+1}-t_k)[z(t_{j+1}) - z(\tau)]^{2\alpha}} d\tau,$$

$$\approx \frac{[w_{j+1}]^{-1}}{\Gamma(2-2\alpha)} \sum_{k=0}^{j} \frac{w_{k+1}u_{k+1}^i - w_k u_k^i}{(t_{k+1}-t_k)} \left[(z_{j+1} - z_k)^{1-2\alpha} - (z_{j+1} - z_{k+1})^{1-2\alpha}\right]. \quad (5.2)$$

Similarly for second term,

$$\left[\frac{{}^*\partial^{\alpha} u(x,t)}{{}^*\partial t^{\alpha}}\right]_{(x_i,t_{j+1})} = \frac{[w(t)]^{-1}}{\Gamma(1-\alpha)} \int_0^{t_{j+1}} \frac{\frac{\partial}{\partial \tau}[w(\tau)u(x_i,\tau)]}{[z(t_{j+1}) - z(\tau)]^{\alpha}} d\tau.$$

$$\approx \frac{[w(t_{j+1})]^{-1}}{\Gamma(1-\alpha)} \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} \frac{w(t_{k+1})u(x_i, t_{k+1}) - w(t_k)u(x_i, t_k)}{(t_{k+1} - t_k)[z(t_{j+1}) - z(\tau)]^{\alpha}} d\tau,$$

$$\approx \frac{[w_{j+1}]^{-1}}{\Gamma(2-\alpha)} \sum_{k=0}^{j} \frac{w_{k+1}u_{k+1}^{i} - w_k u_k^{i}}{(t_{k+1} - t_k)} \left[(z_{j+1} - z_k)^{1-\alpha} - (z_{j+1} - z_{k+1})^{1-\alpha}\right]. \tag{5.3}$$

For second order space derivative,

$$\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right]_{(x_i, t_{j+1})} \approx \frac{u(x_{i+1}, t_{j+1}) - 2u(x_i, t_{j+1}) - u(x_{i-1}, t_{j+1})}{(\Delta x)^2}$$

$$= \frac{u_{j+1}^{i+1} - 2u_{j+1}^{i} + u_{j+1}^{i-1}}{(\Delta x)^2}. \tag{5.4}$$

Hence, considering approximations from Eq. (5.2), Eq. (5.3), and Eq. (5.4); Eq. (5.1) convert into the scheme as

$$\frac{[w_{j+1}]^{-1}}{\Gamma(2-2\alpha)} \sum_{k=0}^{j} \frac{w_{k+1}u_{k+1}^{i} - w_k u_k^{i}}{(t_{k+1} - t_k)} \left[(z_{j+1} - z_k)^{1-2\alpha} - (z_{j+1} - z_{k+1})^{1-2\alpha}\right]$$

$$+ \frac{[w_{j+1}]^{-1}}{\Gamma(2-\alpha)} \sum_{k=0}^{j} \frac{w_{k+1}u_{k+1}^{i} - w_k u_k^{i}}{(z_{k+1} - z_k)} [(z_{j+1} - z_k)^{1-\alpha} - (z_{j+1} - z_{k+1})^{1-\alpha}]$$

$$= \left[\frac{u_{j+1}^{i+1} - 2u_{j+1}^{i} - u_{j+1}^{i-1}}{(\Delta x)^2}\right] + f_{j+1}^{i}. \tag{5.5}$$

For simplicity, Eq. (5.5) can be written as

$$\mu(u_{j+1}^{i+1} - 2u_{j+1}^{i} - u_{j+1}^{i-1}) = \sum_{k=0}^{j} (a_k^j u_{k+1}^{i} - b_k^j u_k^{i}) + 2\lambda \sum_{k=0}^{j} (s_k^j u_{k+1}^{i} - v_k^j u_k^{i}) - f_{j+1}^{i}, \tag{5.6}$$

for $0 \leq j \leq M-1$, and $0 \leq i \leq N-1$, where

$$a_k^j = \frac{w_{j+1}^{-1} w_{k+1}}{\Gamma(2-2\alpha)(z_{k+1}-z_k)}[(z_{j+1}-z_k)^{1-2\alpha} - (z_{j+1}-z_{k+1})^{1-2\alpha}], \tag{5.7}$$

$$b_k^j = \frac{w_{j+1}^{-1} w_{k}}{\Gamma(2-2\alpha)(z_{k+1}-z_k)}[(z_{j+1}-z_k)^{1-2\alpha} - (z_{j+1}-z_{k+1})^{1-2\alpha}], \tag{5.8}$$

$$s_k^j = \frac{w_{j+1}^{-1} w_{k+1}}{\Gamma(2-\alpha)(z_{k+1}-z_k)}[(z_{j+1}-z_k)^{1-\alpha} - (z_{j+1}-z_{k+1})^{1-\alpha}], \tag{5.9}$$

$$v_k^j = \frac{w_{j+1}^{-1} w_{k}}{\Gamma(2-\alpha)(z_{k+1}-z_k)}[(z_{j+1}-z_k)^{1-\alpha} - (z_{j+1}-z_{k+1})^{1-\alpha}], \tag{5.10}$$

$$\mu = \frac{c^2}{\Delta x^2}. \tag{5.11}$$

For $j \geq 1$, Eq. (5.6) can be written as

$$\mu u_{j+1}^{i-1} + (-2\mu - a_j^j - 2\lambda s_j^j) u_{j+1}^i + \mu u_{j+1}^{i+1} =$$

$$- b_j^j u_j^i - 2\lambda v_j^j u_j^i + \sum_{k=0}^{j-1} (a_k^j u_{k+1}^i - b_k^j u_k^i) + 2\lambda \sum_{k=0}^{j-1} (s_k^j u_{k+1}^i - v_k^j u_k^i) - f_{j+1}^i. \tag{5.12}$$

Using $k_j^i = -2\mu - a_j^j - 2\lambda s_j^j$, Eq. (5.12) can be written in the following matrix form

$$A_{j+1} U_{j+1} = F_{j+1}, \quad 0 \leq j \leq M-1, \tag{5.13}$$

where

$$A_{j+1} = \begin{pmatrix} k_j^1 & \mu & & & & & \\ \mu & k_j^2 & \mu & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \mu & k_j^{N-2} & \mu \\ & & & & \mu & k_j^{N-1} \end{pmatrix},$$

$$U_{j+1} = [u_{j+1}^1, u_{j+1}^2, ..., u_{j+1}^i, ..., u_{j+1}^{N-1}]^T, \ u_j^0 = u_j^N = 0, \qquad (5.14)$$

$$F_{j+1} = [F_{j+1}^1, F_{j+1}^2, ..., F_{j+1}^i, ..., F_{j+1}^{N-1}]^T, \qquad (5.15)$$

and

$$F_{j+1}^i = \begin{cases} -b_j^j u_j^i - 2\lambda v_j^j u_j^i + \sum_{k=0}^{j-1}(a_k^j u_{k+1}^i - b_k^j u_k^i) + \\ +2\lambda \sum_{k=0}^{j-1}(s_k^j u_{k+1}^i - v_k^j u_k^i) - f_{j+1}^i, & 1 \le j \le M-1. \\ -b_0^0 - 2\lambda v_0^0 u_0^i, & j = 0 \end{cases} \qquad (5.16)$$

5.3 Stability and Convergence Analysis

The properties of coefficients $a_k^j,\ b_k^j,\ s_k^j$, and v_k^j can be easily checked in the following lemma, which is useful in proving the stability of the finite difference scheme

Lemma 5.3.1. If $0 < \alpha < 1/2$, the weight function $w(t)$ is positive and increasing, the scale function $z(t)$ is non-negative and strictly increasing. Under these conditions, the following results hold:

(i) $a_k^j > b_k^j > 0$ and $w_k a_k^j = w_{k+1} b_k^j$,

(ii) $s_k^j > v_k^j > 0$ and $w_k s_k^j = w_{k+1} v_k^j$, and

(iii) If $w(t)$ is a non-zero constant, then $a_k^j = b_k^j$, $s_k^j = v_k^j \quad \forall k = 0, 1, 2, \cdots, j$.

Proof. (i) Since $z(t)$ and $w(t)$ are both positive and increasing functions, for any $t_k < t_{k+1}$, we have $z(t_k) > 0$ and $z(t_k) < z(t_{k+1})$, which implies

$$\begin{aligned} z_{j+1} - z_k &> z_{j+1} - z_{k+1}, \\ (z_{j+1} - z_k)^{1-2\alpha} &> (z_{j+1} - z_{k+1})^{1-2\alpha} \end{aligned} \tag{5.17}$$

Hence, $a_k^j > 0$. Similarly $b_k^j > 0, s_k^j > 0$, and $v_k^j > 0$.

From Eq. (5.7) and Eq. (5.8), $\frac{a_k^j}{b_k^j} = \frac{w_{k+1}}{w_k} \geq 1$. Thus, $w_k a_k^j = w_{k+1} b_k^j$.

(ii) If $w(t)$ is a non-zero constant, then from $w_k a_k^j = w_{k+1} b_k^j$, we get that $a_k^j = b_k^j$, and similarly, $s_k^j = v_k^j$. □

Theorem 5.3.1. If the scale function $z(t)$ is positive and increasing and, weight function $w(t)$ is positive and non-decreasing, then the numerical scheme (5.6) is stable.

Proof. Here the need is only to prove the stability of the homogeneous part of the iteration scheme (5.6). So, let the numerical solution of Eq. (5.1) is of the form $u_{j+1}^m = \delta_{j+1} e^{i\theta m y}$, where $i = \sqrt{-1}$ (unit of complex numbers), $\theta \in \mathbb{R}$, and $1 \leq m \leq N-1$. So, the homogeneous part of Eq. (5.6) can be written as,

$$\begin{aligned} &\mu\delta_{j+1}e^{i\theta(m-1)y} + (-2\mu - a_j^j - 2\lambda s_j^j)\delta_{j+1}e^{i\theta m y} + \mu\delta_{j+1}e^{i\theta(m+1)y} \\ &\qquad = -b_j^j\delta_j e^{i\theta m y} - 2\lambda v_j^j \delta_j e^{i\theta m y} + \sum_{k=0}^{j-1}(a_k^j \delta_{k+1} e^{i\theta m y} - b_k^j \delta_k e^{i\theta m y}) \\ &\qquad + 2\lambda\sum_{k=0}^{j-1}(s_k^j \delta_{j+1} e^{i\theta m y} - v_k^j \delta_j e^{i\theta m y}). \end{aligned} \tag{5.18}$$

This implies,

$$\mu\delta_{j+1}e^{-i\theta y} + (-2\mu - a_j^j - 2\lambda s_j^j)\delta_{j+1} + \mu\delta_{j+1}e^{i\theta y}$$

$$= -(b_j^j + 2\lambda v_j^j)\delta_j + \sum_{k=0}^{j-1}(a_k^j\delta_{j+1} - b_k^j\delta_j) + 2\lambda\sum_{k=0}^{j-1}(s_k^j\delta_{j+1} - v_k^j\delta_j).$$

$$-\mu\delta_{j+1}e^{-i\theta y} + (2\mu + a_j^j + 2\lambda s_j^j)\delta_{j+1} - \mu\delta_{j+1}e^{i\theta y}$$

$$= b_j^j\delta_j + 2\lambda v_j^j\delta_j - \sum_{k=0}^{j-1}(a_k^j\delta_{j+1} - b_k^j\delta_j) - 2\lambda\sum_{k=0}^{j-1}(s_k^j\delta_{j+1} - v_k^j\delta_j),$$

$$\mu(2 - e^{-i\theta y} - e^{i\theta y})\delta_{j+1} + (a_j^j + 2\lambda s_j^j)\delta_{j+1}$$

$$= (b_j^j + 2\lambda v_j^j)\delta_j - \sum_{k=0}^{j-1}(a_k^j\delta_{j+1} - b_k^j\delta_j) - 2\lambda\sum_{k=0}^{j-1}(s_k^j\delta_{j+1} - v_k^j\delta_j),$$

$$\mu(2 - 2\cos\theta y)\delta_{j+1} + (a_j^j + 2\lambda s_j^j)\delta_{j+1}$$

$$= (b_j^j + 2\lambda v_j^j)\delta_j - \sum_{k=0}^{j-1}(a_k^j\delta_{j+1} - b_k^j\delta_j) - 2\lambda\sum_{k=0}^{j-1}(s_k^j\delta_{j+1} - v_k^j\delta_j). \tag{5.19}$$

From Lemma (5.3.1), $a_k^j > b_k^j > 0$, $s_k^j > v_k^j > 0 \quad \forall\ k = 0, 1, 2, \cdots, j$, and $j = 0, 1, 2, \cdots, M-1$. Hence, from Eq. (5.19)

$$\delta_{j+1} = \frac{(b_j^j + 2\lambda v_j^j)\delta_j - \sum_{k=0}^{j-1}(a_k^j\delta_{j+1} - b_k^j\delta_j) - 2\lambda\sum_{k=0}^{j-1}(s_k^j\delta_{j+1} - v_k^j\delta_j)}{\mu(2 - 2\cos(\theta y)) + (a_j^j + 2\lambda s_j^j)}$$

$$= \frac{(b_j^j + 2\lambda v_j^j)\delta_j}{\mu(2-2\cos(\theta y)) + (a_j^j + 2\lambda s_j^j)} - \frac{\sum_{k=0}^{j-1}(a_k^j\delta_{j+1} - b_k^j\delta_j)}{\mu(2-2\cos(\theta y)) + (a_j^j + 2\lambda s_j^j)}$$

$$- \frac{2\lambda\sum_{k=0}^{j-1}(s_k^j\delta_{j+1} - v_k^j\delta_j)}{\mu(2-2\cos(\theta y)) + (a_j^j + 2\lambda s_j^j)}$$

$$\leq \frac{(b_j^j + 2\lambda v_j^j)\delta_j}{(a_j^j + 2\lambda s_j^j)} - \frac{\sum_{k=0}^{j-1}(a_k^j\delta_{j+1} - b_k^j\delta_j)}{\mu(2-2\cos(\theta y)) + (a_j^j + 2\lambda s_j^j)} - \frac{2\lambda\sum_{k=0}^{j-1}(s_k^j\delta_{j+1} - v_k^j\delta_j)}{\mu(2-2\cos(\theta y)) + (a_j^j + 2\lambda s_j^j)}$$

$$\leq \delta_j. \tag{5.20}$$

Since, each term in summation is non-negative which implies that $\delta_{j+1} \leq \delta_j \leq \cdots \leq \delta_1 \leq \delta_0$, therefore $\delta_{j+1} = |u_{j+1}^m| \leq \delta_0 = |u_0^m| = |u_0|$. Hence, $\|u_{j+1}\|_{l^2} \leq \|u_0\|_{l^2}$, and the stability of numerical scheme Eq. (5.6) is proved. □

The consistency of the numerical scheme (5.6) is easy to prove, and so the numerical scheme (5.6) is convergent (from Lax-Richtmyer Theorem [206], Chapter 4, Theorem 4.2.1).

5.4 Numerical Examples

This section discusses some examples to validate our numerical scheme which verifies the stability and convergence of the numerical method. Some numerical experiments are also provided to observe the effect of the scale and weight functions on the solution of chosen GTFTE.

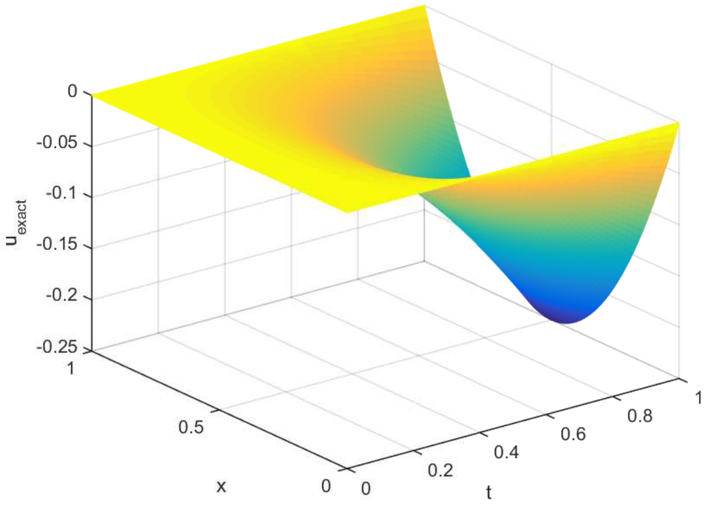


FIGURE 5.1: The analytical solution of Example 5.4.1.

Example 5.4.1. Consider Eq. (5.1) with force term $f(x,t) = \frac{2x(x-1)t^{2-2\alpha}}{\Gamma(3-2\alpha)} + \frac{4\lambda x(x-1)t^{2-\alpha}}{\Gamma(3-\alpha)} - 2c^2t^2$, and initial and boundary conditions as $u(x,0) = 0,\ x \in [0,1],\ u(0,t) = u(1,t) = 0,\ t > 0$.

With $z(t) = t,\ w(t) = 1$, the exact solution of the Eq. (5.1) will be $u(x,t) = x(x-1)t^2$. The Eq. (5.1) is solved by the numerical finite difference scheme given in Eq. (5.12) with step sizes $\Delta x = 0.01,\ \Delta t = 0.001$. The analytical and the numerical solutions at $\alpha = 0.2,\ 0.3,\ 0.4$ are shown in Fig 5.2: F-1 to F-6. The maximum absolute error (MAE) and the order of convergence (CO) are calculated for different step sizes. The obtained results for $\alpha = 0.2$ and $\alpha = 0.4$ are given in Table 5.1 and Table 5.2, respectively. From Table 5.1 and Table 5.2, it can be assumed that the numerical scheme is stable and CO is $(\Delta t)^{(2-2\alpha)}$.

Example 5.4.2. Consider Eq. (5.1) with force term $f(x,t) = \frac{2x(x-1)t^{2-2\alpha}}{\Gamma(3-2\alpha)} + \frac{4\lambda x(x-1)t^{2-\alpha}}{\Gamma(3-\alpha)} - 4c^2\pi^2 \sin(2\pi x) - 2c^2t^2$, and initial and boundary conditions as $u(x,0) = 0,\ x \in [0,1],\ u(0,t) = u(1,t) = 0,\ t > 0$.

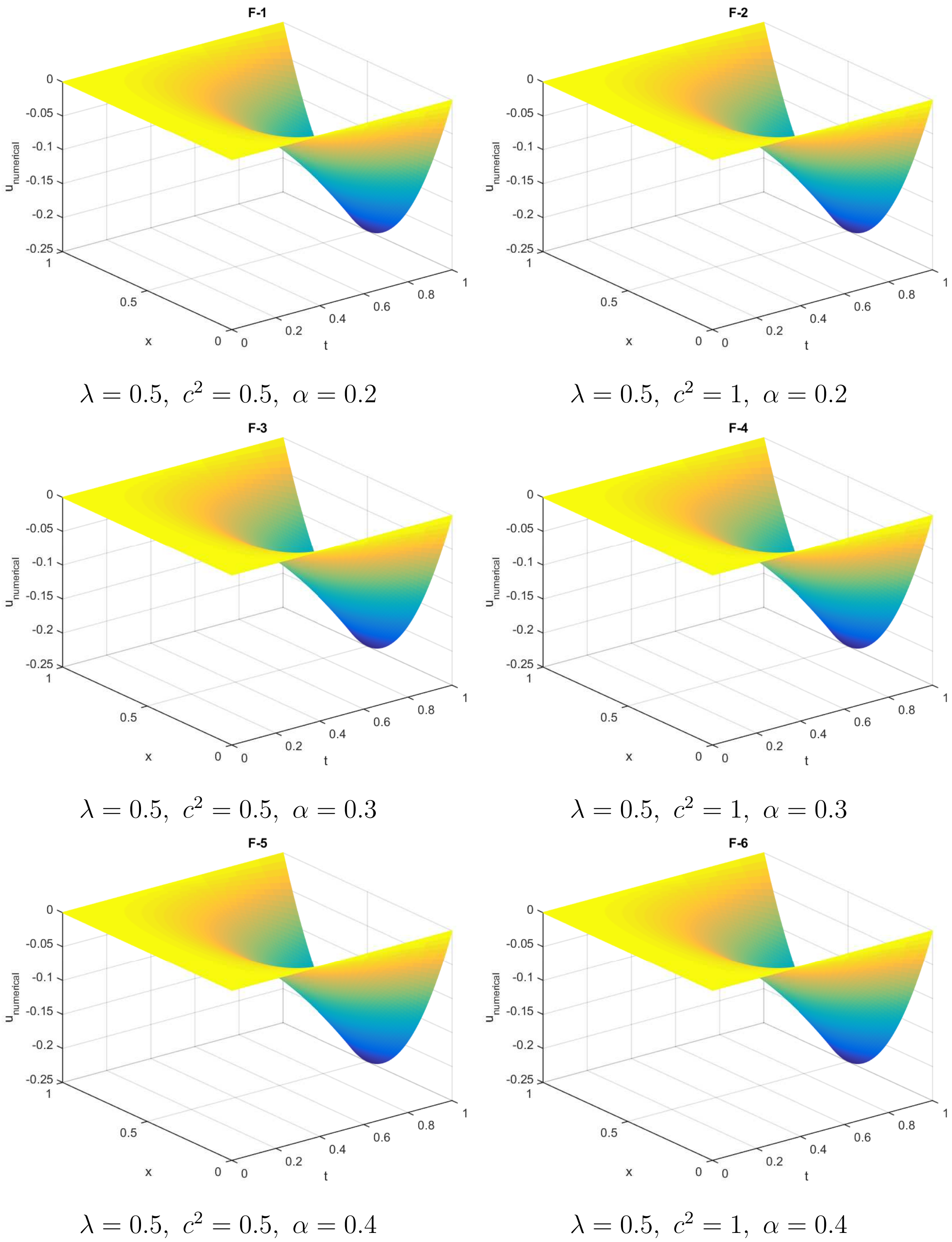


FIGURE 5.2: The numerical solutions of Example 5.4.1 for different parameters.

TABLE 5.1: MAE and CO for Example 5.4.1 with $\lambda = 0.5,\ c^2 = 1,\ \alpha = 0.2$.

Δt	Δx	MAE	CO
$\frac{1}{10}$	$\frac{1}{10}$	0.00026	-
$\frac{1}{20}$	$\frac{1}{20}$	0.000085	1.6130
$\frac{1}{40}$	$\frac{1}{40}$	0.000028	1.6020
$\frac{1}{80}$	$\frac{1}{80}$	0.0000092	1.6057

TABLE 5.2: MAE and CO for Example 5.4.1 with $\lambda = 0.5,\ c^2 = 1,\ \alpha = 0.4$.

Δt	Δx	MAE	CO
$\frac{1}{10}$	$\frac{1}{10}$	0.0013	-
$\frac{1}{20}$	$\frac{1}{20}$	0.00056	1.2150
$\frac{1}{40}$	$\frac{1}{40}$	0.00024	1.2224
$\frac{1}{80}$	$\frac{1}{80}$	0.0001	1.2630

TABLE 5.3: MAE and CO for Example 5.4.2 with $\lambda = 0.5,\ c^2 = 0.5,\ \alpha = 0.4,\ \Delta x = 1/512$.

Δt	MAE	CO
$\frac{1}{8}$	0.0031	-
$\frac{1}{16}$	0.0013	1.2538
$\frac{1}{32}$	0.00056	1.2150
$\frac{1}{64}$	0.00024	1.2224

TABLE 5.4: MAE and CO for Example 5.4.2 with $\lambda = 0.5,\ c^2 = 0.5,\ \alpha = 0.4,\ \Delta t = 1/512$.

Δx	MAE	CO
$\frac{1}{8}$	0.0505	-
$\frac{1}{16}$	0.0124	2.0259
$\frac{1}{32}$	0.0031	2.0000
$\frac{1}{64}$	0.00078	1.9907

With $z(t) = t,\ w(t) = 1$, the exact solution of the Eq. (5.1) is $u(x,t) = x(x-1)t^2 + \sin(2\pi x)$. Solve Eq. (5.1) by the FDS (5.12) with step sizes $\Delta x = 0.0001,\ \Delta t = 0.01$; $\Delta x = 1/512,\ \Delta t = 1/20$; $\Delta x = 1/256,\ \Delta t = 1/20$ and $\Delta x = 1/128,\ \Delta t = 1/20$, and other different parameters. The analytical solution and numerical solutions are shown in Fig 5.3 and Fig 5.4: G-1 to G-6, . MAE and CO are given in Table 5.3 and Table 5.3 for fixed $\Delta x = 1/512$, and $\Delta t = 1/512$, respectively. Numerical results for this Example 5.4.2 verify that this scheme is stable and it can be concluded from the Table 5.3 and Table 5.4 that the CO of the scheme reaches $(\Delta t)^{2-2\alpha}$ and $(\Delta x)^2$ as the step sizes decreases in the temporal direction and spatial direction respectively.

5.4.1 Influence of scale function on the solution of GTFTE

The influence of scale function $z(t)$ on the solutions of GTFTE is observed . Consider $z(t)$ to be monotonic function, and $w(t) = 1$ as a constant function. Note that due to the scale function $z(t)$, the time domain $(0, T)$ is shifted to $(z(0), z(T))$ or $(z(T), z(0))$ depending upon whether it is monotonic increasing or monotonic decreasing in behaviour, respectively. For numerical descriptions, the values taken for Example 5.4.1 are $\Delta x = 0.01, \Delta t = 0.001,\ z(t) = t^2,\ t^3,\ t^5,\ t^{0.5},\ t^{0.9}$, and

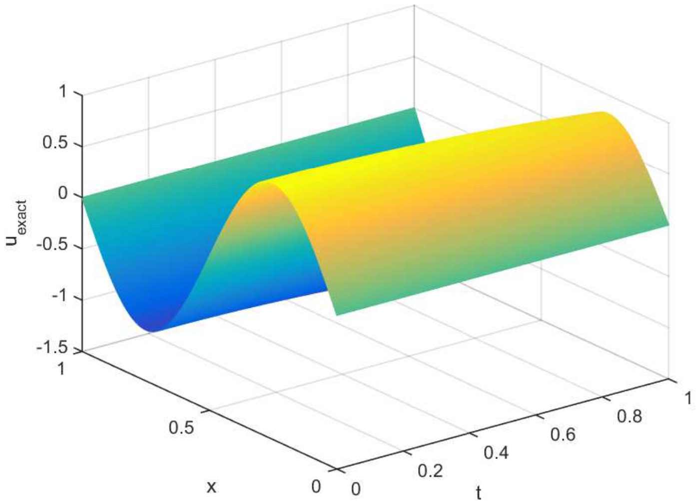


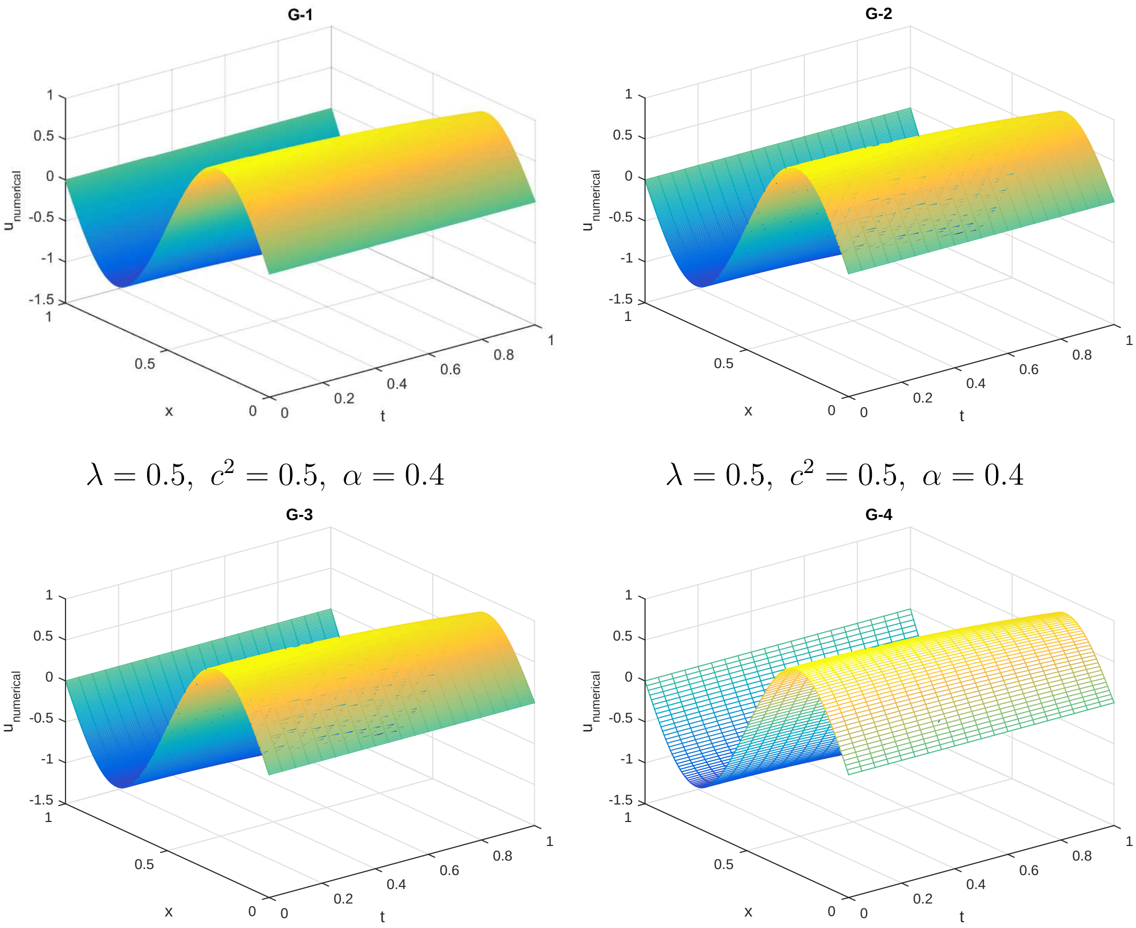
FIGURE 5.3: The analytical solution of Example 5.4.2.

$20t$. It is clear that the scale function $z(t) = 20t$ represents the linear function, and the other ones are non-linear functions. The results for the above details are demonstrated in Fig 5.5: Z-1 to Z-12.

Case 1: When a contracting scale function is taken, the behaviour of the the solution is stretching. In Fig 5.5: Z-1, Z-2, Z-3, Z-7, Z-8, and Z-9, which are numerical simulations of Example 5.1, all scale functions are contracting function, but solutions show stretch in behaviour.

Case 2: Here, the scale function is chosen as a stretching function, so from Fig 5.5: Z-4, Z-5, Z-10, and Z-11, it is clear that the behaviour of the solution is contracting.

Hence, the non-linear scale function affects inversely to the solution of GTFTE. It is also observed that for linear stretching function $z(t) = 20t$, the solution of GTFTE is stretching in behaviour (Fig 5.5: Z-6 and Z-12). Fig 5.5: Z-4, Z-5, Z-10, and Z-11, also present that when scale function $z(t) = t^{\beta},\ 0 < \beta \leq 1$ approach to $z(t) = t$, then it reaches to the exact solution of GTFTE.



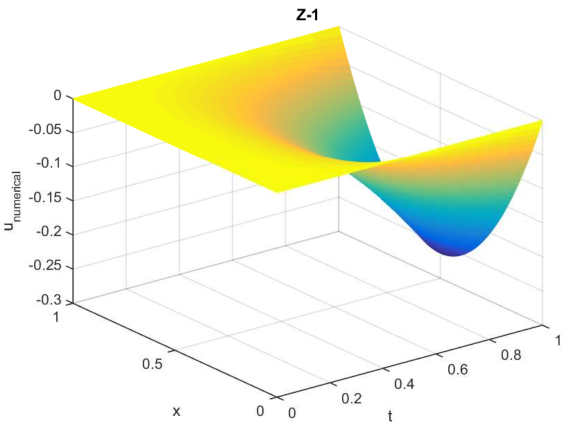
$\lambda = 0.5, \ c^2 = 0.5, \ \alpha = 0.4$ $\qquad$ $\lambda = 0.5, \ c^2 = 0.5, \ \alpha = 0.4$

$\lambda = 0.5, \ c^2 = 1, \ \alpha = 0.2$ $\qquad$ $\lambda = 0.5, \ c^2 = 1, \ \alpha = 0.2$

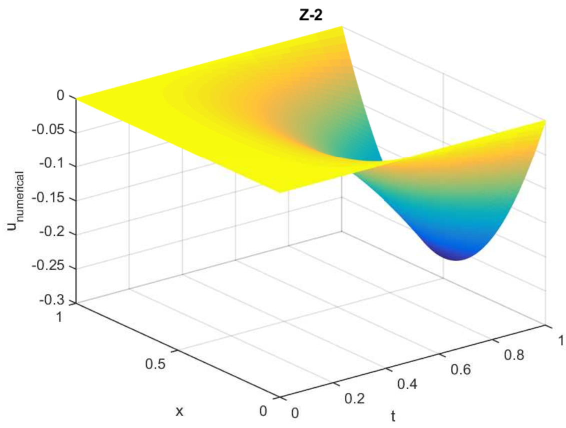
FIGURE 5.4: The numerical solutions of Example 5.4.2 for different parameters.

5.4.2 Influence of weight function on the solution of GTFTE

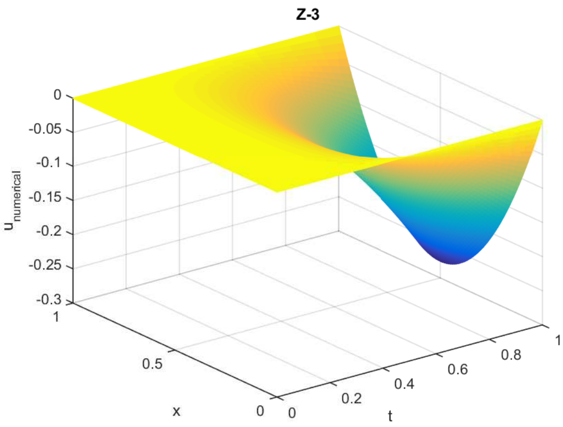
To study the action of weight function w(t) on the solution of GTFTE, fix the scale function $z(t) = t$ and take $\Delta x = 0.01, \ \Delta t = 0.001$, and $w(t) = \exp(t), \ \exp(4t)$, $\exp(-t)$, and $\exp(-4t)$. From numerical simulations of Example 5.4.1, it is observed that the numerical solution of GTFTE is affected directly by the behaviour of weight function which means if the weight function is increasing, the solution shifts in upward direction, and if the weight function is decreasing, the solution shifts in downward direction. This can



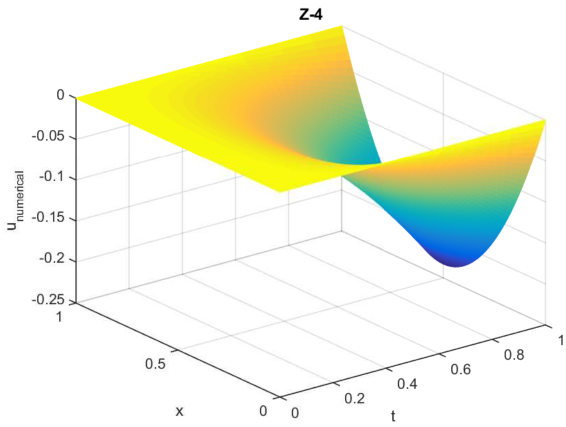
$z(t) = t^2, \ \lambda = 0.5, \ c^2 = 0.5, \ \alpha = 0.2$



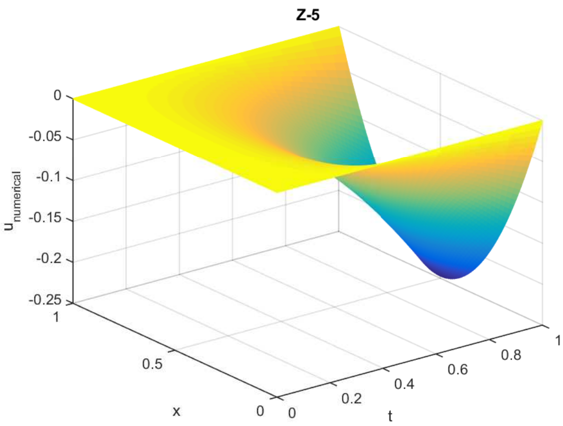
$z(t) = t^3, \ \lambda = 0.5, \ c^2 = 0.5, \ \alpha = 0.2$



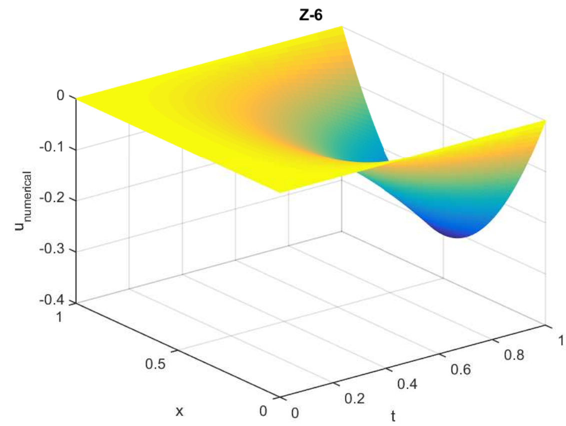
$z(t) = t^5, \ \lambda = 0.5, \ c^2 = 0.5, \ \alpha = 0.2$



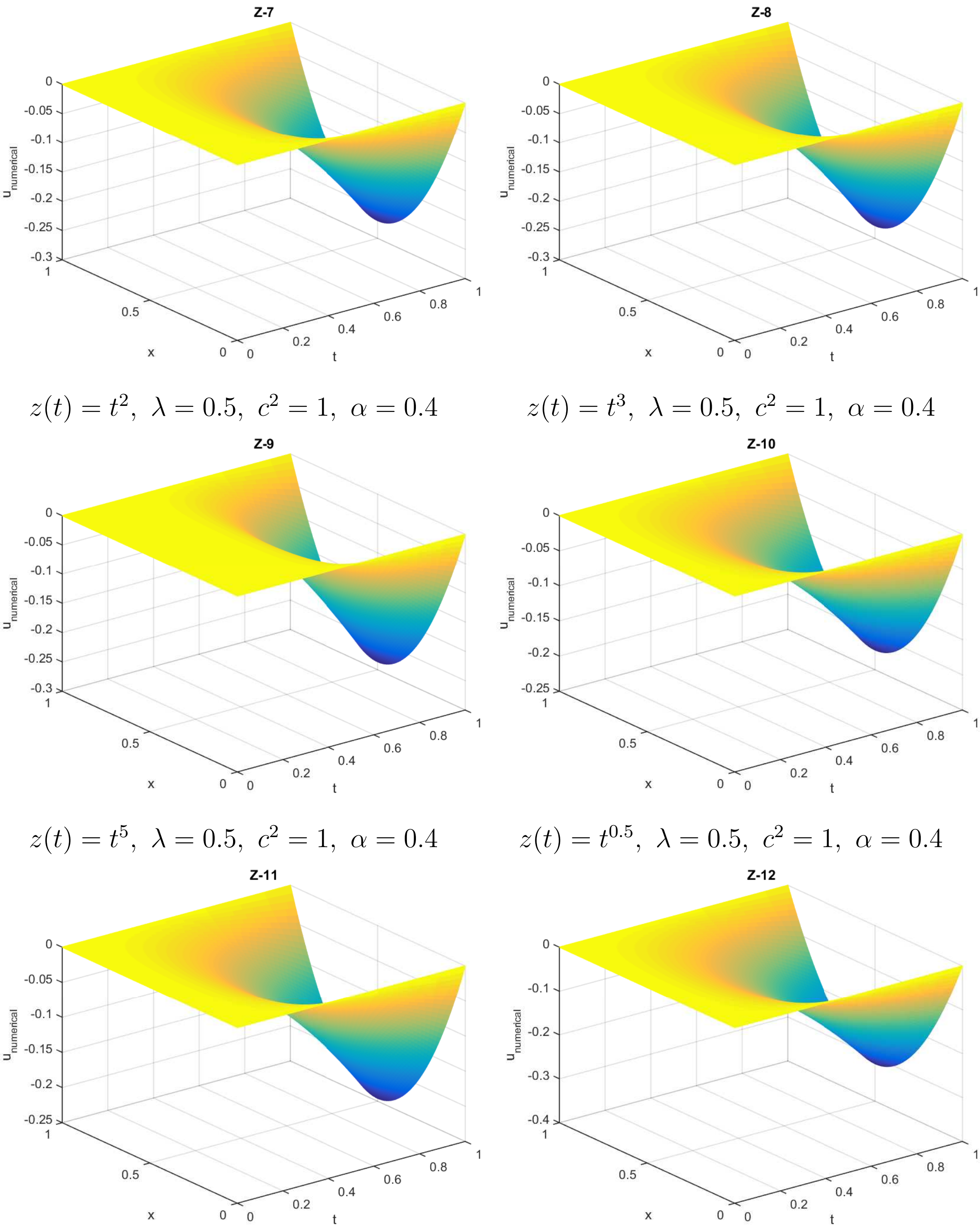
$z(t) = t^{0.5}, \ \lambda = 0.5, \ c^2 = 0.5, \ \alpha = 0.2$



$z(t) = t^{0.9}, \ \lambda = 0.5, \ c^2 = 0.5, \ \alpha = 0.2$



$z(t) = 20t, \ \lambda = 0.5, \ c^2 = 0.5, \ \alpha = 0.2$

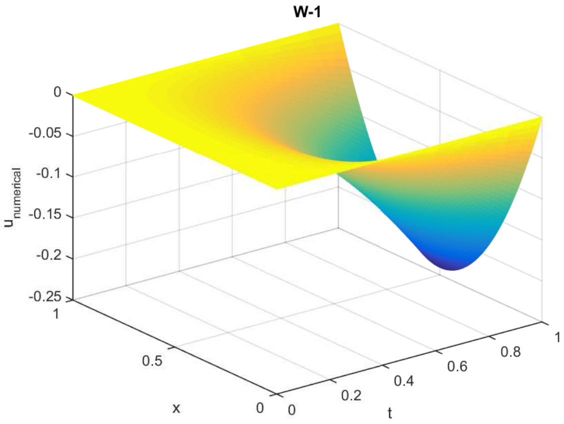


$z(t) = t^2,\ \lambda = 0.5,\ c^2 = 1,\ \alpha = 0.4$ $z(t) = t^3,\ \lambda = 0.5,\ c^2 = 1,\ \alpha = 0.4$

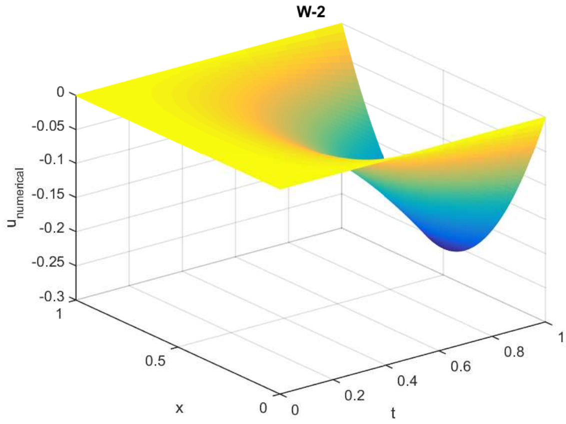
$z(t) = t^5,\ \lambda = 0.5,\ c^2 = 1,\ \alpha = 0.4$ $z(t) = t^{0.5},\ \lambda = 0.5,\ c^2 = 1,\ \alpha = 0.4$

$z(t) = t^{0.9},\ \lambda = 0.5,\ c^2 = 1,\ \alpha = 0.4$ $z(t) = 20t,\ \lambda = 0.5,\ c^2 = 1,\ \alpha = 0.4$

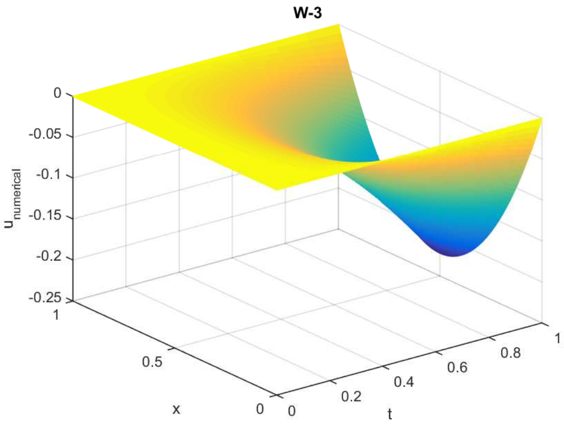
FIGURE 5.5: The numerical solutions of Example 5.4.1 for different parameters and scale functions.



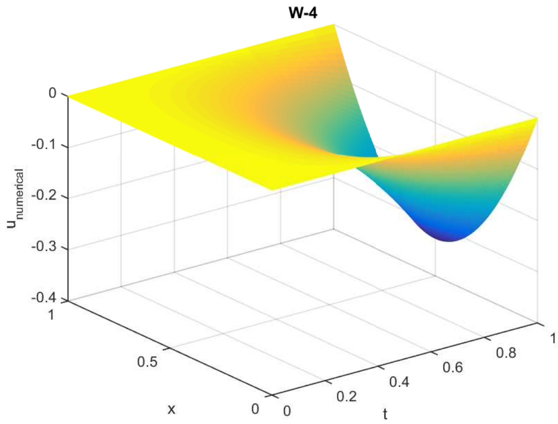
$w(t) = \exp(t),\ c^2 = 0.5,\ \alpha = 0.2$



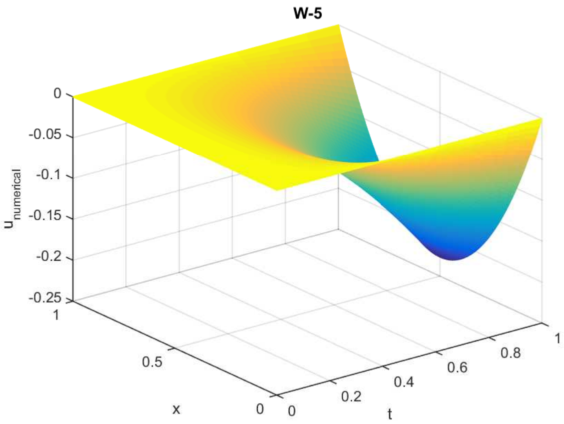
$w(t) = \exp(-t),\ c^2 = 0.5,\ \alpha = 0.2$



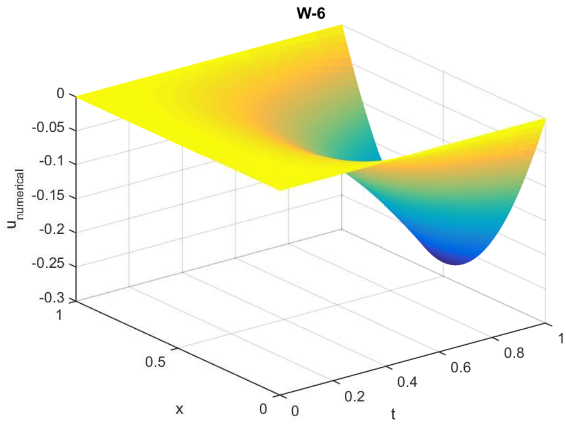
$w(t) = \exp(4t),\ c^2 = 0.5,\ \alpha = 0.2$



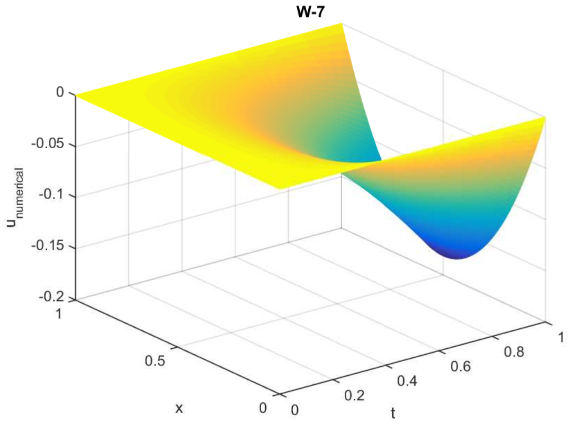
$w(t) = \exp(-4t),\ c^2 = 0.5,\ \alpha = 0.2$



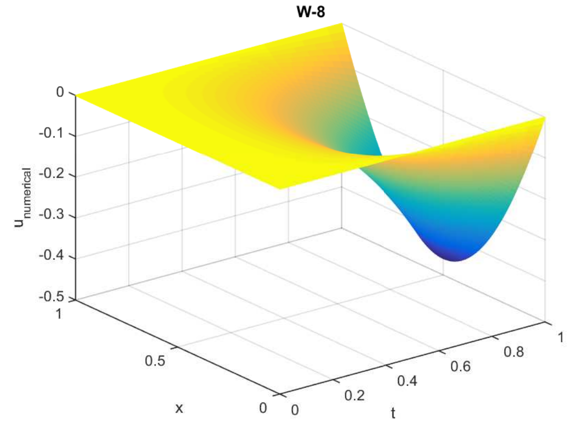
$w(t) = \exp(t),\ c^2 = 0.5,\ \alpha = 0.4$



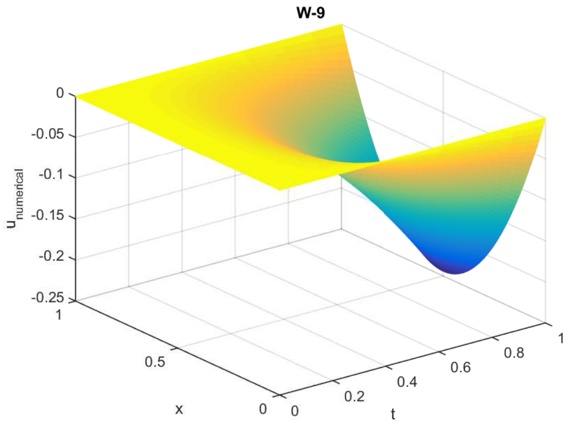
$w(t) = \exp(-t),\ c^2 = 0.5,\ \alpha = 0.4$



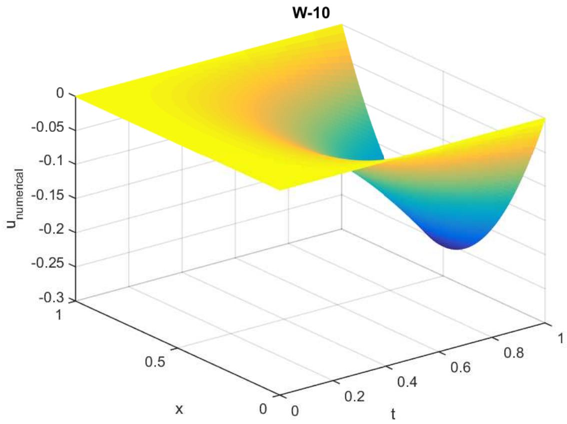
$w(t) = \exp(4t),\ c^2 = 0.5,\ \alpha = 0.4$



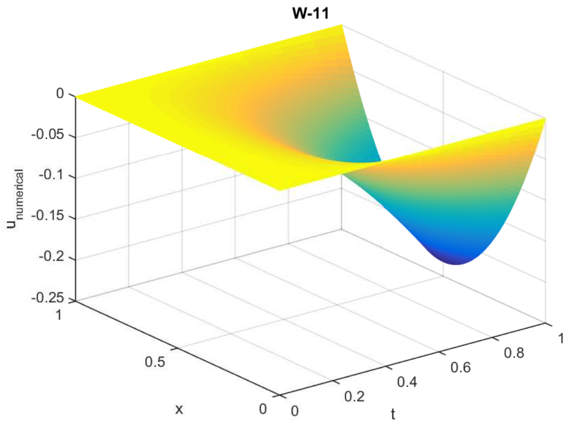
$w(t) = \exp(-4t),\ c^2 = 0.5,\ \alpha = 0.4$



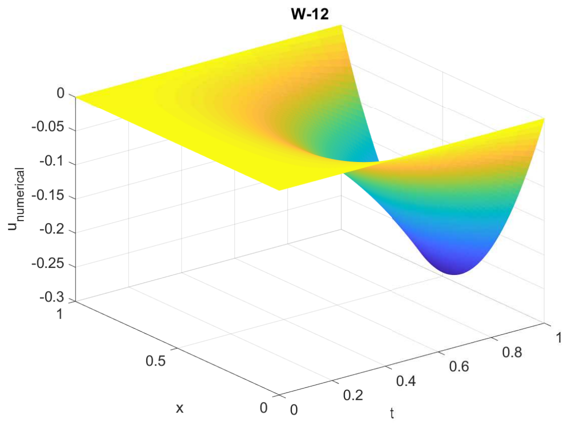
$w(t) = \exp(t),\ c^2 = 1,\ \alpha = 0.2$



$w(t) = \exp(-t),\ c^2 = 1,\ \alpha = 0.2$



$w(t) = \exp(4t),\ c^2 = 1,\ \alpha = 0.2$



$w(t) = \exp(-4t),\ c^2 = 1,\ \alpha = 0.2$

FIGURE 5.6: The numerical solutions of Example 5.4.1 for different parameters and weight functions.

be observed from Fig 5.6: W-1 to W-12 the results which are obtained for different parameters and functions ($\lambda = 0.5$).

5.5 Conclusion

In this chapter, the finite difference scheme is discussed for GTFTE. The GFD is defined with respect to the scale function $z(t)$ and the weight function $w(t)$ in the sense of Caputo type. The proposed scheme is stable and convergent, and for verifying its validity, two examples with the different parameters are considered. The order of convergence and absolute errors of the scheme are calculated. The simulation results showed that numerical scheme is of order $(2 - 2\alpha)$. The stability of the finite difference scheme is derived using the Fourier series method. The effects of the scale and weight functions on the solution of GTFTE in the sense of monotonicity are also observed. When scale function $z(t)$ is taken as $t^{\beta}, 0 < \beta \leq 1$, and for β tends to 1, the solution of GTFTE converge to the exact solution. It is observed that the effect of the weight function is directly while that of the scale function is inversely proposal to the solution of GTFTE.