Chapter 3

Numerical Approximation of Fractional Burgers Equation with Atangana-Baleanu Caputo Derivative

This chapter describes another application of the approximated ABC derivative. Brief introduction is provided in Section 3.1. The numerical scheme is proposed in Section 3.2. Some essential results to prove the stability of the scheme are studied in this section. Section 3.3 illustrates some numerical examples. The conclusion of the chapter is given in Section 3.4.

3.1 Introduction

Fractional derivatives depend on the previous values of the function i.e. they are good enough to provide mechanism to incorporate the hereditary properties of many physical phenomena. With these properties, fractional derivatives are found to be useful in various fields such as physics [183], biology [184], control theory [177], bioengineering [185], thermodynamics [186] and consequently fractional PDEs are important as well [187, 188, 174].

A non-linear PDE Burgers equation models the traffic flow and contains non-linear propagation with diffusion effects. The generalization of the Burgers equation is proposed to describe the non-linear phenomena more accurately. After Momani [29] generalized the Burgers equation by using space and time-fractional derivatives, many other researchers have worked on fractional Burgers equation. Chen and An [189] provided numerical solutions of coupled fractional Burgers equation using the Adomian decomposition method. Bhrawy et al. [190] applied the Legendre spectral-collocation method for the numerical approximation of fractional Burgers equation. Inc [191] used a variational iteration method to approximate the solution of space and time-fractional Burgers equation. Liu and Hou [192] solved fractional coupled Burgers equation using a generalized differential transform method. Li et al. [193] applied a finite difference scheme on the modified form of fractional Burgers equation.

This chapter provides a numerical scheme using the finite difference method to obtain the solution of the fractional Burgers equation. The derivative in temporal direction is considered as Atangana-Baleanu fractional derivative in Caputo sense (ABC derivative) [31]. Atangana and Koca showed the relationship of this derivative with the other integral transform operators [72]. Gómez presented the Irving-Mullineux oscillator using the Atangana-Baleanu derivative, which modified the model to describe the memory without singularity [153]. A new formulation of fractional optimal control problem was proposed by Baleanu et al. [151] involving this derivative. Some other recent works with this definition can be found in [194, 80, 195, 196]

3.2 Formation of Numerical Scheme

Here, focus is on the fractional Burgers equation, whose time-derivative is ABC derivative i.e.

$$ {}^{ABC}_{\;\;0}D_t^{\alpha}u(x,t) + u(x,t)\frac{\partial u(x,t)}{\partial x} = V\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 \le x \le L, \quad 0 \le t \le T, \tag{3.1} $$

with boundary conditions

$$ u(0,t) = u(L,t) = 0, \qquad u(x,0) = u_0(x), \tag{3.2} $$

where $\alpha \in (0, 1/2]$.

For discretization, the FDM is used. The derivatives are replaced by corresponding difference equations using uniform mesh.

Let $0 = t_0 < t_1 < ... < t_M = T$, $t_k = k\tau$ and $0 = x_0 < x_1 < ... < x_N = L$, $x_i = ih$, where $\tau = \frac{T}{M}$, and $h = \frac{L}{N}$. The following calculation can be done similarly as in the previous chapter

$$\begin{aligned}
{}^{ABC}_{0}D_t^{\alpha} f(t)|_{t=t_k} &= \frac{M(\alpha)}{1-\alpha} \int_0^{t_k} f'(s) E_\alpha \left[\frac{-\alpha}{1-\alpha}(t_k - s)^\alpha \right] ds \\
&= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f'(s) E_\alpha \left[\frac{-\alpha}{1-\alpha}(t_k - s)^\alpha \right] ds \\
&= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{f(t_{j+1}) - f(t_j)}{\tau} E_\alpha \left[\frac{-\alpha}{1-\alpha}(t_k - s)^\alpha \right] ds + R_k \\
&= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k} C_j^k f(t_j) + R_k, \qquad (3.3)
\end{aligned}$$

where, if $E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}(t_k - t_i)^\alpha\right]$ is written as E_i^k then the coefficients C_j^k are define as

$$C_j^k = \begin{cases} (k-1)E_1^k - kE_0^k, & j = 0 \\ (k-j+1)E_{j-1}^k - 2(k-j)E_j^k + (k-j-1)E_{j+1}^k, & 0 < j < k \\ E_{k-1}^k, & j = k \end{cases}, \qquad (3.4)$$

and, the truncation error R_k is

$$\begin{aligned}
R_k &= {}^{ABC}_{0}D_t^{\alpha} f(t)|_{t=t_k} - \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k} C_j^k f(t_j) \\
&\leq \frac{M(\alpha)}{1-\alpha} \frac{\tau^2}{2} \left[\max_{0 \leq t \leq t_{k-1}} f''(t) \right] c_1, \qquad (3.5)
\end{aligned}$$

where c_1 is a constant.

So, at point (x_i, t_k),

$${}^{ABC}_{0}D_t^{\alpha} u(x,t)|_{(x_i,t_k)} \approx \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k} C_j^k u_i^k,$$

where $u_i^k = u(x_i, t_j)$. The space derivatives at point (x_i, t_k) are approximated as

$$\frac{\partial u}{\partial x}\bigg|_{(x_i,t_k)} = (u_i^k)_{\hat{x}} = \frac{u_{i+1}^k - u_{i-1}^k}{2h}, \qquad \frac{\partial^2 u}{\partial x^2}\bigg|_{(x_i,t_k)} = (u_i^k)_{x\bar{x}} = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2}.$$

$u\frac{\partial u}{\partial x}$ can be written as $\frac{1}{3}[u\frac{\partial u}{\partial x} + \frac{\partial u^2}{\partial x}]$. So for simplicity of calculation and proving the boundedness and stability of the scheme, the non-linear term can be approximated as

$$u\frac{\partial u}{\partial x}\bigg|_{(x_i,t_k)} = \frac{1}{3}\left[u_i^{k-1}\frac{u_{i+1}^k - u_{i-1}^k}{2h} + \frac{u_{i+1}^{k-1}u_{i+1}^k - u_{i-1}^{k-1}u_{i-1}^k}{2h}\right]. \tag{3.6}$$

Now, the numerical scheme for the fractional Burgers equation (3.1) is

$$\frac{M(\alpha)}{1-\alpha}\sum_{j=0}^{k} C_j^k u_i^j + \frac{1}{3}\left[u_i^{k-1}\frac{u_{i+1}^k - u_{i-1}^k}{2h} + \frac{u_{i+1}^{k-1}u_{i+1}^k - u_{i-1}^{k-1}u_{i-1}^k}{2h}\right]$$

$$= V\left(\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2}\right) + f_i^k, \tag{3.7}$$

$$\mu C_k^k u_i^k + \mu\sum_{j=1}^{k-1} C_j^k u_i^j + \mu C_0^k u_i^k + \frac{1}{3}\left[u_i^{k-1}(u_i^k)_{\hat{x}} + (u_i^{k-1}u_i^k)_{\hat{x}}\right] = V(u_i^k)_{x\bar{x}} + f_i^k, \tag{3.8}$$

where $\mu = \frac{M(\alpha)}{1-\alpha}$ and $f_i^k = f(x_i, t_k)$.

Lemma 3.2.1. If R_i^k be the local truncation error (LTE) of the numerical scheme (3.8) then there exists a constant c_2 such that

$$|R_i^k| \leq c_2(\tau + h^2). \tag{3.9}$$

Remark 1: The order of LTE has decreased from 2 to 1 in temporal direction since here $\frac{1}{3}[u\frac{\partial u}{\partial x} + \frac{\partial u^2}{\partial x}]$ is used in place of $u\frac{\partial u}{\partial x}$ to make the method linear and convenient [193].

Lemma 3.2.2. If, $a_n = nE_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}(n\tau)^\alpha\right] - (n-1)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}((n-1)\tau)^\alpha\right]$ then $a_k,\ a_1 \geq 0$, and a_n is a decreasing function which implies $C_j^k = a_{k-j+1} - a_{k-j} \leq 0 \ \forall \ 0 \leq j \leq k-1$.

Proof. As $E_{\alpha,2}$ is an increasing function, so from the definition of $a_{k-j},\ a_k, a_1 \geq 0$, and

$$a_{k-j} = (k-j)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}(t_k - t_j)^\alpha\right] - (k-j-1)E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}(t_k - t_{j+1})^\alpha\right]$$

$$= (k-j)\sum_{i=0}^{\infty}\frac{((\frac{-\alpha}{1-\alpha})(k\tau - j\tau)^\alpha)^i}{\Gamma(\alpha i+2)} - (k-j-1)\sum_{i=0}^{\infty}\frac{((\frac{-\alpha}{1-\alpha})(k\tau - (j+1)\tau)^\alpha)^i}{\Gamma(\alpha i+2)}$$

$$= \frac{1}{\tau}\sum_{i=0}^{\infty}\frac{(\frac{-\alpha}{1-\alpha})^i}{\Gamma(\alpha i+2)}\left[(k\tau - j\tau)^{\alpha i+1} - (k\tau - (j+1)\tau)^{\alpha i+1}\right],$$

differentiating with respect to j,

$$a'_{k-j} = \sum_{i=0}^{\infty}\frac{(\frac{-\alpha}{1-\alpha})^i}{\Gamma(\alpha i+1)}\left[-(k\tau - j\tau)^{\alpha i} + (k\tau - (j+1)\tau)^{\alpha i}\right]$$

$$= E_{\alpha,1}\left[\frac{-\alpha}{1-\alpha}(t_k - t_{j+1})^\alpha\right] - E_{\alpha,1}\left[\frac{-\alpha}{1-\alpha}(t_k - t_j)^\alpha\right] \geq 0,$$

as $E_{\alpha,1}$ is an increasing function. □

3.2.1 Stability analysis of numerical scheme

To prove the stability of a numerical scheme, it is sufficient to establish the stability of the original problem without source term. The numerical method for the modified problem

$$ {}^{ABC}_{\ 0}D_t^{\alpha}u(x,t) + u(x,t)\frac{\partial u(x,t)}{\partial x} = V\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{3.10} $$

becomes

$$ \mu C_k^k u_i^k + \mu\sum_{j=1}^{k-1} C_j^k u_i^j + \mu C_0^k u_i^k + \frac{1}{3}\left[u_i^{k-1}(u_i^k)_{\hat{x}} + (u_i^{k-1}u_i^k)_{\hat{x}}\right] = V(u_i^k)_{x\bar{x}}. \tag{3.11} $$

Now, to show that the numerical scheme (3.11) with the boundary conditions (3.2) is stable.

Theorem 3.2.1. Let $u_0 \in H_0^2[0,L]$, then the solution of the proposed numerical scheme (3.11) is bounded, i.e., there exists a constant K such that $\|u^k\| \leq K, \quad k = 1,2,...,M$.

Proof. Multiplying (3.11) by hu_i^k and summing up for i from 1 to $N-1$,

$$ \mu C_k^k\|u^k\|^2 + \mu\sum_{j=1}^{k-1} C_j^k(u^j,u^k) + \mu C_0^k(u^0,u^k) + \frac{h}{3}\sum_{i=0}^{N-1}\left(u_i^{k-1}(u_i^k)_{\hat{x}} + (u_i^{k-1}u_i^k)_{\hat{x}}\right)u_i^k $$
$$ = V(u_{x\bar{x}}^k, u^k). $$

Noting that $(u^k_{x\bar{x}}, u^k) = -((u^k)_x, (u^k)_x) = -\|(u^k)_x\|$, and

$$\frac{h}{3}\sum_{i=0}^{N-1}\left(u_i^{k-1}(u_i^k)_{\hat{x}} + (u_i^{k-1}u_i^k)_{\hat{x}}\right)u_i^k$$

$$=\frac{1}{6}\sum_{i=0}^{N-1}\left(u_i^{k-1}(u_{i+1}^k - u_{i-1}^k) + (u_{i+1}^{k-1}u_{i+1}^k - u_{i-1}^{k-1}u_{i-1}^k)\right)u_i^k$$

$$=\frac{1}{6}\sum_{i=0}^{N-1}\left(u_i^{k-1}u_i^k u_{i+1}^k - u_{i-1}^k u_i^{k-1}u_i^k + u_i^k u_{i+1}^{k-1}u_{i+1}^k - u_{i-1}^{k-1}u_{i-1}^k u_i^k\right)$$

$$=\frac{1}{6}\left(-u_0^{k-1}u_0^k u_1^k + u_{N-1}^k u_{N+1}^{k-1}u_{N+1}^k\right) = 0,$$

which implies,

$$\mu C_k^k\|u^k\|^2 + V\|(u^k)_x\|^2 = \mu\sum_{j=1}^{k-1}(-C_j^k)(u^j, u^k) - \mu C_0^k(u^0, u^k)$$

$$\leq \frac{1}{2}\mu\sum_{j=1}^{k-1}(-C_j^k)(\|u^j\|^2 + \|u^k\|^2) + \frac{1}{2}\mu(-C_0^k)(\|u^0\|^2 + \|u^k\|^2)$$

$$= \frac{1}{2}\mu\sum_{j=1}^{k-1}(-C_j^k)\|u^j\|^2 + \frac{1}{2}\mu(-C_k^k)\|u^k\|^2 + \frac{1}{2}\mu(-C_0^k)\|u^0\|^2,$$

$$\Rightarrow \qquad \frac{1}{2}\mu C_k^k\|u^k\|^2 + V\|(u^k)_x\|^2 \leq \frac{1}{2}\mu\sum_{j=1}^{k-1}(-C_j^k)\|u^j\|^2 + \frac{1}{2}\mu(-C_0^k)\|u^0\|^2. \quad (3.12)$$

Since, $\|u^1\|^2 \leq \|u^0\|^2$,

so, let $\|u^j\|^2 \leq \|u^0\|^2$ holds for $j < k$ then

$$\begin{aligned} \frac{1}{2}\mu C_k^k\|u^k\|^2 + V\|(u^k)_x\|^2 &\leq \frac{1}{2}\mu\sum_{j=1}^{k-1}(-C_j^k)\|u^0\|^2 + \frac{1}{2}\mu(-C_0^k)\|u^0\|^2 \\ &= \frac{1}{2}\mu E_{k-1}^k\|u^0\|^2. \end{aligned} \tag{3.13}$$

$$\begin{aligned} \Rightarrow \quad \|u^k\|^2 + \frac{2V}{\mu}\|(u^k)_x\|^2 &\leq \|u^0\|^2, \\ \|u^k\|^2 &\leq \|u^0\|^2. \end{aligned} \tag{3.14}$$

Hence, by mathematical induction, $\|u^k\|^2 \leq \|u^0\|^2 \quad \forall\ k = 1, 2, ..., M$.

Since, $\exists\ K > 0$ such that $K \geq \|u^0\|^2$, which proves the theorem

$$\|u^k\| \leq K.$$

□

Theorem 3.2.1 implies that the numerical solution of the considered problem (3.10) with initial function $u_0(x)$ is bounded with respect to the time. Now, consider the given problem with different initial condition $\bar{u}_0(x)$.

Definition: Let u^k and $\bar{u}^k$ are the numerical solutions of the considered problem with initial conditions as $u_0(x)$ and $\bar{u}_0(x)$, respectively, then a numerical scheme is said to be **stable globally** if there exists a constant K such that

$$\|u^k - \bar{u}^k\| \leq K\|u_0(x) - \bar{u}_0(x)\|.$$

Using the above definition and the boundedness of the numerical scheme (3.11) (Lemma 3.2.2), it can be concluded directly that the scheme (3.11) is globally stable.

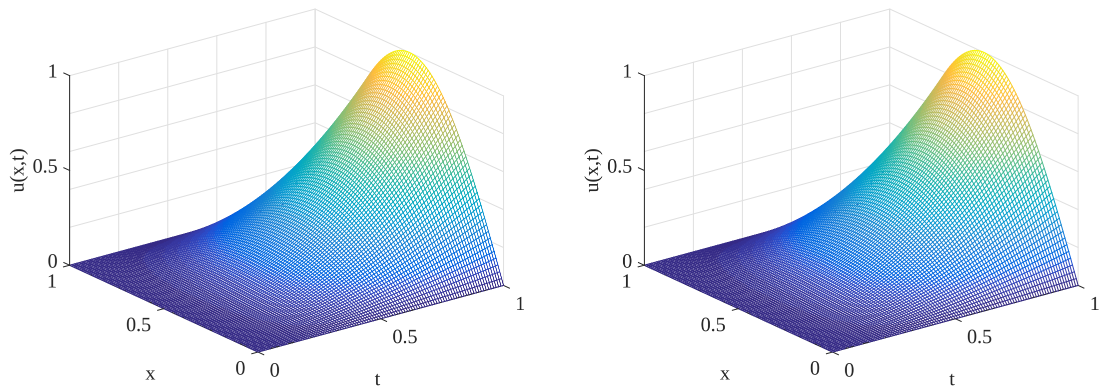
3.3 Numerical Examples

This section presents some numerical examples to validate the theoretical findings discussed in the previous section. The maximum absolute errors (MAE) and the convergence order (CO) of the scheme are calculated using the following formulas

$$\text{MAE}(j) = \max_{\substack{1\leq i<N \\ 1\leq k\leq M}} \left(U(x_i, t_k) - u_i^k\right), \tag{3.15}$$

$$\text{CO}(j+1) = \log_2\left(\frac{\text{MAE}(j)}{\text{MAE}(j+1)}\right), \tag{3.16}$$

where $U(x,t)$ is the exact solution of the problem, and j is the number of iterations.



(a) Numerical solution (b) Analytical solution

FIGURE 3.1: A comparison between the solution obtained numerically by the scheme to the exact solution of Example 3.3.1.

Example 3.3.1. Consider the PDE (3.1) with $0 \leq x \leq 1,\ 0 \leq t \leq 1$, initial condition $u_0(x) = 0$ and source term, $f(x,t) = \frac{2}{1-\alpha} t^2 \sin(\pi x) E_{\alpha,3}\left[\frac{-\alpha}{1-\alpha} t^\alpha\right] + \pi t^4 \cos(\pi x) \sin(\pi x) + 2\pi^2 t^2 \sin(\pi x)$.

Let $V = 2,\ M(\alpha) = 1$ and $\alpha \in (0, 0.5]$, then the solution $U(x,t) = t^2 \sin(\pi x)$ satisfies the Eq. (3.1) together with the initial and boundary conditions. The numerical scheme (3.8) is successfully applied to the above problem. The Figure 3.1 shows the similarity between the analytical solution and the numerical solution obtained by the derived scheme at $\alpha = 0.25,\ h = \tau = 0.01$. The calculated MAE and CO of Example 3.3.1 for different values of fractional order α are given in Table 3.1 and Table 3.2. In Table 3.1, $\frac{1}{\tau} = M = 2^{12}$ is fixed and $\frac{1}{h} = N$ takes different values to present the data in temporal direction, while in Table 3.2, $N = 2^9$ is fixed and M changes to present the data in spatial direction.

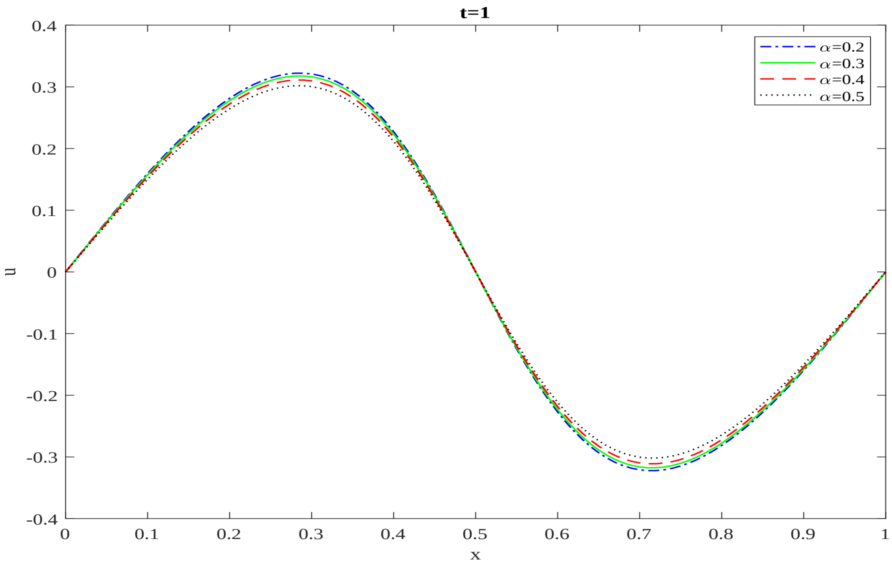


FIGURE 3.2: Numerical solution of Example 3.3.2 with fixed time $t = 1$.

Example 3.3.2. Consider the PDE (3.1) with $0 \leq x \leq 1,\ 0 \leq t \leq T$, initial condition $u_0(x) = \sin(2\pi x)$, source term $f(x,t) = 0$, and let $M(\alpha) = 1$.

In this case, the exact solution to the problem is not known. So numerical solutions are obtained by applying the proposed scheme. The solutions at $V = 0.05,\ T = 5$ are plotted in the Figure 3.2, and Figure 3.3 at different time and different fractional-order α. The computational results are taken at $\tau = h = 0.01$. The variation in the solutions can be observed clearly from Figure 3.2, and Figure 3.3 as the value of fractional order α changes. For calculating the errors and convergence orders, $V = 1,\ T = 1,\ \alpha = 0.2, 0.3,$ and 0.5 are supposed values. For Table 3.3 and Table 3.4, let $M = 2^{12}$, $N = 2^7$ and $M = 2^7$, $N = 2^{11}$ as reference solutions, respectively. It can be observed from tables that the scheme has first-order convergence in time and second-order convergence in space, as stated in Lemma 3.2.1.

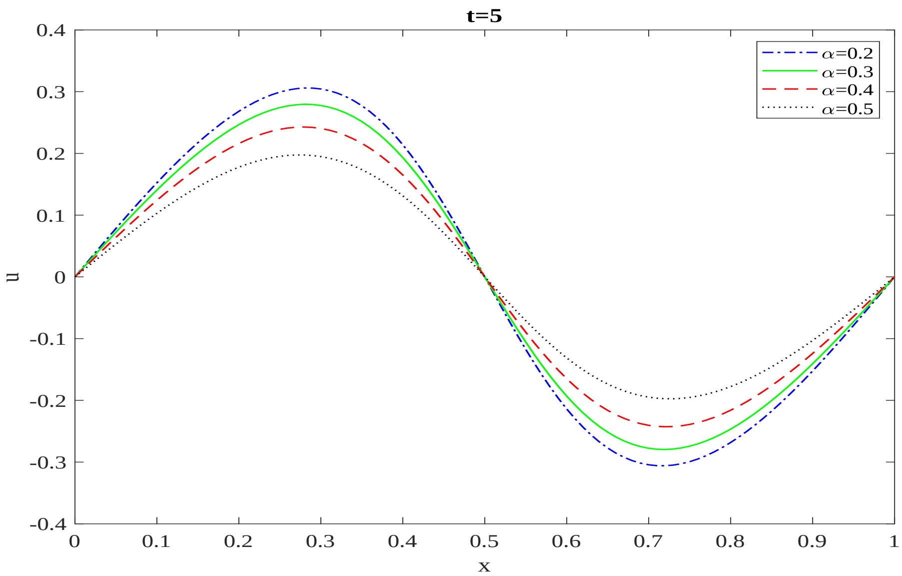


FIGURE 3.3: Numerical solution of Example 3.3.2 with fixed time $t = 5$.

Example 3.3.3. Consider the PDE (3.1) with $0 \leq x \leq 1,\ 0 \leq t \leq T$, initial condition $u_0(x) = e^{(2-\alpha x)}x(1-x)$, source term $f(x,t) = 0$, and let $M(\alpha) = 1$.

Another example with different initial condition is given here. Figure 3.4 and Figure 3.5 show that the decay is faster as α increases for the values of $V = 0.05,\ T = 5,\ M = N = 100$. The errors and the convergence orders are calculated at $V =$

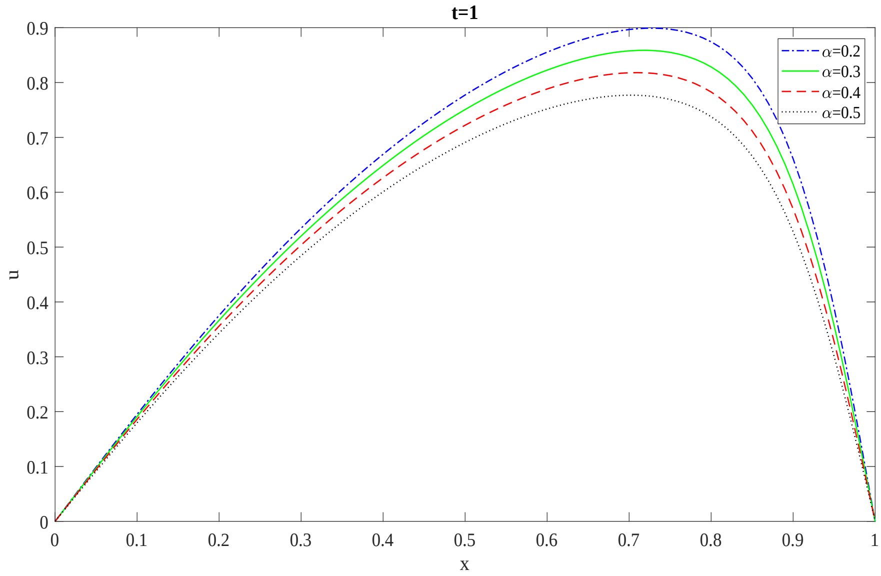


FIGURE 3.4: Numerical solution of Example 3.3.3 with fixed time $t = 1$.

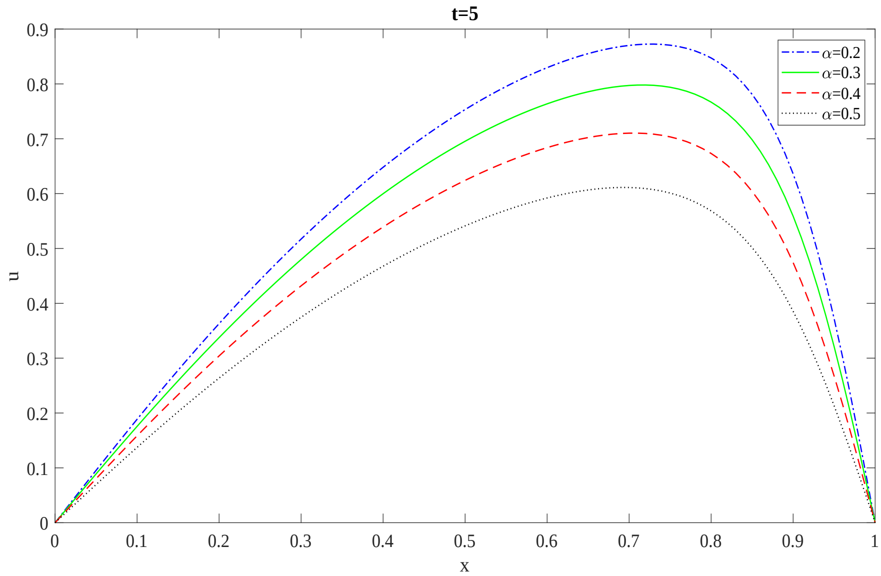


FIGURE 3.5: Numerical solution of Example 3.3.3 with fixed time $t = 5$.

1, $T = 1$ which are given in Table 3.5 and Table 3.6. The reference solutions for the Tables 3.5 and Table 3.6 are the numerical solutions which are taken at $M = 2^{12}$ and $N = 2^9$ and $M = 2^9$ and $N = 2^{11}$, respectively.

Table 3.1: MAE and CO of Example 3.3.1 in spatial direction with $M = 2^{12}$.

α	N	MAE	CO
	2^3	0.012401356	-
	2^4	0.003127835	1.987262
0.2	2^5	0.000784221	1.995831
	2^6	0.000199548	1.974526
	2^7	5.47402E-05	1.866062
	2^3	0.012374169	-
	2^4	0.003121275	1.987124
0.3	2^5	0.000782583	1.995819
	2^6	0.000199139	1.974472
	2^7	5.46367E-05	1.865829
	2^3	0.012335105	-
	2^4	0.003111847	1.986927
0.4	2^5	0.000780229	1.995801
	2^6	0.000198550	1.974394
	2^7	5.44879E-05	1.865497
	2^3	0.012282369	-
	2^4	0.003099112	1.986662
0.5	2^5	0.000777049	1.995778
	2^6	0.000197755	1.974291
	2^7	5.42862E-05	1.865055

Table 3.2: MAE and CO of Example 3.3.1 in temporal direction with $N = 2^9$.

α	M	MAE	CO
0.2	2^3	0.004738484	-
	2^4	0.002453112	0.949812
	2^5	0.001248404	0.974528
	2^6	0.000630558	0.985385
	2^7	0.000317781	0.988597
0.3	2^3	0.004750034	-
	2^4	0.002455055	0.952182
	2^5	0.001248355	0.975728
	2^6	0.000630264	0.985999
	2^7	0.000317562	0.988917
0.4	2^3	0.004766887	-
	2^4	0.002457806	0.955677
	2^5	0.001248241	0.977474
	2^6	0.000629823	0.986879
	2^7	0.000317240	0.989371
0.5	2^3	0.004791947	-
	2^4	0.002461967	0.960800
	2^5	0.001248152	0.980017
	2^6	0.000629224	0.988149
	2^7	0.000316796	0.990020

TABLE 3.3: MAE and CO of Example 3.3.2 in temporal direction with $N = 2^9$.

α	M	MAE	CO
0.2	2^8	0.000143207	
	2^9	0.000120093	0.253959460
	2^{10}	9.32020E-05	0.365715235
	2^{11}	5.74102E-05	0.699052438
0.3	2^8	0.000256917	
	2^9	0.000202854	0.340859418
	2^{10}	0.000147592	0.458824779
	2^{11}	8.47863E-05	0.799716211
0.4	2^8	0.000521922	
	2^9	0.000360044	0.535660463
	2^{10}	0.000227030	0.665289791
	2^{11}	0.000111931	1.020273380

TABLE 3.4: MAE and CO of Example 3.3.2 in spatial direction with $M = 2^7$.

α	N	MAE	CO
0.2	2^7	5.32848E-06	
	2^8	1.31633E-06	2.017198
	2^9	3.13407E-07	2.070420
	2^{10}	6.26812E-08	2.321930
0.3	2^7	5.92483E-06	
	2^8	1.46366E-06	2.017197
	2^9	3.48482E-07	2.070421
	2^{10}	6.96961E-08	2.321935
0.4	2^7	8.00778E-06	
	2^8	1.97823E-06	2.017194
	2^9	4.70997E-07	2.070417
	2^{10}	9.41997E-08	2.321924

TABLE 3.5: MAE and CO of Example 3.3.3 in temporal direction with $N = 2^9$.

α	M	MAE	CO
0.2	2^8	0.0007898	
	2^9	0.0006530	0.274213064
	2^{10}	0.0004951	0.399440086
	2^{11}	0.0002919	0.762471581
0.3	2^8	0.0013487	
	2^9	0.0010504	0.360612016
	2^{10}	0.0007480	0.489839581
	2^{11}	0.0004069	0.878391001
0.4	2^8	0.0023406	
	2^9	0.0015866	0.560983413
	2^{10}	0.0009730	0.705373909
	2^{11}	0.0004493	1.114893116

TABLE 3.6: MAE and CO of Example 3.3.3 in spatial direction with $M = 2^9$.

α	M	MAE	CO
0.2	2^7	7.82116E-06	
	2^8	1.93229E-06	2.017072059
	2^9	4.60058E-07	2.070423785
	2^{10}	9.19991E-08	2.322122046
0.3	2^7	8.29842E-06	
	2^8	2.05021E-06	2.017063893
	2^9	4.88142E-07	2.070400285
	2^{10}	9.76219E-08	2.322023761
0.4	2^7	9.88088E-06	
	2^8	2.44118E-06	2.017059039
	2^9	5.81225E-07	2.070410844
	2^{10}	1.16239E-07	2.322007854

3.4 Conclusion

A numerical scheme is presented here to solve fractional Burgers equation numerically, whose time derivative is ABC derivative of order $\alpha \in (0, 1/2]$. The numerical scheme is a linear and implicit finite difference scheme. It is often difficult to prove the stability of a non-linear fractional problem, but the proposed method is established to be unconditionally stable. Some examples are also given to validate the theory presented here. The convergence order of the scheme is estimated numerically as $O(\tau + h^2)$.