Chapter 2

Numerical Approximations of Atangana-Baleanu Caputo Derivative and Its Application

In this chapter, the discretization of the Atangana-Baleanu Caputo (ABC) derivative is discussed. Section 2.1 provides some background information for forming the base of the chapter. Section 2.2 is the main part that presents the numerical approximation of ABC derivative using different methods. Section 2.3 shows the application of the approximated ABC derivative. Numerical examples are given in Section 2.4 for the support of the theories. Section 2.5 concludes the chapter.

2.1 Introduction

Fractional operators model many physical problems more accurately than classical differential operators [176, 177, 178, 179, 65]. The researchers worked on fractional operators and investigated many definitions of fractional derivative like Grünwald-Letnikov, Riemann-Liouville, Caputo, Riesz, Hadamard, etc. [30, 3, 63, 180]. These definitions contain singular kernel, which is a restriction in modelling the behaviour of the viscoelastic material, electromagnetic systems, etc. Atangana-Baleanu derivatives overcome such types of difficulties involving Mittag-Leffler function as a non-local and non-singular kernel.

In 2018, Atangana et al. [181] presented the numerical approximation to the fractional advection-diffusion equation (FADE) whose fractional derivatives are Atangana-Baleanu derivative of Riemann-Liouville type. This chapter provides the discretization of the ABC derivative by two methods. In the first method, the finite difference method (FDM) is applied similar to the manner discussed by Atangana et al. in [181]. In the second method, the Taylor expansion of function is used along with the FDM. These methods can be implemented to solve the models in which time derivatives are taken as ABC derivative. Here, the fractional advection-diffusion equation (FADE) is considered as an example. Advection-diffusion equation (ADE) appears in modelling the problems of biology, physics, and chemistry, which involve diffusion phenomena [11, 10, 7]. The order of convergence of the schemes is obtained as $O(\tau^2 + h^2)$ and $O(\tau^3 + h^2)$, where τ is the step size in time and h is the step size in space.

2.2 Numerical Approximation to ABC Derivative

FDM is one of the simplest numerical methods for solving ordinary and partial differential equations (ODEs and PDEs). In this method, derivatives are approximated using difference equations, converting ODEs and PDEs into a linear or non-linear algebraic equations systems according to the differential equations [182].

Let, for time domain $[0,1]$, $0 = t_0 < t_1 < \cdots < t_n = 1$, $t_k = k\tau$ where $\tau = \frac{1}{n}$. Numerical approximations of the ABC derivative are presented here in the following two ways:

2.2.1 Method 1: Discretization using FDM

Discretize the first derivative of a function $f(t)$ as

$$f'(t) = \frac{f(t_{j+1}) - f(t_j)}{\tau} - \frac{\tau}{2} f''(t_j) + O(\tau^2), \qquad t \in (t_j, t_{j+1}). \tag{2.1}$$

Therefore,

$$
\begin{aligned}
&{}^{ABC}_{\ \ 0}D^{\alpha}_{t} f(t)|_{t=t_k} \\
&= \frac{M(\alpha)}{1-\alpha} \int_0^{t_k} f'(s) E_\alpha \left[\frac{-\alpha}{1-\alpha} (t_k - s)^\alpha \right] ds \\
&= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f'(s) E_\alpha \left[\frac{-\alpha}{1-\alpha} (t_k - s)^\alpha \right] ds \\
&= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{f(t_{j+1}) - f(t_j)}{\tau} E_\alpha \left[\frac{-\alpha}{1-\alpha} (t_k - s)^\alpha \right] ds + R_k \\
&= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} \frac{f(t_{j+1}) - f(t_j)}{\tau} \left\{ (t_k - t_j) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (t_k - t_j)^\alpha \right] \right. \\
&\left. - (t_k - t_{j+1}) E_{\alpha,2} \left[\frac{-\alpha}{1-\alpha} (t_k - t_{j+1})^\alpha \right] \right\} + R_k.
\end{aligned}
$$

$$ {}^{ABC}_{\ \ 0}D_t^{\alpha} f(t)|_{t=t_k} = \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k} C_j^k f(t_j) + R_k. \tag{2.2}$$

Represent $E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}(t_k - t_j)^{\alpha}\right]$ as E_j^k, then

$$ C_j^k = \begin{cases} (k-1)E_1^k - kE_0^k, & j=0 \\ (k-j+1)E_{j-1}^k - 2(k-j)E_j^k + (k-j-1)E_{j+1}^k, & 0<j<k \\ E_{k-1}^k, & j=k \end{cases}, \tag{2.3}$$

and R_k is the truncation error defined as

$$\begin{aligned} R_k &= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{\tau}{2} f''(t_j) E_{\alpha}\left[\frac{-\alpha}{1-\alpha}(t_k - s)^{\alpha}\right] ds \\ &= \frac{M(\alpha)}{1-\alpha}\frac{\tau^2}{2} \sum_{j=0}^{k-1} f''(t_j)\{(k-j)E_j^k - (k-j-1)E_{j+1}^k\} \\ &\le \frac{M(\alpha)}{1-\alpha}\frac{\tau^2}{2}\left[\max_{0\le t\le t_{k-1}} f''(t)\right] c_1, \end{aligned} \tag{2.4}$$

where c_1 is a constant.

2.2.2 Method 2: Discretization using Taylor expansion

This method uses the Taylor expansion for the derivative of function $f(t)$ to achieve the higher order of convergence. Let the function is five times differentiable, so the

Taylor expansion for $f'(t)$ for $t \in (t_j, t_{j+1})$ around t_j is given by

$$f'(t) = f'(t_j) + f''(t_j)(t - t_j) + \frac{f^{(3)}(t_j)}{2!}(t - t_j)^2 + O((t - t_j)^3)$$

$$= \frac{f(t_{j+1}) - f(t_{j-1})}{2\tau} + \frac{f(t_{j+1}) - 2f(t_j) + f(t_{j-1})}{\tau^2}(t - t_j)$$

$$- \frac{f^{(3)}(t_j)}{3!}\tau^2 + \frac{f^{(3)}(t_j)}{2!}(t - t_j)^2 + O((t - t_j)^3), \qquad t \in (t_j, t_{j+1}), \tag{2.5}$$

since

$$f'(t_j) = \frac{f(t_{j+1}) - f(t_{j-1})}{2\tau} - \frac{f^{(3)}(t_j)}{3!}\tau^2 + O(\tau^4), \qquad t \in (t_j, t_{j+1}), \tag{2.6}$$

and

$$f''(t_j) = \frac{f(t_{j+1}) - 2f(t_j) + f(t_{j-1})}{\tau^2} - \frac{f^{(4)}(t_j)}{12}\tau^2 + O(\tau^4), \qquad t \in (t_j, t_{j+1}). \tag{2.7}$$

Therefore,

$${}^{ABC}_{\ \ 0}D_t^{\alpha} f(t)|_{t=t_k}$$

$$= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f'(s) E_\alpha \left[\frac{-\alpha}{1-\alpha}(t_k - s)^\alpha \right] ds$$

$$= \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \Bigg(\frac{f(t_{j+1}) - f(t_{j-1})}{2\tau}$$

$$+ \frac{f(t_{j+1}) - 2f(t_j) + f(t_{j-1})}{\tau^2}(s - t_j) \Bigg) E_\alpha \left[\frac{-\alpha}{1-\alpha}(t_k - s)^\alpha \right] ds + R_k$$

$$= \sum_{j=0}^{k-1} \left[\{f(t_{j+1}) - f(t_{j-1})\} A_j^k + \{f(t_{j+1}) - 2f(t_j) + f(t_{j-1})\} B_j^k \right] + R_k$$

$$= \sum_{j=0}^{k-1} \left[a_j^k f(t_{j-1}) + b_j^k f(t_j) + c_j^k f(t_{j+1}) \right] + R_k, \tag{2.8}$$

where taking $E_{\alpha,2}\left[\frac{-\alpha}{1-\alpha}(t_k - t_i)^\alpha\right]$, and $E_{\alpha,3}\left[\frac{-\alpha}{1-\alpha}(t_k - t_i)^\alpha\right]$ as ${}_1E_j^k$ and ${}_2E_j^k$, respectively

$$\begin{aligned}
A_j^k &= \frac{1}{2}\frac{M(\alpha)}{1-\alpha}\{(k-j)\ {}_1E_j^k - (k-j-1)\ {}_1E_{j+1}^k\}, \\
B_j^k &= \frac{M(\alpha)}{1-\alpha}\{-(k-j-1)\ {}_1E_{j+1}^k - (k-j-1)^2\ {}_2E_{j+1}^k + (k-j)^2\ {}_2E_j^k\}, \\
a_j^k &= -A_j^k + B_j^k, \\
b_j^k &= -2B_j^k, \\
c_j^k &= A_j^k + B_j^k,
\end{aligned}$$

and

$$R_k = \frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left[-\frac{f^{(3)}(t_j)}{3!}\tau^2 + \frac{f^{(3)}(t_j)}{2!}(s - t_j)^2 \right] E_\alpha \left[\frac{-\alpha}{1-\alpha}(t_k - s)^\alpha \right] ds$$

$$= \frac{M(\alpha)}{1-\alpha}\sum_{j=0}^{k-1}\left[-\frac{f^{(3)}(t_j)}{3!}\tau^2\left(-(t_k-t_{j+1})\ {}_1E_{j+1}^k+(t_k-t_j)\ {}_1E_{j+1}^k\right)\right.$$

$$+\frac{f^{(3)}(t_j)}{2!}\left(-\tau^2(t_k-t_{j+1})\ {}_1E_{j+1}^k-2\tau(t_k-t_{j+1})^2\ {}_1E_{j+1}^k\right.$$

$$\left.\left.-\ 2(t_k-t_{j+1})^3\ {}_2E_{j+1}^k+2(t_k-t_j)^3\ {}_2E_j^k\right)\right], \tag{2.9}$$

$$\Rightarrow\quad |R_k|\le\frac{M(\alpha)}{1-\alpha}\tau^3\left[\max_{0\le t\le t_{k-1}}f^{(3)}(t)\right]c_2, \tag{2.10}$$

where c_2 is a constant.

2.3 Application: Solution of Fractional Advection-Diffusion Equation

The definition of the ABC derivative is for $\alpha\in(0,1)$, but this chapter restricts itself to the case when $\frac{\alpha}{1-\alpha}\le 1$, i.e. $\alpha\le\frac{1}{2}$ for approximating ABC derivative.

Consider the following FADE in which time derivative is taken as ABC derivative:

$${}^{ABC}_{\ \ 0}D_t^\alpha u(x,t)=V\frac{\partial^2u(x,t)}{\partial x^2}-K\frac{\partial u(x,t)}{\partial x}+f(x,t),\qquad 0<x<1,\quad 0<t<1, \tag{2.11}$$

with the following initial and boundary conditions

$$\begin{gathered} u(x,0)=u_0(x),\\ u(0,t)=\phi_1(t),\quad\text{and}\quad u(1,t)=\phi_2(t), \end{gathered} \tag{2.12}$$

where $\alpha \in (0, \frac{1}{2}]$, and $V > 0$ are real parameters, representing the order of fractional derivative and diffusion coefficient, respectively. K is the average velocity, and $f(x,t)$ is the source term.

The first and second-order spatial derivatives are approximated using finite difference, so for space domain [0,1], taking $0 = x_0 < x_1 < \cdots < x_m = 1,\ \ x_i = ih$, where $h = \frac{1}{m}$

$$\frac{\partial u(x_i,t_k)}{\partial x} = \frac{u_{i+1}^k - u_{i-1}^k}{2h} + O(h^2), \tag{2.13}$$

$$\frac{\partial^2 u(x_i,t_k)}{\partial x^2} = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2} + O(h^2), \tag{2.14}$$

for $0 \le i \le m$ and $1 \le k \le n$.

So, for Method 1, using Eq. (2.2), Eq. (2.13), and Eq. (2.14) in Eq. (2.11)

$$\frac{M(\alpha)}{1-\alpha}\sum_{j=0}^{k} C_j^k u_i^j = \frac{V}{h^2}(u_{i+1}^k - 2u_i^k + u_{i-1}^k) - \frac{K}{2h}(u_{i+1}^k - u_{i-1}^k) + f(x_i,t_k),$$

$$\left(-\frac{V}{h^2} - \frac{K}{2h}\right)u_{i-1}^k + \left(\frac{M(\alpha)}{1-\alpha}C_k^k + \frac{2V}{h^2}\right)u_i^k + \left(-\frac{V}{h^2} + \frac{K}{2h}\right)u_{i+1}^k$$

$$= -\frac{M(\alpha)}{1-\alpha}\sum_{j=0}^{k-1} C_j^k u_i^j + f(x_i,t_k), \tag{2.15}$$

for $1 \leq k \leq n$.

The Eq. (2.15) can be written in the following matrix form

$$Q^k U^k = -\frac{M(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} C_j^k U^j + G^k + F^k, \tag{2.16}$$

where

$$Q^k = \text{tri}\left[-\frac{V}{h^2} - \frac{K}{2h}, \quad \frac{M(\alpha)}{1-\alpha} C_k^k + \frac{2V}{h^2}, \quad -\frac{V}{h^2} + \frac{K}{2h}\right], \quad 1 \leq k \leq n, \tag{2.17}$$

and

$$\begin{aligned} U^k &= [u_1^k, \ldots, u_i^k, \ldots, u_{m-1}^k]^T, \\ G^k &= \left[\left(\frac{V}{h^2} + \frac{K}{2h}\right) u_0^k, \ldots, \left(\frac{V}{h^2} - \frac{K}{2h}\right) u_m^k\right]^T, \\ F^k &= [f_1^k, \ldots, f_i^k, \ldots, f_{m-1}^k]^T. \end{aligned} \tag{2.18}$$

Now, for Method 2, using Eq. (2.8), Eq. (2.13), and Eq. (2.14) in Eq. (2.11)

$$\sum_{j=0}^{k-1} \left[a_j^k u_i^{j-1} + b_j^k u_i^j + c_j^k u_i^{j+1}\right] = \frac{V}{h^2}(u_{i+1}^k - 2u_i^k + u_{i-1}^k) - \frac{K}{2h}(u_{i+1}^k - u_{i-1}^k) + f(x_i, t_k). \tag{2.19}$$

For $k = 1$,

$$\left(-\frac{V}{h^2} - \frac{K}{2h}\right) u_{i-1}^1 + \left(c_0^1 + \frac{2V}{h^2}\right) u_i^1 + \left(-\frac{V}{h^2} + \frac{K}{2h}\right) u_{i+1}^1 = -(a_0^1 + b_0^1) u_i^0 + f(x_i, t_1). \tag{2.20}$$

For $k = 2$,

$$\left(-\frac{V}{h^2} - \frac{K}{2h}\right) u_{i-1}^2 + \left(c_1^2 + \frac{2V}{h^2}\right) u_i^2 + \left(-\frac{V}{h^2} + \frac{K}{2h}\right) u_{i+1}^2$$

$$= -(a_0^2 + b_0^2 + a_1^2)u_i^0 - (c_0^2 + b_1^2)u_i^1 + f(x_i, t_2). \tag{2.21}$$

For $2 < k \leq n$,

$$\left(-\frac{V}{h^2} - \frac{K}{2h}\right) u_{i-1}^k + \left(c_{k-1}^k + \frac{2V}{h^2}\right) u_i^k + \left(-\frac{V}{h^2} + \frac{K}{2h}\right) u_{i+1}^k = -(a_0^k + b_0^k)u_i^0 - c_0^k u_i^1$$

$$- \sum_{j=1}^{k-2} \left\{a_j^k u_i^{j-1} + b_j^k u_i^j + c_j^k u_i^{j+1}\right\} - a_{k-1}^k u_i^{k-2} - b_{k-1}^k u_i^{k-1} + f(x_i, t_k). \tag{2.22}$$

As $u_i^{-1} = u_i^0 - \tau \frac{\partial u_i^0}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 u_i^0}{\partial t^2} + O(\tau^3)$, but here, the considered case is $\frac{\partial u(x,0)}{\partial t} = \frac{\partial^2 u(x,0)}{\partial t^2} = 0$, so $u_i^{-1} = u_i^0$.

The Eq. (2.20), Eq. (2.21), and Eq. (2.22) can be written in the following matrix form

$$Q^k U^k = H^k + G^k + F^k, \tag{2.23}$$

where

$$Q^k = \text{tri}\left[-\frac{V}{h^2} - \frac{K}{2h}, \quad c_{k-1}^k + \frac{2V}{h^2}, \quad -\frac{V}{h^2} + \frac{K}{2h}\right], \quad 1 \leq k \leq n, \tag{2.24}$$

$$H^k = \begin{cases} -(a_0^1 + b_0^1)U^0, & k = 1 \\ -(a_0^2 + b_0^2 + a_1^2)U^0 - (c_0^2 + b_1^2)U^1, & k = 2 \\ -\sum_{j=1}^{k-2} \left[a_j^k U^{j-1} + b_j^k U^j + c_j^k U^{j+1}\right] & \\ \quad - a_{k-1}^k U^{k-2} - b_{k-1}^k U^{k-1} - (a_0^k + b_0^k)U^0 - c_0^k U^1, & 2 < k \leq n \end{cases} . \tag{2.25}$$

G^k, F^k, and U^k are same as stated in Eq. (2.18).

2.4 Numerical Examples

This section deals with some examples to support the theoretical analysis. If U is the exact solution, then maximum absolute error (MAE) and the order of convergence (CO) of the scheme are calculated by the following formulas

$$\text{MAE}(j) = \max_{\substack{1\le i<m \\ 1\le k\le n}} |U(x_i,t_k) - u_i^k|,$$

$$\text{CO}(j+1) = \log\left(\frac{\text{MAE}(j)}{\text{MAE}(j+1)}\right) / \log\left(\frac{N(j+1)}{N(j)}\right), \tag{2.26}$$

where j depends on the number of iteration.

The first example shows that the CO of Method 1 and Method 2 for ABC derivative is close to 2 and 3, respectively, which validate the theoretical results presented in section 2.3. Second and third examples verify the schemes for the considered FADE with ABC time derivative. All cases are solved by considering the normalization function $M(\alpha) = 1$.

Example 2.4.1. Consider the function $f(t) = t^4$.

Clearly, $t^4 \in H^1(0,1)$, and its ABC derivative is $24\left(\frac{M(\alpha)}{1-\alpha}\right) t^4 E_{\alpha,5}\left[\frac{-\alpha}{1-\alpha}t^\alpha\right]$. The obtained MAE and the CO for Method 1 and Method 2 are shown in Table 2.1 and Table 2.2, respectively.

Example 2.4.2. Consider the following equation

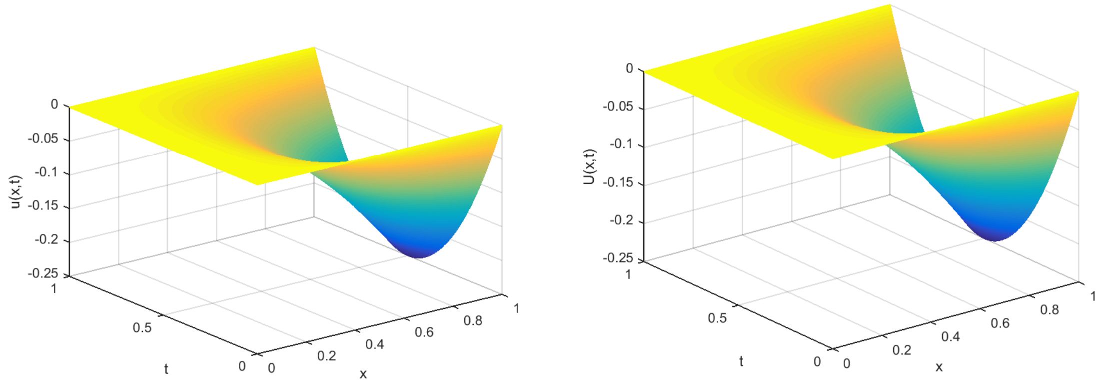
$${}^{ABC}_{\ \ 0}D_t^\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad 0<x<1, \quad 0<t<1, \tag{2.27}$$

TABLE 2.1: MAE and CO of Example 2.4.1 with different values of $\tau = \frac{1}{n}$ by Method 1.

α	n	MAE	CO
0.2	10	0.001044082	-
	20	0.000295655	1.820250289
	40	8.13837E-05	1.861100418
	80	2.19740E-05	1.888945490
	160	5.85020E-06	1.909236185
0.3	10	0.002175936	-
	20	0.000604314	1.848265813
	40	0.000163246	1.888249930
	80	4.32886E-05	1.914990833
	160	1.13287E-05	1.934003674
0.4	10	0.003791651	-
	20	0.001034804	1.873469193
	40	0.000274936	1.912190284
	80	7.17960E-05	1.937120264
	160	1.85295E-05	1.954079238
0.5	10	0.006161249	-
	20	0.001655205	1.896213069
	40	0.000433476	1.932984448
	80	0.000111780	1.955297046
	160	2.8541E-05	1.969546360

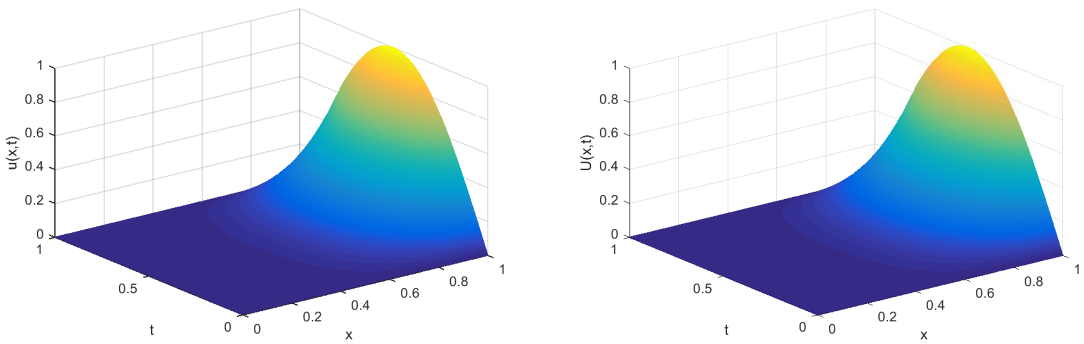
TABLE 2.2: MAE and CO of Example 2.4.1 with different values of $\tau = \frac{1}{n}$ by Method 2.

α	n	MAE	CO
	10	0.000123761	-
	20	1.74571E-05	2.825670055
0.2	40	2.38145E-06	2.873896240
	80	3.18324E-07	2.903275235
	160	4.19761E-08	2.922856294
	10	0.000258452	-
	20	3.58219E-05	2.850985296
0.3	40	4.80365E-06	2.898638807
	80	6.31585E-07	2.927082036
	160	8.19886E-08	2.945482183
	10	0.000450591	-
	20	6.14900E-05	2.873393456
0.4	40	8.12562E-06	2.919803510
	80	1.05406E-06	2.946525541
	160	1.35181E-07	2.962992906
	10	0.000731003	-
	20	9.83843E-05	2.893376094
0.5	40	1.28414E-05	2.937627746
	80	1.64822E-06	2.961820734
	160	2.09513E-07	2.975796628



(a) Numerical solution (b) Analytical solution

FIGURE 2.1: Comparison of the numerical solution obtained by Method 1 to the analytical solution of Example 2.4.2.



(a) Numerical solution (b) Analytical solution

FIGURE 2.2: Comparison of the numerical solution obtained by Method 1 to the analytical solution of Example 2.4.3.

with the initial and boundary conditions $u(x,0) = u(0,t) = u(1,t) = 0$, and the source term $f(x,t) = 2\left(\frac{M(\alpha)}{1-\alpha}\right) x(x-1)t^2 E_{\alpha,3}\left[\frac{-\alpha}{1-\alpha}t^\alpha\right] - 2t^2 + (2x-1)t^2$.

The exact solution of the above example is $x(x-1)t^2$. A comparison between exact solution and numerical solution, obtained by Method 1, taking step sizes as $h = \tau = 0.001$, is presented in Figure 2.1 for $\alpha = 0.4$. Fixing $h = \frac{1}{m} = 0.001$, and vary $\tau = \frac{1}{n}$, then the results obtained by Method 1 and Method 2 are provided in Table 2.3 and Table 2.5, respectively. Table 2.4 presents the results obtained by Method 1 when step sizes τ and h both vary.

TABLE 2.3: MAE and CO of Example 2.4.2 with different values of $\tau = \frac{1}{n}$, and fixed $h = \frac{1}{1000}$ by Method 1.

α	n	MAE	CO
0.2	10	6.29089E-06	-
	20	1.69949E-06	1.888166062
	40	4.53017E-07	1.907459519
	80	1.19488E-07	1.922695207
	160	3.12514E-08	1.934880530
0.3	10	1.34627E-05	-
	20	3.58496E-06	1.908944525
	40	9.41934E-07	1.928258301
	80	2.44961E-07	1.943071838
	160	6.31993E-08	1.954573069
0.4	10	2.38602E-05	-
	20	6.27827E-06	1.926164892
	40	1.63031E-06	1.945222205
	80	4.19281E-07	1.959154613
	160	1.07066E-07	1.969412862
0.5	10	3.90135E-05	-
	20	1.01634E-05	1.940583396
	40	2.61414E-06	1.958980182
	80	6.66564E-07	1.971522766
	160	1.68953E-07	1.980125119

TABLE 2.4: MAE and CO of Example 2.4.2 with different values of $\tau = \frac{1}{n}$, and $h = \frac{1}{m}$ by Method 1.

α	$n = m$	MAE	CO
0.2	10	6.30203E-06	-
	20	1.70133E-06	1.889155874
	40	4.52969E-07	1.909176733
	80	1.19502E-07	1.922377832
	160	3.12501E-08	1.935099332
0.3	10	1.34866E-05	-
	20	3.58884E-06	1.909933591
	40	9.41832E-07	1.929977781
	80	2.44988E-07	1.942760823
	160	6.31982E-08	1.954754361
0.4	10	2.39024E-05	-
	20	6.28509E-06	1.927152757
	40	1.63013E-06	1.946946105
	80	4.19330E-07	1.958828865
	160	1.07068E-07	1.969561794
0.5	10	3.90827E-05	-
	20	1.01745E-05	1.941570692
	40	2.61386E-06	1.960707916
	80	6.66644E-07	1.971192408
	160	1.68955E-07	1.980273364

TABLE 2.5: MAE and CO of Example 2.4.2 with different values of $\tau = \frac{1}{n}$, and fixed $h = \frac{1}{1000}$ by Method 2.

α	n	MAE	CO
0.2	10	1.47713E-06	-
	20	3.30673E-07	2.159314380
	40	7.37788E-08	2.164127115
	80	1.64124E-08	2.168418660
	160	3.64137E-09	2.172234587
0.3	10	2.93743E-06	-
	20	6.26097E-07	2.230097755
	40	1.32347E-07	2.242058549
	80	2.77813E-08	2.252143273
	160	5.79758E-09	2.260590049
0.4	10	4.88269E-06	-
	20	9.89395E-07	2.303057906
	40	1.97508E-07	2.324633438
	80	3.89626E-08	2.341751234
	160	7.61493E-09	2.355187142
0.5	10	7.56034E-06	-
	20	1.45314E-06	2.379279671
	40	2.72997E-07	2.412213876
	80	5.04250E-08	2.436673334
	160	9.19907E-09	2.454580816

Table 2.6: MAE and CO of Example 2.4.3 with different values of $\tau = \frac{1}{n}$, and fixed $h = \frac{1}{1000}$ by Method 1.

α	n	MAE	CO
0.2	10	0.000139981	-
	20	4.08956E-05	1.775214940
	40	1.19251E-05	1.777941708
	80	3.79060E-06	1.653504636
	160	1.56917E-06	1.272422075
0.3	10	0.000287302	-
	20	8.16643E-05	1.814789261
	40	2.28252E-05	1.839076189
	80	6.63756E-06	1.781904211
	160	2.29644E-06	1.531255342
0.4	10	0.000493326	-
	20	0.000137198	1.846284896
	40	3.73104E-05	1.878605287
	80	1.03383E-05	1.851583641
	160	3.22472E-06	1.680750781
0.5	10	0.00078925	-
	20	0.00021535	1.873801387
	40	5.73409E-05	1.909044348
	80	1.53859E-05	1.897956775
	160	4.47913E-06	1.780318285

TABLE 2.7: MAE and CO of Example 2.4.3 with different values of $h = \frac{1}{m}$, and fixed $\tau = \frac{1}{500}$ by Method 1.

α	m	MAE	CO
0.2	10	0.007622189	-
	20	0.001915395	1.992563186
	40	0.000478550	2.000901471
	80	0.000119693	1.999324187
	160	2.99395E-05	1.999223018
0.3	10	0.007569951	-
	20	0.001902765	1.992186298
	40	0.000475413	2.000842977
	80	0.000118918	1.999219732
	160	2.97599E-05	1.998523576
0.4	10	0.007494668	-
	20	0.001884556	1.991639999
	40	0.000470889	2.000766894
	80	0.000117795	1.999103804
	160	2.94972E-05	1.997633680
0.5	10	0.007390490	-
	20	0.001859347	1.990873841
	40	0.000464623	2.000662501
	80	0.000116240	1.998949703
	160	2.91333E-05	1.996370865

TABLE 2.8: MAE and CO of Example 2.4.3 with different values of $\tau = \frac{1}{n}$, and fixed $h = \frac{1}{1000}$ by Method 2.

α	n	MAE	CO
	10	2.37553E-05	-
	12	1.47300E-05	2.621308066
0.2	14	9.88649E-06	2.586525974
	16	7.05371E-06	2.528362944
	18	5.28569E-06	2.449836421
	10	4.80873E-05	-
	12	2.93671E-05	2.704790398
0.3	14	1.93703E-05	2.699533829
	16	1.35476E-05	2.677498563
	18	9.92624E-06	2.640700988
	10	8.20972E-05	-
	12	4.96967E-05	2.753188855
0.4	14	3.24726E-05	2.760551279
	16	2.24787E-05	2.754618490
	18	1.62834E-05	2.737407318
	10	0.000130897	-
	12	7.87232E-05	2.788875253
0.5	14	5.11044E-05	2.802881117
	16	3.51357E-05	2.805726483
	18	2.52667E-05	2.799479631

TABLE 2.9: MAE and CO of Example 2.4.3 with different values of $h = \frac{1}{m}$, and fixed $\tau = \frac{1}{500}$ by Method 2.

α	m	MAE	CO
0.2	10	0.007622166	-
	12	0.005287692	2.005679449
	14	0.003894577	1.983752401
	16	0.002987978	1.984464842
	18	0.002363415	1.990857685
0.3	10	0.007569908	-
	12	0.005251531	2.005583991
	14	0.003869140	1.981745760
	16	0.002968374	1.984685966
	18	0.002347853	1.991059408
0.4	10	0.007494601	-
	12	0.005199419	2.005446359
	14	0.003832472	1.978821524
	16	0.002940116	1.985009024
	18	0.002325422	1.991354055
0.5	10	0.007390391	-
	12	0.005127722	2.004805360
	14	0.003781711	1.975242584
	16	0.002900997	1.985464802
	18	0.002294369	1.991769605

Example 2.4.3. Consider the following equation

$$ {}^{ABC}_{0}D_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), \qquad 0 < x < 1, \quad 0 < t < 1, \quad (2.28) $$

with the initial and boundary conditions $u(x,0) = u(0,t) = u(1,t) = 0$, and the source term $f(x,t) = 120\left(\frac{M(\alpha)}{1-\alpha}\right) t^5 \sin(\pi x) E_{\alpha,6}\left[\frac{-\alpha}{1-\alpha} t^{\alpha}\right] + \pi t^5(\pi \sin(\pi x) + \cos(\pi x))$.

The exact solution of the above example is $t^5 \sin(\pi x)$. Figure 2.2 shows comparison between exact solution and numerical solution, obtained by Method 1 taking step sizes as $h = \tau = 0.001$ for $\alpha = 0.4$. Table 2.6 and Table 2.8 present the MAE and the CO obtained by Method 1 and 2, respectively for $h = 0.001$. Table 2.7 presents the result obtained by Method 1 when step size $\tau = 0.001$ is fixed, and Table 2.9 presents the result obtained by Method 2 for step size $\tau = 0.002$.

2.5 Conclusion

This chapter discussed two approximation methods for Atangana-Baleanu derivative in Caputo sense of order α, $\alpha \in (0, \frac{1}{2}]$ of a function $f(t) \in H^1(0,1)$. These methods were based on FDM and Taylor expansion of the function. Further, the approximation of ABC derivative is used to solve the fractional advection-diffusion equation defined in terms of ABC derivative. The presented method performed well and achieved good accuracy in numerical results. The convergence orders for Method 1 and Method 2 are obtained as $O(\tau^2 + h^2)$ and $O(\tau^3 + h^2)$, respectively. Numerical examples verify the theoretical findings. The numerical approach presented in the chapter can also be applied to various other physical interest models defined in terms of ABC derivatives.