Chapter 1

Introduction

"The great elegance that can be secured by the proper use of fractional operators and the power they have in simplifying the solution of complicated functional equations should more than justify a more general recognition and use."

-Herold T. Davis [1]

This chapter provides the introduction of the thesis. Section 1.1 discusses the background of fractional calculus and fractional partial differential equations. In Section 1.2, basic definitions that are being used throughout this thesis are collectively provided. Section 1.3 presents the survey and recent works on FPDEs. The challenges and motivation behind the topic are explained in Section 1.4. Section 1.5 defines the problem statement and lists the thesis objectives. The outline of the thesis is given in Section 1.6.

1.1 Background

1.1.1 Fractional calculus

Calculus is a powerful tool of mathematics which builds an interdisciplinary bridge to almost all the other fields of science, engineering, economics, demography, rheology, etc. It provides the enormous applications of mathematics by solving many problems real-world such as Newton's second law of motion, Einstein's theory of general relativity, determination of radioactive decay, etc. [2]. However, there are some phenomena in nature having memory effect such as human behaviour, business, viscoelasticity, etc. which can not be modelled with the help of classical calculus [3].

Fractional calculus came into the picture by the end of 17^{th} century, but at that time, most of the scientists and researchers related to this field were more focused on the classical calculus. So, this part of mathematics was left as less interested. Abel's tautochrone (isochrone) problem [4] was the first known problem represented in term of fractional operators. Euler, Lagrange, Laplace, Lacroix, Fourier were among those scientists who had worked in this area. Oldham and Spanier's book titled *"The fractional calculus theory and applications of differentiation and integration to arbitrary order"* [5] is the first book entirely dedicated to the fractional operators.

1.1.2 Fractional partial differential equation

Partial differential equations (PDEs) are used to formulate the physical and other problems involving function of several variables mathematically, such as the propagation of heat or sound, fluid flow, elasticity, electrostatics, electrodynamics, etc. [6]

A fractional partial differential equation (FPDE) is a PDE that involves derivatives of non-integer order, i.e., fractional derivatives. The non-locality of a fractional derivative is the main advantage over the integer-order derivative. The former provides a mechanism for the internalization of memory and hereditary properties of various phenomena. FPDEs are widely used in fluid mechanics, optical fibres, electrochemistry, plasma physics, mathematical biology, and viscoelasticity [3]. The following FPDEs are the base of this thesis and help in the study of other FPDEs as well:

1.1.2.1 Fractional advection-diffusion equation

An advection-diffusion equation (ADE) simulates mass or energy transportation by the fluid in a movement. This type of equation occurs in the problems of physics, chemistry, and biology involving diffusion or dispersion [7, 8, 9, 10, 11, 12].
A standard ADE is given by

$$\frac{\partial u(x,t)}{\partial t} + a(x,t)\frac{\partial u(x,t)}{\partial x} = v^2\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{1.1}$$

where $v > 0$ is real parameter. Eq. (1.1) appears in describing solute transport in aquifers, and u represents the solute concentration. The parameters a and v represent the average fluid velocity and dispersion coefficient, respectively [13].

Fractional advection-diffusion equation (FADE) is obtained by making some or all derivatives of an ADE arbitrary order. Fractional generalization of the diffusion equation is introduced to describe anomalous kinetics of a simple dynamical system with chaotic motion. FADE is used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium [11]. FADE, in which time derivative is of fractional order, arises from power-law particle residence time distribution and describes particle motion with memory in time [14].

1.1.2.2 Fractional telegraph equation

The telegraph equation (TE), also known as telegrapher's equation, was first studied by Kirchhoff in 1857, and then by Heaviside. It was brought to attention by Poincaré in 1893. The original form of the equation is as follows:

$$KL\frac{\partial^2 u}{\partial t^2} + (RK + SL)\frac{\partial u}{\partial t} + RSu = \frac{\partial^2 u}{\partial x^2}, \tag{1.2}$$

where u is electric potential or current, L is self-inductance, K is electrostatic capacity, R is resistance, and S is leakage conductance [15]. To investigate the TE connecting with the diffusion equation, Eq. (1.2) is considered without leakage in the following form:

$$\frac{\partial^2 u}{\partial t^2} + \lambda\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{1.3}$$

where λ and c are positive constants.

A TE is used in the field of dynamics of population and hydrology [16], heat transfer theory in thermodynamics [17, 18], signal analysis [19], wave propagation [20], transportation charged particles [21], diffusion process of chemicals [22, 23], etc. There are some processes in nature that are governed by time-fractional telegraph equations (TFTE). Cascaval et al. [24] studied various aspects of the fractional telegraph equations to understand the anomalous diffusion processes occurring in blood flow experiments. Orsingher and Beghin [25] discussed the following TFTE

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + 2\lambda \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < \alpha \leq 1. \tag{1.4}$$

1.1.2.3 Fractional Burgers equation

A Burgers equation is a non-linear PDE whose solution gives a travelling wave containing front sharpening. This equation is a mathematical model of traffic flow and provides non-linear propagation with diffusion effects. The Burgers equation in 1D is defined as follows:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = V \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad 0 < t < T, \tag{1.5}$$

where u is a vector of conserved quantity as mass, momentum or energy, and V is kinematic viscosity. It is the famous Navier Stokes equation for incompressible flow with no pressure gradient. Burgers equation is generalized by some researchers in different ways to describe the non-linear phenomena more accurately, which can be seen as generalized Burger-Huxley equation, generalized Burgers-Fisher equation, etc. [26, 27]. Further, Wazwaz [28] developed a two-mode Burgers equation and drove multiple kink solutions. In 2016, Momani [29] widespread the Burgers equation by using space and time-fractional derivatives to model the unidirectional propagation of non-linear acoustic waves through a pipe filled with gas.

1.2 Definitions of Fractional Derivatives

This section provides some definitions of fractional derivatives and integrals used throughout this thesis. All the definitions given below are left-sided. A similar calculation can be done for right-sided definitions [3, 30, 31]. The integrals presented here are assumed to be finite.

1.2.1 Basic definitions

Let $[a, b] \subset \mathbb{R}$ be a finite interval, and function $f \in L^1[a, b]$.

- The Riemann-Liouville fractional integral of the function f of order $\alpha \in \mathbb{R}^+$ is defined as

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a. \tag{1.6}$$

- The Riemann-Liouville fractional derivative of the function f of order $\alpha \in \mathbb{R}^+$, such that $m-1 < \alpha < m$, $m \in \mathbb{N}$ is defined as

$${}_a^R D_t^{\alpha} f(t) = I_{a^+}^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \quad t > a. \tag{1.7}$$

- The Caputo fractional derivative of the function f of order $\alpha \in \mathbb{R}^+$, such that $m-1 < \alpha < m$, $m \in \mathbb{N}$ is defined as

$${}_a^C D_t^{\alpha} f(t) = I_{a^+}^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \quad t > a. \tag{1.8}$$

- The Caputo fractional derivative of order $\alpha \in (0, 1)$ of the function f is defined as

$${}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \quad t > a. \tag{1.9}$$

1.2.2 Atangana-Baleanu derivative

The definitions of fractional derivative, like Grünwald-Letnikov, Riemann-Liouville, Caputo, Riesz, Hadamard, etc. [3, 30] have a prominent place in the field of fractional calculus, but still, have limitations. They contain singular and local kernel, which is a restriction in modelling the viscoelastic material's behaviour, electromagnetic systems, etc. To overcome such type of difficulties, Caputo and Fabrizio [32] proposed a new definition of fractional derivative based on the exponential kernel.

- If a function $f \in H^1(0,1)$ then, Caputo-Fabrizio derivative of Caputo type for $\alpha \in (0,1)$ is defined as

$$ {}^{CFC}_{\ \ 0}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t f'(s) \exp\left[\frac{-\alpha}{1-\alpha}(t-s)\right] ds, \tag{1.10}$$

  where $M(\alpha)$ is a normalization function which satisfies $M(0) = M(1) = 1$.

Atangana and Baleanu [31] provided another definition for fractional derivative, which generalized the Caputo-Fabrizio definition. Atangana-Baleanu derivative contains the Mittag-Leffler function as a non-local and non-singular kernel.

- If a function $f \in H^1(0,1)$ then, Atangana-Baleanu derivative of Caputo type for $\alpha \in (0,1)$ is defined as

$$ {}^{ABC}_{\ \ 0}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t f'(s) E_{\alpha}\left[\frac{-\alpha}{1-\alpha}(t-s)^{\alpha}\right] ds, \tag{1.11}$$

  where $E_{\alpha}(z)$ is the Mittag-Leffler function defined as

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{1.12}$$

1.2.3 Generalized fractional derivative

Generalized fractional derivatives contain a scale function $z(t)$ and a weight function $w(t)$ which are described in following ways [33, 34]:

- Left/forward generalized fractional integral of order $\alpha > 0$ of a function f on $[a, b]$ with respect to scale function $z(t)$ and weight $w(t)$ is defined as

$$ {}_aI^{\alpha}_{t,(z,w)}f(t) = \frac{[w(t)]^{-1}}{\Gamma(\alpha)} \int_a^t \frac{w(s)z'(s)f(s)}{[z(t) - z(s)]^{1-\alpha}} ds, \qquad t > a. \tag{1.13}$$

- Left/forward generalized fractional derivative of order m is defined as

$$ {}_aD^{m}_{t,(z,w)}f(t) = [w(t)]^{-1} \left[\left(\frac{1}{z'(t)} D_t \right)^m (w(t)f(t)) \right], \qquad t > a, \tag{1.14}$$

  where $m - 1 < \alpha < m$, and $D_t = d/dt$.

- Left/forward generalized derivative of type 1 of order $\alpha > 0$ of a function f is defined as

$$ {}^1_aD^{\alpha}_{t,(z,w)}f(t) = {}_aD^{m}_{t,(z,w)} \left({}_aI^{m-\alpha}_{t,(z,w)} f \right) (t). \tag{1.15}$$

- Left/forward generalized derivative of type 2 of order $\alpha > 0$ is defined as

$$ {}^2_aD^{\alpha}_{t,(z,w)}f(t) = {}_aI^{m-\alpha}_{t,(z,w)} \left({}_aD^{m}_{t,(z,w)} f \right) (t). \tag{1.16}$$

The generalized derivative of type 1 (Eq. (1.15)) may be referred as generalized Riemann derivative, and generalized derivative of type 2 (Eq. (1.16)) as generalized Caputo derivative, then the notations used for them are ${}^1_aD^{\alpha}_{t,(z,w)} = {}^R_aD^{\alpha}_{t,(z,w)}$, and ${}^2_aD^{\alpha}_{t,(z,w)} = {}^C_aD^{\alpha}_{t,(z,w)}$.

The functions $z(t)$ and $w(t)$ are 'good enough'. The scale function $z(t)$ allows the change in the considered domain. There are some cases where the results appear for a few seconds only, in that situation the stretching scale function is used. On the other hand, there are also cases where the process takes several decades, and here contracting scale function is required. The weight function $w(t)$ allows extension in the kernel in fractional operators, allowing a higher degree of flexibility in modelling. In the case of modelling memory of some diffusion process, it may require a heavy-weight at the current time point, whereas other diffusion processes may require more weight on past events. A weight function can help with such tasks. Selecting a different type of scale and weight function, one can obtain various generalized fractional derivatives and integrals. These generalized fractional derivatives and integrals in fractional differential equations make the model more attractive and useful [33, 34].

1.3 Literature Review

In last few decades, researchers and scientists contributed much to the area of fractional calculus. Caputo, Grünwald, Liouville, Riemann, Riesz, Weyl, etc. put their significant contribution to this field, especially in forming the definition of fractional integrals and derivatives. Classification of fractional operators given by Baleanu and Fernandez [35] would be helpful. Moshrefi-Torbati and Hammond [36] provided an interpretation of fractional operators in the time domain. Many advances have been observed as the applications of fractional operators such as in economic growth [37], electrical circuits [38], chaos and statistics [39], Earth system dynamics [40], control theory [41], geo-hydrology [42, 43], medical [44, 45], and more [46, 47, 48, 49]. Yildiz et al. [50] presented some aspects of fractional optimal control problems and derived the optimality system for this problem. The role of fractional operators in modern mechanics is studied intensively by Xu and Tan [51]. Sun et al. [52] presented a survey on applications of fractional calculus in various fields of science and engineering. Machado et al. [53] analysed the evolution dedicated to the theory and applications of fractional operations and their generalizations. Yang et al. [54] discussed the role of fractional calculus in image processing. Comparative study of an autonomous dynamical financial system using contemporary tools from fractional calculus is provided by Yusuf et al. [55] Ionescu et al. [56] provided a review with the latest scientific results of fractional calculus in modelling biological phenomena. There are some other surveys [57, 58, 59] and reviews [60, 61, 62] throwing the light on chronological developments in fractional calculus and its applications. Some text-books (for reference [3, 30, 63, 64, 65, 66, 67, 68]) are also available for further and deep studies.

1.3.1 Literature review on fractional partial differential equations

The research work on FPDEs has been the attraction for scientists and engineers [69, 70, 71, 72]. Time-fractional diffusion equation with distributed order between 0 and 1 is investigated by Mainardi [73]. Momani and Odibat [74] discussed the linear FPDEs arising in fluid mechanics. Jafari and Seifi [75] worked on a system of non-linear FPDEs and provided solutions using homotopy analysis method. Purohit [76] solved FPDEs of quantum mechanics. Bhrawy [77] studied some time-space FPDEs with sub-diffusion and super-diffusion and provided spectral collocation algorithm for solving them. Feng and Meng [78] presented an improved fractional Jacobi elliptic equation method to find the exact solutions of space-time FPDEs arising in mathematical physics. The work of Ara et al. [79] on some FPDEs is considered as an application to financial modelling. Arqub [80] introduced the reproducing kernel algorithm for time-fractional PDEs subject to Robin boundary conditions with parameters derivative concurring in fluid flows, heat conduction and electric circuit etc. Goswami et al. [81] studied time-fractional regularized long wave equations which describe the nature of shallow water waves in oceans and ion-acoustic waves in plasma. Wang and Zheng [82] derived the well-posedness and regularity of the variable-order time-fractional partial differential equations with the applications in modelling local and non-local dynamics of multi-physics phenomena. A survey paper by Luchko and Yamamoto [83] discussed FPDEs generated by the general fractional derivative introduced by A. Kochubei. Recently, Li and Chen [84] reviewed some numerical methods for solving FPDEs. There is also a book by Guo et al. [85] for the same. Some more recent work on different types of FPDEs can be found in [86, 87, 88, 89, 90, 91, 92].

1.3.1.1 Literature review on fractional advection-diffusion equation

FADE is an important FPDE treated by several authors. This equation presents an approach to describe the transport dynamics in complex systems governed by anomalous diffusion [93]. Liu et al. [94, 95, 96] studied Lévy motion with $\alpha-$stable densities using a FADEs. Meerschaert et al. [97] developed practical numerical methods for solving the space FADE with variable coefficients on a finite domain and presented a practical application of the results in modelling a radial flow problem. Momani et al. [98] constructed a reliable algorithm using the Adomian decomposition method to find the numerical solutions of the space-time FADE in the form of a rapidly convergent series with easily computable components. Wang and Wang [99] developed a fast characteristic finite difference method for the solution of space-fractional transient ADEs. Parvizi et al. [100] studied FADE with non-linear source term. Fundamental solutions of ADE with time-fractional Caputo–Fabrizio derivative are obtained by Mirza and Vieru [101]. Kundu [102] studied suspension concentration distribution in turbulent flows using FADE. Chen et al. [103] developed a fully-discrete numerical scheme for multi-term time-space variable-order FADE which describes transient dispersion observed at a field tracer test. McLean et al. [104] established well-posedness of time-fractional advection-diffusion-reaction equations with Riemann–Liouville fractional derivative. Tang [105] considered an optimal control problem of an FADE with Caputo time-fractional derivative. Some more work in which effective methods to solve FADE numerically is provided in [106, 107, 108, 109, 110, 111] with applications to various areas.

1.3.1.2 Literature review on fractional Burgers equation

Fractional Burgers equation is a basic and important non-linear FPDE which is studied by many authors [112, 113, 114, 115, 116, 117]. Miškinis [118] discussed the relation between integer ordered Burgers equation and its properties of fractional generalization. Earlier, Sugimoto [119] examined the initial-value problems asymptotically and numerically for the Burgers equation with fractional derivative. Yang et al. [120] investigated the local fractional Burgers equation and its non-linear dynamics arising in fractal flow. Esen et al. [121] studied fractional diffusion equation and fractional Burgers-Fisher equation with Haar wavelet method where time-fractional derivatives are Caputo type. Saad and Al-Sharif [122] computed variational iteration method solutions for the fractional Burgers equation and shown the behaviour of the solutions as the fractional derivative parameter changed. Yokus and Kaya [123] gave an exact solution of time-fractional Burgers equation using the expansion method and the Cole-Hopf transformation. Liu and Chang [124] solved a time-fractional Burgers equation with unknown space–time-dependent source term. Torebek [125] obtained sufficient conditions for the non-existence of fractional Burgers equation's time-global solutions. Yang and Machado [126] considered the non-linear Burgers equation engaging local fractional derivative with the use of travelling-wave transformation and used results to describe the propagation of the acoustic signals in the stratified fractal media. Akram et al. [127] used a finite difference scheme based on an extended cubic B-spline for the second-order derivative to solve Caputo type time-fractional Burgers equation. Recently, Li and Li [128] discussed Cole–Hopf transformation and the method of separation of variables for the exact and numerical solutions of the fractional Burgers equation.

1.3.1.3 Literature review on fractional telegraph equation

Some major advances in solving fractional TE and its applications can be seen in [129, 130, 131, 132, 133, 134, 135]. Orsingher and Beghin [25] examined the solutions of Eq. (1.4) with $f(x,t) = 0$ and $\alpha \in [0, 1/2]$ using Fourier transform. They also examined the Eq. (1.4) with special type of rational order [22]. Dehghan and Shokri [23] proposed a numerical method to solve the hyperbolic TE using collocation points. Chen et al. [21] studied the analytical solution of time-FTE using the concept of separating variables. Povstenko [136] considered generalized telegraph equations with time-space fractional derivatives and formulated the corresponding theories of thermal stresses. Space-FTE is studied by Kumar [137] using fractional homotopy analysis transform method. Chen et al. [138] provided the unconditional stable difference schemes for Riesz space FTE. Hesameddini and Asadolahifard [139] discussed a novel spectral Galerkin method for dealing with the hyperbolic TE in two dimension. Sharifi and Rashidinia [140] considered cubic B-spline collocation method for the numerical solution of hyperbolic TE. Ferreira et al. [141] obtained the fundamental solution of multidimensional time-FTE of Caputo type. Tawfik [142] solved the space-time FTE and space-time FADE analytically, which are mathematical models of energetic particle transport for both uniform and non-uniform large-scale magnetic field. Madhukar et al. [143] studied the heat conduction in biological materials whose mathematical modelling yield the time-FTE. Some recent work on FTE includes: the numerical solutions of Caputo type FTE using Crank-Nicholson difference schemes by Modanli and Akgül [144]; the discussion of well-posedness and regularity of the solutions of time-fractional damped wave equations including time-FTEs by Zhou and He [145]; the numerical solution of time-FTE by a local meshless method [146].

1.4 Challenges and Motivation

Unlike classical calculus, the definition of fractional derivatives are not clear. Theoretical and experimental studies are being conducted for the last few decades to provide the precise definition that satisfies the basic rules of derivatives. Caputo derivative and Riemann-Liouville derivative are the most popular and profoundly used definitions [30]. Some authors recently extended these definitions to model some important physical phenomena like viscoelasticity, electromagnetic systems, etc. Katugampola [147, 148] presented new definitions to generalize the Riemann–Liouville and Hadamard fractional integrals and derivatives into a single form. Jarad et al. [149] proposed a generalized proportional fractional derivative.

Caputo and Fabrizio [32] discussed fractional derivative in which the kernel of the integral contains an exponential term. Using the Mittag-Leffler function in place of an exponential function, Atangana and Baleanu [31] modified this definition. As the Atangana-Baleanu derivative is non-singular and non-local, it does not lead to inclusion of artificial singularity into the mathematical model. Also, it has ability to describe two different waiting times distribution which is an ideal waiting time distribution as observed in many biological phenomena such as in spread of cancer. This definition has vast applications in many fields of science and engineering [150, 151, 152, 153, 154, 155, 156, 157, 158]. Atangana [72], Abdeljawad [159], Baleanu [160], Gómez-Aguilar [69], etc. are among the authors who developed the analytical and numerical theory for Atangana-Baleanu derivative.

Agrawal [33, 34] gave generalized definitions of fractional integrals and derivatives in 2012. These new definitions were proposed using scale function and weight function. Agrawal and Xu [161, 162] worked on the fractional differential equations using generalized derivatives. In their papers, they gave stable numerical schemes whose order of convergence is estimated to one. Xu and Zheng [163] presented spectral collocation method, Kumar et al. [164, 165] and Cao et al. [166] developed finite difference schemes for the numerical solution of fractional differential/integral equations with generalized fractional operators. Ding and Wong [167] derived high order schemes inspired by the classical result of the Grünwald-Letnikov formula.

Fractional partial differential equations with the new derivatives are the new possibilities in studying natural phenomena with memory effect more thoroughly and accurately. The motivation behind the thesis is to understand the nature of different types of FPDEs generated by the Atangana-Baleanu derivative and the generalized fractional derivative proposed by Agrawal. The fractional advection-diffusion equation and fractional telegraph equation are the basic linear FPDEs, and the fractional Burgers equation is a non-linear FPDE. These equations help in studying other complex FPDEs. The presence of non-integer order derivatives makes the differential equations too complex to be solved analytically, so numerical simulation is required. It is a challenge to provide more accurate numerical results. Finite difference method (FDM) [168, 106, 169], finite element method (FEM) [170, 171, 100], short memory principle [172, 173], homotopy analysis method [75, 174], meshless method of radial basis functions [175] are some methods which are being used to solve FPDEs.

1.5 Problem Statement and Thesis Objective

FADE, fractional TE, and fractional Burgers equations are examples of linear and non-linear fractional partial differential equations. Significant work has been done when the fractional derivatives are of Riemann-Liouville and Caputo type. However, numerical methods and analysis of newly proposed fractional derivatives are quite limited. Also, the stability and convergence of a numerical scheme for FPDEs are difficult to derive.

The aim of this research work is to develop some high order numerical schemes for these types of equations using Atangana-Baleanu and generalized derivatives. FDM is used to maintain the simplicity and to provide the accuracy of the scheme. For achieving a high order of convergence, Taylor series expansion is used along with FDM.

The objectives of the thesis are:

1. To develop high order convergence schemes for the generalized and Atangana-Baleanu definitions of fractional derivative using FDM and Taylor series expansion.
2. To apply the schemes on FPDEs like the advection-diffusion equation, telegraph equation, and Burgers equation to get the approximate solutions.
3. To study the stability of numerical schemes developed for FPDEs.
4. To provide the experimental analysis for supporting the theoretical statements.

1.6 Outline of the Thesis

The outline of the thesis is as follows:

Chapter 1 provides the introduction of the thesis. Basic definitions that are being used throughout this thesis are collectively provided. It also presents the literature survey and recent works on FPDEs. The motivation behind choosing the topic and problem statement of the thesis are explained.

Chapter 2 presents numerical schemes for the Atangana-Baleanu Caputo derivative in two ways and uses the same for solving the FADE, whose time derivative is Atangana-Baleanu Caputo derivative. In the first method, FDM is used, and in second method, the Taylor expansion series of the function is used along with FDM. The stability of the schemes is established numerically. The convergence orders for Method 1 and Method 2 are obtained as $O(\tau^2 + h^2)$ and $O(\tau^3 + h^2)$, respectively. Numerical examples are provided to support the theory.

Chapter 3 is based on a numerical technique using an FDM to solve the fractional Burgers equation whose time-derivative is Atangana-Baleanu fractional derivative. Some examples are considered to perform numerical simulations. The stability of the scheme is proved, and the order of convergence is estimated numerically, which is $O(\tau + h^2)$.

Chapter 4 presents an approximation of generalized Caputo derivative of order $\alpha \in (0, 1)$ using Taylor's expansion. This approximation is used to develop a high order numerical scheme for solving the generalized fractional advection-diffusion equation, which is formed using the generalized Caputo derivative in respect of time. Some examples are provided to show the effects of different parameters on the diffusion process of the equation.

Chapter 5 discusses a FDM for the generalized time-fractional telegraph equation (GTFTE) via generalized fractional derivative. It also presents the behaviour of the solution of GTFTE by changing the weight and scale functions in the generalized fractional derivative. The convergence and the stability of the finite difference scheme are also studied.

Chapter 6 concludes the thesis and explains possible future work in solving FPDEs formed by using newly proposed derivatives.