Chapter 5

Effects of infinite occurrence of hybrid impulses on quasi-synchronization of parameter mismatched neural networks

5.1 Introduction

This chapter investigate the effects of hybrid impulsive sequence on quasi synchronization of two non identical neural networks with mixed time-varying delays. In the real world's problems, there exists an impulsive sequence in which impulses occur infinite number of times and the length of impulsive interval increases gradually. Such type of impulses does not essentially adversely affect the synchronization of the coupled neural networks. Inspired from this fact, different from the definitions of average impulsive interval considered in previous chapters, authors in [108] proposed two new concepts of average impulsive interval $T_a = \lim_{t\to\infty} \frac{t-s}{N_\zeta(t,s)}$ and average impulsive gain $\mu = \lim_{t\to\infty} \frac{|\mu_1|+|\mu_2|+\ldots+|\mu_{N_\zeta(t,s)}|}{N_\zeta(t,s)}$, and derived the unified synchronization criteria for an array of coupled neural networks with hybrid impulses. When the number of impulsive points $N_\zeta(t,s)$ will be infinite in the time span (s,t) then $T_a = \infty$, otherwise $T_a < \infty$.

Using the new concepts of average impulsive interval and average impulsive gain, we have derived sufficient criteria for achieving synchronization between nonidentical neural networks with mixed time-varying delays. In the last of this chapter, we considered two numerical examples and elaborately discussed the efficiency of obtained results for two different cases: one is $T_a = \infty$ and another is $T_a < \infty$.

5.2 Problem formulation and some preliminaries

Consider a neural network with mixed time-varying delays whose state equation is described as follows:

$$\begin{aligned}\dot{x}_i(t) =& -\bar{a}_i x_i(t) + \sum_{j=1}^{n} \bar{b}_{ij} f_{1j}(x_j(t)) + \sum_{j=1}^{n} \bar{c}_{ij} f_{2j}(x_j(t-\sigma_1(t))) \\ &+ \sum_{j=1}^{n} \bar{d}_{ij} \int_{t-\sigma_2(t)}^{t} f_{3j}(x_j(s))\ ds \ + I_i, \\ x_i(t) =& \phi_i(t) \in C([-\sigma, 0], \mathbb{R}),\ \ i = 1, 2, ..., n. \end{aligned} \tag{5.1}$$

In a compact form it can be re-written as

$$\begin{aligned}\dot{x}(t) =& -\bar{A}x(t) + \bar{B}f_1(x(t)) + \bar{C}f_2(x(t-\sigma_1(t))) + \bar{D}\int_{t-\sigma_2(t)}^{t} f_3(x(s))ds + I, \\ x(t) =& \phi(t) \in C([-\sigma, 0],\ \mathbb{R}^n), \end{aligned} \tag{5.2}$$

where n denotes the number of neurons in the network; $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T \in \mathbb{R}^n$ is the state vector associated with the neurons at time t; $\bar{A} = diag(\bar{a}_1, \bar{a}_2, ..., \bar{a}_n) > 0$; $\bar{B} = (\bar{b}_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ is the connection weights matrix of the neurons at time t; $\bar{C} = (\bar{c}_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ and $\bar{D} = (\bar{d}_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ are the matrices of weights of connections among the neurons with and without delay, respectively; $f_1(x(t)) = [f_{11}(x_1(t)), f_{12}(x_2(t)), ..., f_{1n}(x_n(t))]^T \in \mathbb{R}^n$, $f_2(x(t-\sigma_1(t))) = [f_{21}(x_1(t-\sigma_1(t))), ..., f_{2n}(x_n(t-\sigma_1(t)))]^T \in \mathbb{R}^n$ and $f_3(x(t)) = [f_{31}(x_1(t)), f_{32}(x_3(t)), ..., f_{3n}(x_n(t))]^T \in \mathbb{R}^n$

are activation functions of the neurons at time t, $t-\sigma_1(t)$ and distributed time delay $t-\sigma_2(t)$ respectively, such that $f_1(0)=0$, $f_2(0)=0$ and $f_3(0)=0$; $\sigma=\max\{\sigma_1,\sigma_2\}$, where σ_1 and σ_2 are upper bounds of $\sigma_1(t)$ and $\sigma_2(t)$ respectively; the initial condition of the equation (5.1) is $\phi(s)=[\phi_1(s),\phi_2(s),...,\phi_n(s)]^T$ belongs to the set of continuous functions $C([-\sigma,0],\ \mathbb{R}^n)$; the external input value to the network is denoted by $I=[I_1,I_2,...,I_n]^T$.

Now, we will consider some assumptions, lemmas, and definitions which will be used throughout the article to achieve quasi-synchronization of neural networks under hybrid impulses.

Assumption 5.1. For any $u_1,u_2\in\mathbb{R}$ and $i=1,2,...,n$, the continuous activation functions $f_1(.)$, $f_2(.)$ and $f_3(.)$ satisfy the following conditions:

$$0\leq\frac{f_{1i}(u_1)-f_{1i}(u_2)}{u_1-u_2}\leq l_{f_{1i}},\forall u_1,u_2\in\mathbb{R}\ ,$$

$$0\leq\frac{f_{2i}(u_1)-f_{2i}(u_2)}{u_1-u_2}\leq l_{f_{2i}},\forall u_1,u_2\in\mathbb{R}\ ,$$

$$0\leq\frac{f_{3i}(u_1)-f_{3i}(u_2)}{u_1-u_2}\leq l_{f_{3i}},\forall u_1,u_2\in\mathbb{R}\ ,$$

where $L_{f_1}=\text{diag}(l_{f_{11}},l_{f_{12}},...,l_{f_{1n}})>0$, $L_{f_2}=\text{diag}(l_{f_{21}},l_{f_{22}},...,l_{f_{2n}})>0$, and $L_{f_3}=\text{diag}(l_{f_{31}},l_{f_{32}},...,l_{f_{3n}})>0$ are positive diagonal matrices.

Assumption 5.2. The trajectory of the equation (5.2) is bounded, i.e., there exists $M>0$ such that $\|x(t)\|\leq M,\forall t\in[-\sigma,+\infty]$.

Lemma 5.2.1. For any $X,Y\in\mathbb{R}^n$, we have $2X^TY\leq X^TX+Y^TY$.

Lemma 5.2.2. [109] For any positive constant matrix $P\in\mathbb{R}^{n\times n}$, $P=P^T$, a scalar value $\alpha>0$, and a vector valued function $F:[0,\alpha]\to\mathbb{R}^n$, the following integral inequality is well-defined

$$\left(\int_0^{\alpha} F(s)ds\right)^T P\left(\int_0^{\alpha} F(s)ds\right) \leq \alpha\int_0^{\alpha} F^T(s)PF(s)ds. \tag{5.3}$$

To formulate the problem of this article, we need another state equation of neural network whose connection weights matrices are different from the equation (5.2). Consider the state equation of the slave system as

$$\begin{aligned}\dot{y}(t) =& -\tilde{A}y(t) + \tilde{B}f_1(y(t)) + \tilde{C}f_2(y(t-\sigma_1(t))) + \tilde{D}\int_{t-\sigma_2(t)}^{t} f_3(y(s))ds + U(t) + J,\\ y(t) =& \varphi(t) \in C([-\sigma, 0],\ \mathbb{R}^n),\end{aligned} \tag{5.4}$$

where $y(t) = [y_1(t), y_2(t), ..., y_n(t)]^T$ is the state vector of the networks' neurons with initial condition $y(t) = \varphi(t) = [\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)]^T \in C([-\sigma, 0],\ \mathbb{R}^n)$; $\tilde{A} = diag(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) > 0$; $\tilde{B} = (\tilde{b_{ij}})_{n\times n} \in \mathbb{R}^{n\times n}$, $\tilde{C} = (\tilde{c_{ij}})_{n\times n} \in \mathbb{R}^{n\times n}$ and $\tilde{D} = (\tilde{d_{ij}})_{n\times n} \in \mathbb{R}^{n\times n}$ are the connection weights matrices among the neurons of the network without and with delays; $J = [J_1, J_2, ..., J_n]^T$ is column vector of external input values to the network; $U(t)$ is the hybrid impulsive control function which will be discussed later.

In order to investigate the quasi-synchronization between the systems' network, the error neural network which is defined as $e(t) = y(t) - x(t)$ has been constructed by using equations (5.4) and (5.2). Derivative of the error neural network can be written as

$$\begin{aligned}\dot{e}(t) =& -\tilde{A}e(t) + \tilde{B}\hat{f}_1(e(t)) + \tilde{C}\hat{f}_2(e(t-\sigma_1(t))) + \tilde{D}\int_{t-\sigma_2(t)}^{t} \hat{f}_3(e(s))ds\\ &+ R(x(t), \sigma_1(t), \sigma_2(t)) + U(t),\end{aligned} \tag{5.5}$$

where $\hat{f}_1(e(t)) = f_1(y(t)) - f_1(x(t)), \hat{f}_2(e(t-\sigma_1(t))) = f_2(y(t-\sigma_1(t))) - f_2(x(t-\sigma_1(t))), \hat{f}_3(e(s)) = f_3(y(s)) - f_3(x(s))$, and

$$\begin{aligned} R(x(t),\sigma_1(t),\sigma_2(t)) =& (-\tilde{A}+\bar{A})x(t) + (\tilde{B}-\bar{B})f_1(x(t)) + (\tilde{C}-\bar{C})f_2(x(t-\sigma_1(t))) \\ &+ (\tilde{D}-\bar{D})\int_{t-\sigma_2(t)}^{t} f_3(x(s))ds + (J-I). \end{aligned} \tag{5.6}$$

From Assumption 5.1, we have

$$\|f_1(x(t))\| \leq \sqrt{\lambda_{max}(L_{f_1}^T L_{f_1})}M, \tag{5.7}$$

$$\|f_2(x(t-\sigma_1(t)))\| \leq \sqrt{\lambda_{max}(L_{f_2}^T L_{f_2})}M, \tag{5.8}$$

$$\|f_3(x(s))\| \leq \sqrt{\lambda_{max}(L_{f_3}^T L_{f_3})}M. \tag{5.9}$$

Now, using Assumption 5.2 and taking norm $\|.\|$ on both sides of the equality (5.6), we get

$$\begin{aligned} \|R(x(t),\sigma_1(t),\sigma_2(t))\| \leq& \|\bar{A}-\tilde{A}\|\|x(t)\| + \|\tilde{B}-\bar{B}\|\|f_1(x(t))\| + \|\tilde{C}-\bar{C}\| \\ & \|f_2(x(t-\sigma_2(t)))\| + \|\tilde{D}-\bar{D}\|\int_{t-\sigma_2(t)}^{t}\|f_3(x(s))\|ds + \|J-I\| \\ \leq& (\|\bar{A}\|+\|\tilde{A}\|)M + (\|\tilde{B}\|+\|\bar{B}\|)\times\sqrt{\lambda_{max}(L_{f_1}^T L_{f_1})}M \\ &+ (\|\tilde{C}\|+\|\bar{C}\|)\sqrt{\lambda_{max}(L_{f_2}^T L_{f_2})}M + (\|\tilde{D}\|+\|\bar{D}\|) \\ &\times\sqrt{\lambda_{max}(L_{f_3}^T L_{f_3})}M\sigma_2 + \|J\| + \|I\|. \end{aligned} \tag{5.10}$$

It is clear from the inequality (5.10) that $\|R(x(t),\sigma_1(t),\sigma_2(t))\|$ is bounded $\forall t \geq -\sigma$. Suppose that $\Xi = \sup_{t\geq -\sigma}\|R(x(t),\sigma_1(t),\sigma_2(t))\|$, where $\Xi < \infty$.

For achieving the quasi-synchronization between the systems (5.4) and (5.2), a hybrid impulsive controller is designed as

$$U_i(t) = -\gamma_i e_i(t) + \sum_{k=1}^{\infty} \big(\mu_k e_i(t) - e_i(t)\big)\delta(t - t_k), i = 1, 2, ..., n, \tag{5.11}$$

where $\Gamma = \text{diag}(\gamma_1, \gamma_2, ..., \gamma_n) \geq 0$ and μ_k is impulsive strength at point $t = t_k$ of impulsive sequence $\zeta = \{t_1, t_2, ..., t_k\}$. The impulsive sequence is strictly increasing and unbounded above, i.e., $t_1 < t_2 < t_3 < ... < t_{k-1} < t_k < ...$ and $\lim_{k\to\infty} t_k = +\infty$. The function $\delta(.)$ is the Dirac delta function.

Substituting the control function (5.11) into the error neural network (5.5), we obtain

$$\begin{cases} \dot{e}(t) = & -(\tilde{A} + \Gamma)e(t) + \tilde{B}\hat{f}_1(e(t)) + \tilde{C}\hat{f}_2(e(t - \sigma_1(t))) + \tilde{D}\int_{t-\sigma_2(t)}^{t} \hat{f}_3(e(s))ds \\ & +R(x(t), \sigma_1(t), \sigma_2(t)), \quad t \neq t_k, \\ e(t_k^+) = & \mu_k e(t_k^-), \quad t = t_k, \quad k = 1, 2, ...n. \end{cases} \tag{5.12}$$

The solution of impulsive delayed system (5.12) is assumed to be right-hand continuous and left-hand discontinuous at $t = t_k (k = 1, 2, ..., n)$, i.e., $e(t_k^+) = e(t) \neq e(t_k^-)$. This means that the solution of (5.12) exhibits jump kind of discontinuities from the left side of $t = t_k$. For simplicity, throughout the article we will write $R(x(t), \sigma_1(t), \sigma_2(t))$ as $R(.)$.

Due to the presence of parameter mismatches between the neural networks (5.4) and (5.2), the equilibrium point of the impulsive differential equation (5.12) cannot be zero. Therefore, the complete synchronization is not possible, but it is found that the states of systems can be synchronized up to a small synchronization error bound by using effective controller. This type of synchronization is called quasi-synchronization.

Definition 5.2.1. The synchronization between the systems (5.4) and (5.2) is said to be quasi-synchronization with a small error bound $\bar{e} > 0$ if there exists a compact set $\Delta = \{e \in \mathbb{R}^n : \|e(t)\| \leq \bar{e}\}$ in to which the trajectory of the error system (5.12) converges globally exponentially as time $t \to \infty$. Where $\|(.)\|$ is an Euclidean norm on finite dimensional normed space.

Definition 5.2.2. [110**?**] (Average Impulsive Interval) The average impulsive interval T_a of the impulsive sequence $\zeta = \{t_1, t_2, t_3, ...\}$ is defined as

$$T_a = \lim_{t \to \infty} \frac{t - s}{N_\zeta(t, s)}, \tag{5.13}$$

where $N_\zeta(t, s)$ is the number of impulsive instants of impulsive sequence ζ on the interval (t, s).

Remark 5.2.1. The concept of average impulsive interval was first introduced in [88] to deal with the occurrence of different types of impulses. Most of the results published in the literature based on $\frac{t-s}{T_a} - N_0 \leq N_\zeta(s,t) \leq \frac{t-s}{T_a} + N_0$ are restricted to $T_a < \infty$. the results of this chapter is extended to the case $T_a = \infty$ by using the new concept of average impulsive interval $T_a = \lim_{t\to\infty} \frac{t-s}{N_\zeta(t,s)}$.

Definition 5.2.3. [110**?**] (Average Impulsive Gain) The average impulsive gain of all impulses $\{\mu_1, \mu_2, ...\}$ is designed as

$$\mu = \lim_{t \to \infty} \frac{|\mu_1| + |\mu_2| + ... + |\mu_{N_\zeta(t,s)}|}{N_\zeta(t, s)} > 0, \tag{5.14}$$

where μ_k for all $k = 1, 2, ..., n$ denotes impulsive gain at $t = t_k$.

Lemma 5.2.3. [89] If there exists a positive constant c such that

$$\begin{cases} D^+(u(t)) \leq G(t, u(t), u(t-\sigma_1(t))) + c\int_{t-\sigma_2(t)}^{t} u(s)ds, \ t \neq t_k, \\ u(t_k) \leq I_k(u(t_k^-)), \quad k \in \mathbb{N} \end{cases} \tag{5.15}$$

and

$$\begin{cases} D^+(v(t)) > G(t, v(t), v(t-\sigma_1(t))) + c\int_{t-\sigma_2(t)}^{t} v(s)ds, \ t \neq t_k, \\ v(t_k) \geq I_k(v(t_k^-)), \quad k \in \mathbb{N}, \end{cases} \tag{5.16}$$

where $G(t,\ u,\ \bar{u}_1) : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is non-decreasing in $\bar{u}_1$ for any fixed (t, u) and $I_k(u) : \mathbb{R} \to \mathbb{R}$ is non-decreasing in u. Then $u(t) \leq v(t)$, $\forall t \in [-\sigma, 0]$ implies that $u(t) \leq v(t)$ for $t > 0$, where $D^+(u(t)) = \lim_{h\to 0^+} \sup \frac{u(t+h)-u(t)}{h}$.

5.3 Main Results

In this section, we have studied the influence of hybrid impulses when $T_a < \infty$ on quasi-synchronization of neural networks (5.4) and (5.2) under hybrid impulsive controller (5.11). Furthermore, we have considered the case of $T_a = \infty$ and derived sufficient criteria for quasi-synchronization.

5.3.1 Quasi-synchronization criteria for $T_a < \infty$

Theorem 5.1. *Suppose that the activation functions $f_1(.)$, $f_2(.)$ and $f_3(.)$ satisfy Assumption 5.1. The controlled error neural network (5.12) converges globally exponentially at the convergence rate $\frac{\lambda}{2} > 0$ into a small compact domain $\bar{\Delta}_1$ containing* *the origin if*

$$\xi_1 + 2\frac{\ln \mu}{T_a} + \delta + \xi_2 + \xi_3\sigma_2 < 0, \tag{5.17}$$

where $\xi_1 = \lambda_{max}\Big\{2(-\tilde{A}-\Gamma)+\tilde{B}^T\tilde{B}+\tilde{C}^T\tilde{C}+\tilde{D}^T\tilde{D}+L_{f_1}^T L_{f_1}\Big\}$, $\xi_2 = \lambda_{max}\Big\{L_{f_2}^T L_{f_2}\Big\}$, $\xi_3 = \lambda_{max}\Big\{\sigma_2 L_{f_3}^T L_{f_3}\Big\}$ *and* λ *is a unique solution of equation* $\lambda + 2\frac{\ln\mu}{T_a} + \xi_1 + \delta + \xi_2 e^{\lambda\sigma_1(t)} + \xi_3\frac{e^{\lambda\sigma_2(t)}-1}{\lambda} = 0$. *The compact domain of convergence is*

$$\bar{\Delta}_1 = \left\{ e(t) \in \mathbb{R}^n : \|e(t)\| \leq \frac{\Xi}{\sqrt{-\big(\xi_1 + \frac{2\ln\mu}{T_a}\big) - \xi_2 - \delta - \xi_3\sigma_2}} \right\}.$$

Proof. Suppose the Lyapunov function candidate is constructed as follows:

$$V(e(t)) = e^T(t)e(t), \qquad t \neq t_k, \tag{5.18}$$

where $e(t) = [e_1(t), e_2(t), ..., e_n(t)]^T$ is the state vector of impulsive system (5.12). The Dini derivative of equation (5.18) with respect to t along trajectories of the controlled error neural network (5.12) could be written as

$$D^+(V(e(t))) = 2e^T(t)\dot{e}(t). \tag{5.19}$$

From system (5.12), we have

$$\begin{aligned} D^+(V(e(t))) =& 2e^T(t)(-\tilde{A}-\Gamma)e(t) + 2e^T(t)\tilde{B}\hat{f}_1(e(t)) + 2e^T(t)\tilde{C}\hat{f}_2(e(t-\sigma_1(t))) \\ &+ 2e^T(t)\tilde{D}\int_{t-\sigma_2(t)}^{t}\hat{f}_3(e(s))ds + 2e^T(t)R(.). \end{aligned} \tag{5.20}$$

Using Lemma 5.2.1, we could find these inequalities as

$$2e^T(t)\tilde{B}\hat{f}_1(e(t)) \leq e^T(t)\tilde{B}^T\tilde{B}e(t) + \hat{f}_1^T(e(t))\hat{f}_1(e(t)), \tag{5.21}$$

$$2e^T(t)\tilde{C}\hat{f}_2(e(t-\sigma_1(t))) \leq e^T(t)\tilde{C}^T\tilde{C}e(t) + \hat{f}_2^T(e(t-\sigma_1(t))) \times \hat{f}_2(e(t-\sigma_1(t))), \tag{5.22}$$

$$2e^T(t)\tilde{D}\int_{t-\sigma_2(t)}^{t}\hat{f}_3(e(s))ds \leq e^T(t)\tilde{D}^T\tilde{D}e(t) + \left(\int_{t-\sigma_2(t)}^{t}\hat{f}_3(e(s))ds\right)^T\left(\int_{t-\sigma_2(t)}^{t}\hat{f}_3(e(s))ds\right), \tag{5.23}$$

$$2e^T(t)R(.) \leq e^T(t)e(t) + R^T(.)R(.). \tag{5.24}$$

Substituting the inequalities (5.21), (5.22), (5.23), and (5.24) in the equation (5.20), we get

$$\begin{aligned} D^+(V(e(t))) \leq & 2e^T(t)(-\tilde{A}-\Gamma)e(t) + e^T(t)\tilde{B}^T\tilde{B}e(t) + \hat{f}_1^T(e(t))\hat{f}_1(e(t)) + e^T(t)\tilde{C}^T\tilde{C}e(t) \\ & + \hat{f}_2^T(e(t-\sigma_1(t)))\hat{f}_2(e(t-\sigma_1(t))) + e^T(t)\tilde{D}^T\tilde{D}e(t) \\ & + \left(\int_{t-\sigma_2(t)}^{t}\hat{f}_3(e(s))ds\right)^T\left(\int_{t-\sigma_2(t)}^{t}\hat{f}_3(e(s))ds\right) + e^T(t)e(t) + R^T(.)R(.). \end{aligned} \tag{5.25}$$

From Assumption 5.1 and Lemma 5.2.2, we have

$$\begin{aligned} D^+(V(e(t))) \leq & e^T(t)2(-\tilde{A}-\Gamma)e(t) + e^T(t)\tilde{B}^T\tilde{B}e(t) + e^T(t)L_{f_1}^TL_{f_1}e(t) + e^T(t)\tilde{C}^T\tilde{C}e(t) \\ & + e^T(t-\sigma_1(t))L_{f_2}^TL_{f_2}e(t-\sigma_1(t)) + e^T(t)\tilde{D}^T\tilde{D}e(t) \\ & + \sigma_2L_{f_3}^TL_{f_3}\int_{t-\sigma_2(t)}^{t}e^T(s)e(s)ds + \|R(.)\|^2 \end{aligned} \tag{5.26}$$

$$\begin{aligned} \leq & \lambda_{max}\left\{2(-\tilde{A}-\Gamma) + \tilde{B}^T\tilde{B} + \tilde{C}^T\tilde{C} + \tilde{D}^T\tilde{D} + L_{f_1}^TL_{f_1}\right\}V(e(t)) \\ & + \lambda_{max}\left\{L_{f_2}^TL_{f_2}\right\}V(e(t-\sigma_1(t))) + \lambda_{max}\left\{\sigma_2L_{f_3}^TL_{f_3}\right\}\int_{t-\sigma_2(t)}^{t}V(e(s))ds \\ & + \|R(.)\|^2. \end{aligned} \tag{5.27}$$

Suppose that $\xi_1 = \lambda_{max}\Big\{2(-\tilde{A} - \Gamma) + \tilde{B}^T\tilde{B} + \tilde{C}^T\tilde{C} + \tilde{D}^T\tilde{D} + L_{f_1}^T L_{f_1}\Big\}$, $\xi_2 = \lambda_{max}\Big\{L_{f_2}^T L_{f_2}\Big\}$ and $\xi_3 = \lambda_{max}\Big\{\sigma_2 L_{f_3}^T L_{f_3}\Big\}$. Then, we have

$$D^+(V(e(t))) \leq \xi_1 V(e(t)) + \xi_2 V(e(t - \sigma_1(t))) + \xi_3 \int_{t-\sigma_2(t)}^{t} V(e(s))ds + \|R(.)\|^2. \tag{5.28}$$

For $t = t_k$, the Lyapunov function candidate will be

$$\begin{aligned} V(e(t_k^+)) &= e^T(t_k^+)e(t_k^+) \\ &= \mu_k^2 e^T(t_k^-)e(t_k^-) \\ V(e(t_k^+)) &= \mu_k^2 V(e(t_k^-)). \end{aligned} \tag{5.29}$$

Based on (5.28) and (5.29), the following impulsive system with distributed delay can be derived by using the comparison principle as

$$\begin{cases} \dot{z}(t) = \xi_1 z(t) + \xi_2 z(t - \sigma_1(t)) + \xi_3 \int_{t-\sigma_2(t)}^{t} z(s)ds + \|R(.)\|^2 + \epsilon, & t \neq t_k, \\ z(t_k^+) = \mu_k^2 z(t_k^-), & t = t_k, \\ z(s) = \|\varphi(s) - \phi(s)\|^2, & -\sigma \leq s \leq 0, \end{cases} \tag{5.30}$$

where $z(t)$ is a unique solution for all $\epsilon > 0$. From Lemma 5.2.3, it is concluded that $V(t) \leq z(t)$ for all $t \geq 0$. By employing the extended formula for variation of parameters, $z(t)$ could be written as

$$z(t) = W(t,0)z(0) + \int_0^t W(t,s)\left[\xi_2 z(s - \sigma_1(s)) + \xi_3 \int_{s-\sigma_2(s)}^{s} z(r)dr + \|R(.)\|^2 + \epsilon\right] ds, \quad t \geq 0, \tag{5.31}$$

where $W(t,s)$ is the Cauchy matrix of the following linear impulsive system

$$\begin{cases} \dot{z}(t) = \xi_1 z(t), \ t \neq t_k, \\ z(t_k^+) = \mu_k^2 z(t_k^-), \ t = t_k, k = 1, 2, ..., n. \end{cases}$$

From the Definitions 5.2.2 and 5.2.3, the Cauchy matrix can be calculated as

$$\begin{aligned} W(t,s) =& \mu_1^2 \mu_2^2 ... \mu_{N_\varsigma(t,s)}^2 e^{\xi_1(t-s)} \\ \leq& \Big(\frac{|\mu_1| + |\mu_2| + ... + |\mu_{N_\varsigma(t,s)}|}{N_\varsigma(t,s)} \Big)^{2N_\varsigma(t,s)} e^{\xi_1(t-s)} \\ =& e^{2N_\varsigma(t,s) \ln \frac{|\mu_1|+|\mu_2|+...+|\mu_{N_\varsigma(t,s)}|}{N_\varsigma(t,s)}} e^{\xi_1(t-s)} \\ =& e^{\frac{2\ln \frac{|\mu_1|+|\mu_2|+...+|\mu_{N_\varsigma(t,s)}|}{N_\varsigma(t,s)}}{\frac{t-s}{N_\varsigma(t,s)}}(t-s)} e^{\xi_1(t-s)}. \end{aligned}$$

From (5.13) and (5.14), we have $\lim\limits_{t\to\infty} \frac{2\ln \frac{|\mu_1|+|\mu_2|+...+|\mu_{N_\varsigma(t,s)}|}{N_\varsigma(t,s)}}{\frac{t-s}{N_\varsigma(t,s)}} = \frac{2\ln\mu}{T_a}$. That is, for any $\mu > 1$ or $\mu = 1$ or $\mu < 1$, there exists a sufficiently large $T > 0$ for given $\delta > 0$ such that

$$W(t,s) \leq e^{\Big(\frac{2\ln\mu}{T_a}+\delta+\xi_1\Big)(t-s)}, \quad t > T. \tag{5.32}$$

Substituting inequality (5.32) into the integral equation (5.31), we get

$$\begin{aligned} z(t) \leq& \|\varphi(0) - \phi(0)\|^2 e^{\Big(\frac{2\ln\mu}{T_a}+\delta+\xi_1\Big)t} + \int_0^t e^{\Big(\frac{2\ln\mu}{T_a}+\delta+\xi_1\Big)(t-s)} \\ &\times \Bigg[\xi_2 z(s-\sigma_1(s)) + \xi_3 \int_{s-\sigma_2(s)}^{s} z(r)dr + \|R(.)\|^2 + \epsilon \Bigg] ds \\ \leq& \eta e^{\Big(\frac{2\ln\mu}{T_a}+\delta+\xi_1\Big)t} + \int_0^t e^{\Big(\frac{2\ln\mu}{T_a}+\delta+\xi_1\Big)(t-s)} \times \Bigg[\xi_2 z(s-\sigma_1(s)) \\ &+ \xi_3 \int_{s-\sigma_2(s)}^{s} z(r)dr + \|R(.)\|^2 + \epsilon \Bigg] ds, \end{aligned} \tag{5.33}$$

where $\eta = \sup_{-\sigma \leq s \leq 0} \|\varphi(s) - \phi(s)\|^2$. Define $\psi(\lambda) = \lambda + \frac{2\ln\mu}{T_a} + \delta + \xi_1 + \xi_2 e^{\lambda\sigma_1} + \xi_3 \frac{e^{\lambda\sigma_2}-1}{\lambda}$. It is easy to verify that $\psi(\lambda)$ is continuous function. Furthermore, $\psi(0) = \xi_1 + 2\frac{\ln\mu}{T_a} + \delta + \xi_2 + \xi_3\sigma_2 < 0$, $\psi(+\infty) > 0$ and $\psi'(\lambda) = 1 + \xi_2\sigma_1 e^{\lambda\sigma_1} + \xi_3 \frac{e^{\lambda\sigma_2}(\lambda\sigma_2 - 1) + 1}{\lambda^2} > 0$, this implies that $\psi(\lambda) = 0$ has a unique solution $\lambda > 0$.

Since $\lambda > 0, \epsilon > 0$ and $-\xi_1 - 2\frac{\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2 > 0$, then we have

$$\begin{aligned} z(t) &\leq \sup_{-\sigma \leq t \leq 0} \|\varphi(t) - \phi(t)\|^2, \quad -\sigma \leq t \leq 0 \\ &< \eta e^{-\lambda t} + \frac{\epsilon + \|R(.)\|^2}{-\xi_1 - 2\frac{\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2}, \quad -\sigma \leq t \leq 0. \end{aligned} \tag{5.34}$$

Using the mathematical principle: proof by contradiction, we will proceed to show the inequality (5.34) is true for all $t > 0$. That is,

$$z(t) < \eta e^{-\lambda t} + \frac{\epsilon + \|R(.)\|^2}{-\xi_1 - 2\frac{\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2}, \qquad t > 0. \tag{5.35}$$

Let the inequality (5.35) does not hold $\forall t > 0$, then there exist $t^* > 0$ such that

$$z(t) \geq \eta e^{-\lambda t^*} + \frac{\epsilon + \|R(.)\|^2}{-\xi_1 - 2\frac{\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2}. \tag{5.36}$$

But the inequality (5.35) still holds as

$$z(t) < \eta e^{-\lambda t} + \frac{\epsilon + \|R(.)\|^2}{-\xi_1 - 2\frac{\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2}, \quad t < t^*. \tag{5.37}$$

For the sake of simplicity assume $\bar{\alpha} = \frac{2\ln\mu}{T_a} + \delta + \xi_1$ and $\Omega = -\xi_1 - 2\frac{\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2$. Further, considering the inequalities (5.33), (5.37) and equation $\psi(\lambda) = 0$, we have

$$\begin{aligned}
z(t^*) <&\eta e^{\bar{\alpha}t^*} + \int_0^{t^*} e^{\bar{\alpha}(t^*-s)} \Bigg[\xi_2\Big(\eta e^{-\lambda(s-\sigma_1(s))} + \frac{\epsilon + \|R(.)\|^2}{\Omega}\Big) \\
&+ \xi_3 \int_{s-\sigma_2(s)}^{s} \Big(\eta e^{-\lambda r} + \frac{\epsilon + \|R(.)\|^2}{\Omega}\Big) dr + \|R(.)\|^2 + \epsilon\Bigg] ds \\
<&e^{\bar{\alpha}t^*}\Bigg[\eta + \frac{\epsilon + \|R(.)\|^2}{\Omega} + \xi_2\eta e^{\sigma_1}\int_0^{t^*} e^{-(\bar{\alpha}+\lambda)s} ds + \frac{\epsilon + \|R(.)\|^2}{M}\int_0^{t^*} \xi_2 e^{-\bar{\alpha}s} ds \\
&+ \xi_3\eta \int_0^{t^*} e^{-\bar{\alpha}s}\int_{s-\sigma_2(s)}^{s} e^{-\lambda r} dr ds + \frac{\xi_3(\epsilon + \|R(.)\|^2)}{\Omega}\int_0^{t^*} e^{-\bar{\alpha}s}\int_{s-\sigma_2(s)}^{s} dr ds \\
&+ (\|R(.)\|^2 + \epsilon)\int_0^{t^*} e^{-\bar{\alpha}s} ds\Bigg] \\
\leq&e^{\bar{\alpha}t^*}\Bigg[\eta + \frac{\epsilon + \|R(.)\|^2}{\Omega} + \Big(\xi_2\eta e^{\sigma_1} + \xi_3\eta\frac{e^{\lambda\sigma_2} - 1}{\lambda}\Big)\Big(\frac{1}{\bar{\alpha}+\lambda} - \frac{e^{(\bar{\alpha}+\lambda)t^*}}{\bar{\alpha}+\lambda}\Big) \\
&+ \frac{\epsilon + \|R(.)\|^2}{\Omega}(\xi_2 + \xi_3\sigma_2 + \Omega)\Big(\frac{1}{\bar{\alpha}} - \frac{e^{-\bar{\alpha}t^*}}{\bar{\alpha}}\Big)\Bigg] \\
\leq&\eta e^{\bar{\alpha}t^*} + \eta e^{-\lambda t^*} + \frac{\epsilon + \|R(.)\|^2}{\Omega} - e^{\bar{\alpha}t^*} \\
<&\eta e^{-\lambda t^*} + \frac{\epsilon + \|R(.)\|^2}{-\xi_1 - \frac{2\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2}.
\end{aligned} \tag{5.38}$$

It is obvious that the inequality (5.38) is contradicting the assumption (5.36). Thus, the inequality (5.38) is true for all $t > 0$. Let $\epsilon \to 0$, then we have

$$V(e(t)) = e^T(t)e(t) \leq z(t) < \eta e^{-\lambda t} + \frac{\|R(.)\|^2}{-\xi_1 - \frac{2\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2}. \tag{5.39}$$

Further, the inequality (5.39) could be written as

$$\|e(t)\|^2 \leq \eta e^{-\lambda t} + \frac{\|R(.)\|^2}{-\xi_1 - \frac{2\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2}, \tag{5.40}$$

which implies

$$\|e(t)\| \leq \sqrt{\eta}e^{-\frac{\lambda}{2}t} + \frac{\Xi}{\sqrt{-\xi_1 - \frac{2\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2}}. \tag{5.41}$$

It is clear from (5.41) that trajectory of the impulsive system (5.12) converges globally exponentially at the convergence rate $\frac{\lambda}{2}$ into a small compact set $\bar{\Delta}_1 = \left\{e(t) \in \mathbb{R}^n : \|e(t)\| \leq \bar{e} = \frac{\Xi}{\sqrt{-\xi_1 - \frac{2\ln\mu}{T_a} - \delta - \xi_2 - \xi_3\sigma_2}}\right\}$ as $t \to \infty$. That is, the quasi-synchronization between the systems (5.2) and (5.4) is achieved with a small synchronization error bound $\bar{e}$ under the impulsive control (5.11) with the average impulsive interval $T_a < \infty$. This completes the proof. □

Remark 5.3.1. The time delays in neural network affect the impulsive synchronization negatively. If we do not consider distributed time-varying delay in neural networks (5.2) and (5.4), then for any fixed $\delta > 0$, the inequality (5.17) becomes $\xi_1 + 2\frac{\ln\mu}{T_a} + \delta + \xi_2 < 0$. Furthermore, in the case of delay free systems, the inequality (5.17) could be written as $\xi_1 + 2\frac{\ln\mu}{T_a} + \delta < 0$. Compared the delay free case with the delay case (5.17), it is seen that the less value of coupling strength γ_i in ξ_1 and large value of average impulsive interval T_a are required, that means the cost of control in delay free case is lesser than the delayed case.

5.3.2 Quasi-synchronization criteria for $T_a = \infty$

In this subsection, we will study the case $T_a = \infty$ on quasi-synchronization of neural networks. In fact, this situation will arise when $N_\zeta(t,s) = [\sqrt{t-s}]$ in (5.13) which implies that the impulses occur all the time but their density of occurrence decreases as time goes to infinity. The study of this case on quasi-synchronization is the first time in literature.

Theorem 5.2. *Suppose that Assumption 5.1 holds and $T_a = \infty$. For the constants $\xi_1 = \lambda_{max}\Big\{2(-\tilde{A}-\Gamma)+\tilde{B}^T\tilde{B}+\tilde{C}^T\tilde{C}+\tilde{D}^T\tilde{D}+L_{f_1}^TL_{f_1}\Big\}$, $\xi_2 = \lambda_{max}\Big\{L_{f_2}^TL_{f_2}\Big\}$, and $\xi_3 = \lambda_{max}\Big\{\sigma_2 L_{f_3}^TL_{f_3}\Big\}$, we have two cases as follows:*
Case 1: For any $\delta' > 0$ and $\mu > 1$, the controlled error neural network (5.12) globally exponentially converges at the convergence rate $\frac{\lambda'}{2} > 0$ into a small compact domain containing the origin $\bar{\Delta}_2 = \Big\{e(t) \in \mathbb{R}^n : \|e(t)\| \leq \frac{\Xi}{\sqrt{-\frac{\delta'}{2}-\xi_1-\xi_2-\xi_3\sigma_2}}\Big\}$ if

$$\frac{\delta'}{2}+\xi_1+\xi_2+\xi_3\sigma_2 < 0, \tag{5.42}$$

where λ' is a unique root of the equation $\lambda' + \frac{\delta'}{2}+\xi_1+\xi_2 e^{\lambda'\sigma_1}+\xi_3\frac{e^{\lambda'\sigma_2}-1}{\lambda'} = 0$.
Case 2: For $0 < \mu \leq 1$, the solution of the controlled impulsive system (5.12) globally exponentially converges at the convergence rate $\frac{\lambda''}{2} > 0$ into a small compact domain containing the origin $\bar{\Delta}_3 = \Big\{e(t) \in \mathbb{R}^n : \|e(t)\| \leq \frac{\Xi}{\sqrt{-\xi_1-\xi_2-\xi_3\sigma_2}}\Big\}$ if

$$\xi_1+\xi_2+\xi_3\sigma_2 < 0, \tag{5.43}$$

where λ'' is a unique root of the equation $\lambda''+\xi_1+\xi_2 e^{\lambda''\sigma_1}+\xi_3\frac{e^{\lambda''\sigma_2}-1}{\lambda''} = 0$.
Thus, in both cases, the quasi-synchronization between the neural networks (5.2.2) and (5.4) is achieved with a small synchronization error bounds.

Proof. Following the similar proof as in Theorem 5.1, we get

$$\begin{aligned} W(t,s) =& \mu_1^2\mu_2^2...\mu_{N_\varsigma(t,s)}^2 e^{\xi_1(t-s)} \\ \leq& \Big(\frac{|\mu_1|+|\mu_2|+...+|\mu_{N_\varsigma(t,s)}|}{N_\varsigma(t,s)}\Big)^{2N_\varsigma(t,s)} e^{\xi_1(t-s)} \\ =& e^{2N_\varsigma(t,s)\ln\frac{|\mu_1|+|\mu_2|+...+|\mu_{N_\varsigma(t,s)}|}{N_\varsigma(t,s)}} e^{\xi_1(t-s)} \\ =& e^{\frac{2\ln\frac{|\mu_1|+|\mu_2|+...+|\mu_{N_\varsigma(t,s)}|}{N_\varsigma(t,s)}}{\frac{t-s}{N_\varsigma(t-s)}}(t-s)} e^{\xi_1(t-s)}. \end{aligned} \tag{5.44}$$

Case 1: From Definition 5.2.3 and $T_a = \infty$, when $\mu > 1$, then we have

$$\lim_{t\to\infty} \frac{2\ln \frac{|\mu_1|+|\mu_2|+...+|\mu_{N_\varsigma(t,s)}|}{N_\varsigma(t,s)}}{\frac{t-s}{N_\varsigma(t-s)}} = 0,$$

it means that for any $\delta' > 0$ there exists a sufficiently large $T > 0$ such that

$$\frac{2\ln \frac{|\mu_1|+|\mu_2|+...+|\mu_{N_\varsigma(t,s)}|}{N_\varsigma(t,s)}}{\frac{t-s}{N_\varsigma(t-s)}} \leq \frac{\delta'}{2}, \qquad t > T. \tag{5.45}$$

Using inequality (5.45) in (5.44), we get

$$W(t,s) \leq e^{\left(\frac{\delta'}{2}+\xi_1\right)(t-s)}. \tag{5.46}$$

Substituting inequality (5.46) into equation (5.31), we get

$$z(t) \leq e^{\left(\frac{\delta'}{2}+\xi_1\right)t}\eta + \int_0^t e^{\left(\frac{\delta'}{2}+\xi_1\right)(t-s)}\Bigg[\xi_2 z(s-\sigma_1(s)) + \xi_3\int_{s-\sigma_2(s)}^{s} z(r)dr + \|R(.)\|^2 + \epsilon\Bigg]ds. \tag{5.47}$$

Define a continuous function $\bar{\psi}(\lambda') = \lambda' + \frac{\delta'}{2} + \xi_1 + \xi_2 e^{\lambda'\sigma_1} + \xi_3 \frac{e^{\lambda'\sigma_2-1}}{\lambda'}$. From (5.42), we have $\bar{\psi}(0) < 0$. It can be easily verified that $\bar{\psi}(+\infty) > 0$ and $\bar{\psi}'(\lambda') > 0$ for all $\lambda' > 0$. Thus, $\bar{\psi}(\lambda') = 0$ must possess a unique positive real root λ'.

From now, following the similar proof as in Theorem 5.1, we obtain

$$\|e(t)\| \leq \sqrt{\eta}e^{-\frac{\lambda'}{2}t} + \frac{\Xi}{\sqrt{-\xi_1 - \frac{\delta'}{2} - \xi_2 - \xi_3\sigma_2}}, \quad t \geq 0. \tag{5.48}$$

It is concluded from inequality (5.48) that the solution of the impulsive system

(5.12) is converging globally exponentially into a small compact domain containing the origin $\bar{\Delta}_2 = \left\{ e(t) \in \mathbb{R}^n : \|e(t)\| \leq \frac{\Xi}{\sqrt{-\xi_1 - \frac{\delta'}{2} - \xi_2 - \xi_3\sigma_2}} \right\}$ with the convergence rate $\frac{\lambda'}{2} > 0$ as $t \to \infty$. The proof of the first case is completed.

Case 2: For $0 < \mu \leq 1$ and $T_a = \infty$, from inequality (5.44), we can obtain

$$W(t,s) \leq e^{\xi_1(t-s)}. \tag{5.49}$$

Substituting the inequality (5.49) in equation (5.31), we get

$$z(t) \leq e^{\xi_1 t}\eta + \int_0^t e^{\xi_1(t-s)} \left[\xi_2 z(s-\sigma_1(s)) + \xi_3 \int_{s-\sigma_2(s)}^{s} z(r)dr + \|R(.)\|^2 + \epsilon \right] ds. \tag{5.50}$$

Similar to Case 1, define a continuous function $\tilde{\psi}(\lambda'') = \lambda'' + \xi_1 + \xi_2 e^{\lambda''\sigma_1} + \xi_3 \frac{e^{\lambda''\sigma_2 - 1}}{\lambda''}$. From (5.43), $\tilde{\psi}(0) < 0$. It is easy to verify that $\tilde{\psi}(+\infty) > 0$ and its derivative $\tilde{\psi}'(\lambda'') > 0$ for all $\lambda'' > 0$. This implies that the equation $\tilde{\psi}(\lambda'') = 0$ has a unique positive real root λ''.

Again, if we follow the similar proof as in Theorem 5.1, then we get

$$\|e(t)\| \leq \sqrt{\eta} e^{-\frac{\lambda''}{2}t} + \frac{\Xi}{\sqrt{-\xi_1 - \xi_2 - \xi_3\sigma_2}}, \quad t \geq 0. \tag{5.51}$$

As $t \to \infty$, it is clear from (5.51) that the trajectory of error neural network (5.12) is converging globally exponentially into a small compact domain containing the origin $\bar{\Delta}_2 = \left\{ e(t) \in \mathbb{R}^n : \|e(t)\| \leq \frac{\Xi}{\sqrt{-\xi_1 - \xi_2 - \xi_3\sigma_2}} \right\}$ at the convergence rate $\frac{\lambda''}{2} > 0$. This completes the proof of the second case. □

Corollary 5.3.1. Suppose that the neural networks (5.2) and (5.4) are identical, i.e., $\bar{A} = \tilde{A}, \bar{B} = \tilde{B}, \bar{C} = \tilde{C}, \bar{D} = \tilde{D}$ and $I = J$, then from equation (5.6), we get $R(.) = 0$ implies $\Xi = 0$. Under Assumption 5.1 and $T_a < \infty$, the trajectory of the error neural network (5.12) converges globally exponentially to zero at the rate of convergence $\frac{\lambda}{2} > 0$ if

$$\xi_1 + 2\frac{\ln \mu}{T_a} + \delta + \xi_2 + \xi_3\sigma_2 < 0, \tag{5.52}$$

where $\delta > 0$, $\xi_1 = \lambda_{max}\Big\{2(-\tilde{A} - \Gamma) + \tilde{B}^T\tilde{B} + \tilde{C}^T\tilde{C} + \tilde{D}^T\tilde{D} + L_{f_1}^T L_{f_1}\Big\}$, $\xi_2 = \lambda_{max}\Big\{L_{f_2}^T L_{f_2}\Big\}$, $\xi_3 = \lambda_{max}\Big\{\sigma_2 L_{f_3}^T L_{f_3}\Big\}$ and λ is a unique solution of the equation $\lambda + 2\frac{\ln \mu}{T_a} + \xi_1 + \delta + \xi_2 e^{\lambda\sigma_1(t)} + \xi_3\frac{e^{\lambda\sigma_2(t)}-1}{\lambda} = 0$. That is, the complete synchronization between the neural networks (5.2.2) and (5.4) is achieved under the hybrid impulsive controller (5.11).

Corollary 5.3.2. Suppose that Assumption 5.1, $T_a = \infty$ and $R(.) = 0$ hold.
Case 1: For any $\delta^{'} > 0$ and $\mu > 1$, the solution of the impulsive system (5.12) converges globally exponentially to zero at the rate of convergence $\frac{\lambda^{'}}{2} > 0$ if

$$\frac{\delta^{'}}{2} + \xi_1 + \xi_2 + \xi_3\sigma_2 < 0, \tag{5.53}$$

where $\lambda^{'}$ is a unique root of the equation $\lambda^{'} + \frac{\delta^{'}}{2} + \xi_1 + \xi_2 e^{\lambda^{'}\sigma_1} + \xi_3\frac{e^{\lambda^{'}\sigma_2 - 1}}{\lambda^{'}} = 0$.
Case 2: For $0 < \mu \leq 1$, the solution of the impulsive system (5.12) converges globally exponentially to zero at the convergence rate $\frac{\lambda^{''}}{2} > 0$ if

$$\xi_1 + \xi_2 + \xi_3\sigma_2 < 0, \tag{5.54}$$

where $\lambda^{''}$ is a unique root of the equation $\lambda^{''} + \xi_1 + \xi_2 e^{\lambda^{''}\sigma_1} + \xi_3\frac{e^{\lambda^{''}\sigma_2 - 1}}{\lambda^{''}} = 0$ and $\xi_1 = \lambda_{max}\Big\{2(-\tilde{A} - \Gamma) + \tilde{B}^T\tilde{B} + \tilde{C}^T\tilde{C} + \tilde{D}^T\tilde{D} + L_{f_1}^T L_{f_1}\Big\}$, $\xi_2 = \lambda_{max}\Big\{L_{f_2}^T L_{f_2}\Big\}$, $\xi_3 = \lambda_{max}\Big\{\sigma_2 L_{f_3}^T L_{f_3}\Big\}$ are constants. In both cases, the complete synchronization between the systems (5.2.2) and (5.4) is achieved under the hybrid impulsive controller (5.11) with hybrid impulses which occur infinitely but sparsely.

Remark 5.3.2. It is worth to observe that, in Theorem 5.2, the convergence rates $\frac{\lambda'}{2}$ and $\frac{\lambda''}{2}$ are independent of average impulsive interval T_a. Theoretically, we have shown that the infinite but sparse occurrence of impulses does not have a negative impact on the quasi-synchronization of different neural networks with mixed time-varying delays. Furthermore, since $T_a = \infty$, that is the length of the impulsive interval is very large, which means the control cost for the synchronization of neural networks is very less. Therefore, results obtained in Theorem 5.2 are effective for reducing the cost of control.

Remark 5.3.3. In Theorem 5.2, the inequalities (5.42) and (5.43) are delay dependent sufficient criteria to achieve quasi-synchronization. If σ_2 is very large, then we can adjust the value of coupling strength $\gamma_i > 0$ such that the inequalities $\frac{\delta'}{2} + \xi_1 + \xi_2 + \xi_3\sigma_2 < 0$ and $\xi_1 + \xi_2 + \xi_3\sigma_2 < 0$ hold. That is, the linear feedback term $-\Gamma e(t)$ in the controller (5.11) is playing a meaningful role for the case $T_a = \infty$. On the other hand, the impulsive control without linear feedback term as designed in the previous articles [111, 112, 113, 114, 115] will not work when time-delay is very large. Therefore, the controller (5.11) is perfectly designed to deal with the case $T_a = \infty$.

Remark 5.3.4. In Theorem 5.1 and Theorem 5.2, one can observe that the synchronization error bounds can be reduced by increasing the coupling strength $\gamma_i > 0$ for $i = 1, 2, ...n$ (provided other parameters are given) because ξ_1 depends on the diagonal matrix Γ. Moreover, assume that γ_i and other parameters except $\mu > 0$ are fixed then the synchronization error bound in Theorem 5.1 can be reduced by taking small value of the average impulsive gain μ. In contrast to Theorem 5.1, the radii of the compact spheres $\bar{\Delta}_2$ and $\bar{\Delta}_3$ in Theorem 5.2,i.e., the synchronization error bounds can not be reduced if the coupling strength γ_i is fixed. One more thing is worth mentioning that the synchronization error bounds obtained in the theorems can not be optimized to their minimum values, for the proof we refer to [116].

5.4 Numerical simulation and discussions

In this section, two examples have been considered to validate the theoretical results obtained in Theorem 5.1 and 5.2, where sufficient criteria of the quasi-synchronization are derived for $T_a < \infty$ and $T_a = \infty$.

Example 5.4.1. Consider the neural network (5.2) with the parameters given as

$$\bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \bar{B} = \begin{pmatrix} 2 & -0.1 \\ -5 & 4.5 \end{pmatrix},$$
$$\bar{C} = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{pmatrix}, \qquad \bar{D} = \begin{pmatrix} -0.3 & 0.1 \\ 0.1 & -0.2 \end{pmatrix},$$

$f_1(x(t)) = f_2(x(t)) = f_3(x(t)) =[\tanh x_1(t),\ \tanh x_2(t)]^T$ with $L_{f_1} = L_{f_2} = L_{f_3} = \mathrm{diag}(1,1)$; $\sigma_1 = 1,\ \sigma_2 = 0.2$, and the external input vector $I = [0,0]^T$. The phase portrait of chaotic attractors of the neural network (5.2) for its initial condition $x(s) = [0.01,\ 0.1]^T, -\sigma \leq s \leq 0$ is shown in Fig. (5.2(a)).

Further, the parameters of the neural network (5.4) are given as

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 1.8 & -0.15 \\ -5.2 & 3.5 \end{pmatrix},$$
$$\tilde{C} = \begin{pmatrix} -1.7 & -0.12 \\ -0.26 & -2.5 \end{pmatrix}, \qquad \tilde{D} = \begin{pmatrix} 0.6 & 0.15 \\ -2 & -0.12 \end{pmatrix}.$$

The activation functions and the time delays are $f_1(y(t)) = f_2(y(t)) = f_3(y(t)) = [\tanh y_1(t), \tanh y_2(t)]^T$ and $\sigma_1 = 1,\ \sigma_2 = 0.2$, respectively. The chaotic attractors of the neural network (5.4) for its initial condition $y(s) = [0.02,\ 0.01]^T, -\sigma \leq s \leq 0$ and the external input vector $J = [0,0]^T$ can be observed from Fig.(5.2(b)). First we will

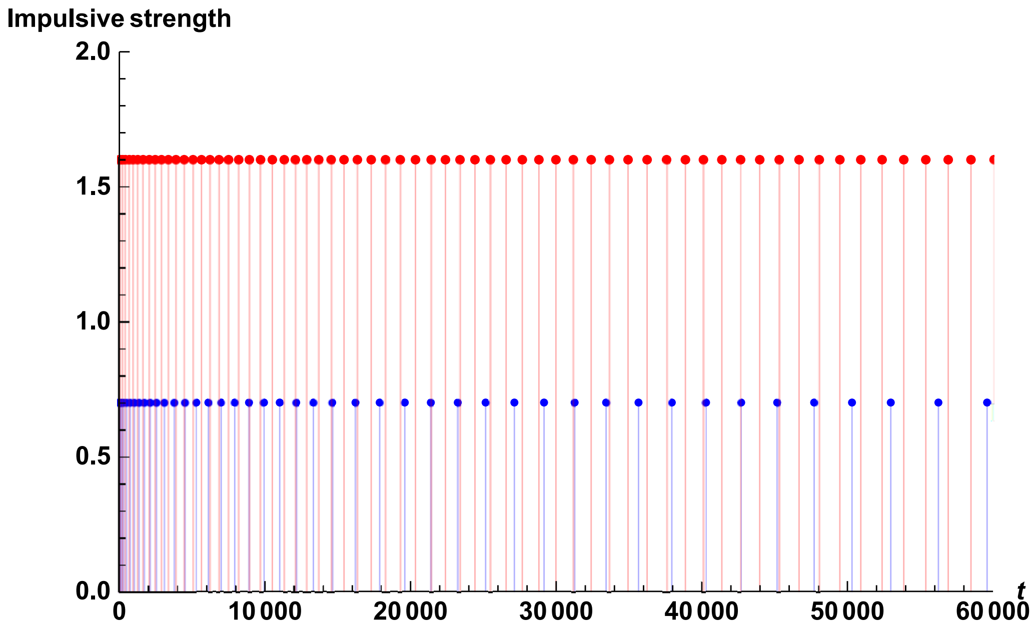
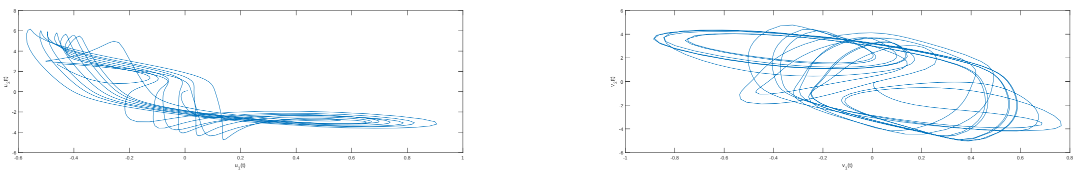
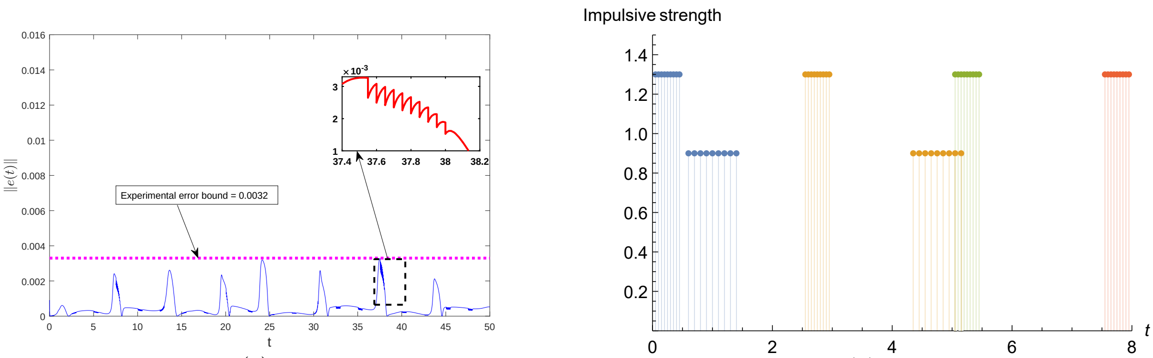


FIGURE 5.1: Phase portrait of hybrid impulses with $T_a = \infty$ for $N_\varsigma(t,s) = [\sqrt[3]{t-s}]$ in Example 5.4.2.



(a) (b)

FIGURE 5.2: Chaotic attractors of equations (5.2) and (5.4) are shown in (a) and (b), respectively.



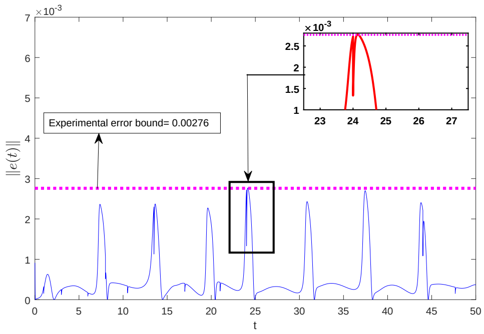
(a) (b)

FIGURE 5.3: Hybrid impulsive sequence with $\mu = 1.1$ and the corresponding time evolution of controlled error neural network (5.12) for $T_a < \infty$ are shown in (a) and (b), respectively.

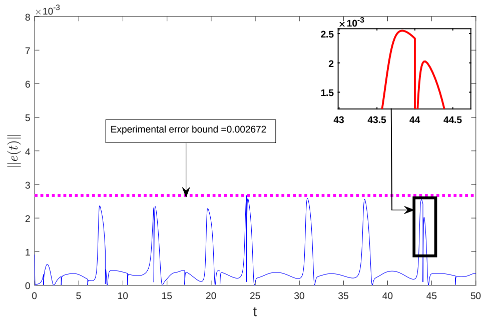
verify the results for the case $T_a < \infty$. For this purpose, set the coupling strength $\gamma_i = 30,\ i = 1, 2$. Using the given data of the parameters, we get $\xi_1 = -13.18,\ \xi_2 = 1$ and $\xi_3 = 0.2$. Let the time-varying impulses are $\mu_1 = 1.3$ and $\mu_2 = 0.9$ with the average impulsive gain $\mu = 1.1$ and the average impulsive interval $T_a = 0.25$. Hence, for fixed value $\delta = 0.5$, the inequality $\xi_1 + 2\frac{\ln \mu}{T_a} + \delta + \xi_2 + \xi_3\sigma_2 = -10.88 < 0$ holds which assured that the trajectories of the error neural network (5.12) converge globally exponentially at the rate of convergence $\frac{\lambda}{2} = 1.135$ into a small compact domain of convergence $\bar{\Delta}_1 = \left\{e(t) \in \mathbb{R}^n : \|e(t)\| \leq 1.96\right\}$. The phase portrait of hybrid impulsive signal with average impulsive interval $T_a = 0.25$ is depicted in Fig.(5.3(a)) and the corresponding error trajectory of system (5.12) is shown in Fig. (5.3(b)) with the experimental error bound 0.0032 which is less than the theoretical error bound 1.96. This implies that the quasi-synchronization between the neural networks (5.2) and (5.4) could be achieved with a given synchronization error bound under the hybrid impulsive controller (5.11). This verifies the Theorem 5.1.

Example 5.4.2. In this example, considering the same systems as in Example 5.4.1, we will verify the results obtained in Theorem 5.2. Let $\mu_1 = 1.6$ and $\mu_2 = 0.7$ are the impulsive strengths at different instants of the impulsive sequence $\zeta = \{t_1, t_2, ..., t_k\}$ with average impulsive gain $\mu = 1.15 > 1$ and the average impulsive interval $T_a = \infty$. For fixed $\delta' = 1.5$, the inequality $\frac{\delta'}{2} + \xi_1 + \xi_2 + \xi_3\sigma_2 = -11.39 < 0$ holds which means that the trajectory of the impulsive system (5.12) converges globally exponentially at the convergence rate $\frac{\lambda'}{2} = 1.15$ in to a small compact domain of convergence $\bar{\Delta}_2 = \left\{e(t) \in \mathbb{R}^n : \|e(t)\| \leq 1.91\right\}$. Fig.5.1 depicts the distribution of hybrid impulses with $T_a = \infty$ for $N_\zeta(t, s) = [\sqrt[3]{t - s}]$ and the corresponding time evolution of the controlled error neural network (5.12) with experimental error bound 0.00276 is shown in Fig.(5.4(a)).

For the Case 2 of Theorem 5.2, set the strengths of the hybrid impulses as $\mu_1 = 0.2$



(a)



(b)

FIGURE 5.4: Time-evolution of error neural network (5.12) with $\mu = 1.15$ and $\mu = 0.35$ for $T_a = \infty$ are shown in (a) and (b), respectively.

and $\mu_2 = 0.5$ with average impulsive gain $\mu = 0.35 < 1$ and average impulsive interval $T_a = \infty$, then the inequality $\xi_1 + \xi_2 + \xi_3\sigma_2 = -12.14 < 0$ holds which implies that the trajectory of error neural network (5.12) converges globally exponentially at the convergence rate $\frac{\lambda''}{2} = 1.19$ into a small compact domain of convergence $\bar{\Delta}_3 = \{e(t) \in \mathbb{R}^n : \|e(t)\| \leq 1.86\}$. For the infinite but sparse occurrence of impulses as in Fig. 5.1, the error trajectory of the system (5.12) is fluctuating within a small compact ball of radius 0.002672 which can be observed from the Fig. (5.4(b)). Therefore, it is verified that in both cases of Theorem 5.2, the quasi-synchronization between neural networks (5.2) and (5.4) could be achieved within a given synchronization error bound under the controller (5.11) with hybrid impulses and infinite average impulsive interval $T_a = \infty$.

5.5 Conclusion

In this chapter, the effects of the hybrid impulses have been deeply investigated for the quasi-synchronization of neural networks with mixed time-varying delays and parameter mismatches. A hybrid impulsive controller with feedback term has been designed so that the quasi-synchronization could easily be achieved, no matter how the simultaneous existence of synchronizing and desynchronizing impulses is affecting the networks' synchronization. By employing the new concept of average impulsive interval, the average impulsive gain, some mathematical techniques, and the extended comparison principle for delayed impulsive system combined with the formula of variation of parameters, delay-dependent quasi-synchronization criteria have been obtained for the neural networks with mixed time-varying delays when $T_a < \infty$ and $T_a = \infty$. Meanwhile, the compact sets containing the origin have been constructed for the cases $T_a < \infty$ and $T_a = \infty$ into which the impulsive error system converges globally exponentially with the exponential convergence rates. Finally, two examples are considered to validate the effectiveness of our theoretical results.

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