Chapter 4

Exponential stability of inertial BAM neural network with time-varying impulses

4.1 Introduction

This chapter investigates the global exponential stability of inertial neural network with mixed time-varying delays under the influence of hybrid impulsive sequence. Inertia in neural networks was first proposed by Babcock and Westervelt [96] in 1986. They had added inertia via an inductor in an RLC circuit which connect the output of a neuron to its input, and analyzed the dynamical behaviors of the network. In consequence of the analysis, they found that the addition of inertial term to the rate equation of a simple electronic neural network consisting of one or two neurons may exhibit complex dynamic behaviors, such as spontaneous oscillation, instability, ringing about the equilibrium point, and chaotic response to a periodic drive. In engineering applications, systems with chaotic nature or complex dynamic behaviors are required. As for instance, in secure communication, chaotic systems increase the security strength of signals passing from a transmitter to a receiver. Therefore, the stability problem of inertial neural networks has been paid much attention to researchers [97, 98, 99, 100, 101, 102, 103].

In previous chapter of this thesis, we have considered fixed impulses for all impulsive points of the sequence. Generally, impulses may vary at different impulsive points of the sequence. Thus, the primary purpose of this chapter is to study the effects of hybrid impulsive sequence on global exponential stability of inertial BAM neural networks.

4.2 Model description and preliminaries

Let us consider the state equation of inertial BAM neural network with mixed time-varying delays as

$$\frac{d^2u_i(t)}{dt^2} = -\gamma_i\frac{du_i(t)}{dt} - \beta_i u_i(t) + \sum_{j=1}^{n} a_{ij}g_j(u_j(t)) + \sum_{j=1}^{n} b_{ij}g_j(u_j(t-\sigma_j(t))) + \sum_{j=1}^{n} d_{ij}\int_{t-\tau_j(t)}^{t} g_j(u_j(s))ds + I_i(t), \tag{4.1}$$

where $i = 1, 2, ..., n$ and the second derivative of the state variable $u_i(t)$ of the $i-$th neuron added in system (4.1) is called the inertial term; γ_i and β_i are positive constants; a_{ij}, b_{ij} and d_{ij} are the connection weights between the $j-$th and $i-$th neurons in the neural network at time t, $t - \sigma_j(t)$ and $t - \tau_j(t)$, respectively; the activation function of the $j-$th neuron in the network is denoted by $g_j(.)$ satisfying $g_j(0) = 0$; $\sigma_j(t)$ and $\tau_j(t)$ are the bounded discrete and distributed time-varying delays, respectively, i.e., $0 \leq \sigma_j(t) \leq \sigma$ and $0 \leq \tau_j(t) \leq \tau$ for all $j = 1, 2, ..., n$ with $\bar{\tau} = \max\{\sigma, \tau\}$; $I_i(t)$ denotes the external input to the $i-$th neuron of the network at time t satisfying $|I_i(t)| \leq I_i$ for all $i = 1, 2, .., n$.

The initial condition of the inertial BAM neural network (4.1) is given as

$$u_i(s) = \psi_i(s), \quad \frac{du_i(s)}{dt} = \varphi_i(s), \ \forall s \in [-\bar{\tau}, 0], \quad i = 1, 2, .., n,$$

where $\psi_i(s)$ and $\varphi_i(s)$ are continuous functions from $[-\bar{\tau}, 0]$ to $\mathbb{R}$.

Consider the following variable transformation

$$v_i(t) = \frac{du_i(t)}{dt} + \zeta_i u_i(t), \quad i = 1, 2, ..., n, \tag{4.2}$$

then the second order integro-differential form of neural network (4.1) can be written as

$$\begin{cases} \frac{du_i(t)}{dt} = & -\zeta_i u_i(t) + v_i(t), \\ \frac{dv_i(t)}{dt} = & -\alpha_i v_i(t) + c_i u_i(t) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(u_j(t - \sigma_j(t))) \\ & + \sum_{j=1}^{n} d_{ij} \int_{t-\tau_j(t)}^{t} g_j(u_j(s))ds + I_i(t), \end{cases} \tag{4.3}$$

together with the initial conditions

$$u_i(s) = \psi_i(s), \quad v_i(s) = \varphi_i(s) + \zeta_i \psi_i(s), \quad \forall s \in [-\bar{\tau}, 0], \; i = 1, 2, ..., n, \tag{4.4}$$

where $\alpha_i = \gamma_i - \zeta_i$, and $c_i = \alpha_i \zeta_i - \beta_i$.

Remark 4.2.1. For the sake of simplicity, we have considered one variable ζ_i in transformation (4.2). By introducing two variables in (4.2) as $v_i(t) = \xi_i \dot{u}_i(t) + \zeta_i u_i(t)$, we could get less conservative results.

In a compact form, system (4.3) can be written as

$$\begin{cases} \frac{du(t)}{dt} = & -\zeta u(t) + v(t), \\ \frac{dv(t)}{dt} = & -\Gamma v(t) + Cu(t) + Ag(u(t)) + Bg(u(t - \sigma(t))) + D \int_{t-\tau(t)}^{t} g(u(s))ds + I(t), \end{cases} \tag{4.5}$$

where $u(t) = [u_1(t), u_2(t), ..., u_n(t)]^T$ and $v(t) = [v_1(t), v_2(t), ..., v_n(t)]^T$ are state vectors, $\zeta = diag\{\zeta_1, \zeta_2, ..., \zeta_n\}$, $\Gamma = diag\{\alpha_1, \alpha_2, ..., \alpha_n\}$, $C = diag\{c_1, c_2, ..., c_n\}$, $A = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}$, $B = [b_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}$, and $D = [d_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}$, $g(u(t)) =$

$[u_1(t), u_2(t), ..., u_n(t)]^T$, $\sigma(t) = \{\sigma_j(t)\}$ and $\tau(t) = \{\tau_j(t)\}$ for $j = 1, 2, ..., n$ and $I(t) = [I_1(t), I_2(t), ..., I_n(t)]^T$.

Assumption 4.1. The activation function $g_j(.)$ satisfies Lipschitz condition, i.e., for any $x_1,\ x_2 \in \mathbb{R}$, there exist constants $l_j > 0$ for all $j = 1, 2, ..., n$ such that

$$|g_j(x_1) - g_j(x_2)| \leq l_j|x_1 - x_2|. \tag{4.6}$$

Assumption 4.2. For any $x_1,\ x_2 \in \mathbb{R}$, there exist constants $l_j > 0,\ j = 1, 2, ..., n$ such that

$$0 \leq \frac{g_j(x_1) - g_j(x_2)}{x_1 - x_2} \leq l_j, \quad \forall j = 1, 2, ..., n. \tag{4.7}$$

Lemma 4.2.1. [100, 104] If $F(x) \in C(\mathbb{R}^n)$, and it satisfies the following conditions as

(1) $F(x)$ is an injective on $\mathbb{R}^n$,

(2) $\|F(x)\| \to \infty$, as $\|x\| \to \infty$, then $F(x)$ is a homeomorphism on $\mathbb{R}^n$.

Definition 4.2.1. The point u^* is said to be an equilibrium point of system (4.1) if

$$-\beta_i u^* + \sum_{j=1}^{n} a_{ij} g_j(u^*) + \sum_{j=1}^{n} b_{ij} g_j(u^*) + \sum_{j=1}^{n} \tau_j(t) d_{ij} g_j(u^*) + I_i(t) = 0, \tag{4.8}$$

for all $i = 1, 2, ..., n$.

The point $(u^*, v^*)^T \in \mathbb{R}^{2n}$ with $u^* = (u_1, u_2, ..., u_n)^T$ and $v^* = (v_1, v_2, ..., v_n)^T$ is said to be an equilibrium point of system (4.5) if

$$\begin{cases} -\zeta u^* + v^* = 0, \\ -\Gamma v^* + C u^* + A g(u^*) + B g(u^*) + D \int_{t-\tau(t)}^{t} g(u^*) ds + I(t) = 0. \end{cases} \tag{4.9}$$

Lemma 4.2.2. Under Assumption 4.1, system (4.1) must have a unique equilibrium point if

$$-\beta_i + \sum_{j=1}^{n} \left(\frac{|a_{ij}|}{2} + \frac{|b_{ij}|}{2} + \tau \frac{|d_{ij}|}{2} \right) l_j + \sum_{j=1}^{n} \left(\frac{|a_{ji}|}{2} + \frac{|b_{ji}|}{2} + \tau \frac{|d_{ji}|}{2} \right) l_i < 0, \quad (4.10)$$

for all $i = 1, 2, ..., n$.

Proof. From Definition 4.2.1, it is clear that $u^* \in \mathbb{R}^n$ be the equilibrium point of system (4.1) if it satisfies equation (4.8). Suppose a continuous function $F(u) = [F_1(u), F_2(u), ..., F_n(u)]^T : \mathbb{R}^n \to \mathbb{R}^n$, defined as

$$F_i(u) = -\beta_i u_i + \sum_{j=1}^{n} a_{ij} g_j(u_j) + \sum_{j=1}^{n} b_{ij} g_j(u_j) + \sum_{j=1}^{n} \tau_j(t) d_{ij} g_j(u_j) + I_i(t), \ \ i = 1, 2, ...n. \quad (4.11)$$

All solutions of $F(u) = 0$ are the equilibrium points of neural network (4.1). If function $F(u)$ is a homeomorphism on $\mathbb{R}^n$, then from Lemma 4.2.1, we will have $\|F(u)\| \to \infty$ as $\|u\| \to \infty$, i.e, function $F(u)$ is proper from $\mathbb{R}^n$ to $\mathbb{R}^n$ which implies that for every compact set K in $\mathbb{R}^n$ there exists a compact set $F^{-1}(K)$ in $\mathbb{R}^n$. Therefore, there must exists a unique equilibrium point $u^* \in \mathbb{R}^n$ such that $F(u^*) = 0$. Hence, in order to show $F(u)$ is a homeomorphism on $\mathbb{R}^n$, we will follow Lemma 4.2.1.

First we will show that $F(u)$ is an injective mapping. For this, suppose there exist $u, \bar{u} \in \mathbb{R}^n$ such that $u \neq \bar{u}$ and $F(u) = F(\bar{u})$, then

$$F_i(u) - F_i(\bar{u}) = 0.$$

$$(u_i - \bar{u}_i)\bigg[- \beta_i(u_i - \bar{u}_i) + \sum_{j=1}^{n} a_{ij}(g_j(u_j) - g_j(\bar{u}_j)) + \sum_{j=1}^{n} b_{ij}(g_j(u_j) - g_j(\bar{u}_j)) + \sum_{j=1}^{n} \tau_j(t) d_{ij}(g_j(u_j) - g_j(\bar{u}_j))\bigg] = 0,$$

for $i = 1, 2, ..., n$. From Assumption 4.1, we get

$$0 \leq -\beta_i |u_i - \bar{u}_i|^2 + \sum_{j=1}^{n} |a_{ij}| l_j |u_j - \bar{u}_j||u_i - \bar{u}_i| + \sum_{j=1}^{n} |b_{ij}| l_j |u_j - \bar{u}_j||u_i - \bar{u}_i| + \sum_{j=1}^{n} \tau |d_{ij}| l_j |u_j - \bar{u}_j||u_i - \bar{u}_i|,$$

for all $i = 1, 2, ..., n$. Now, we have

$$0 \leq \sum_{i=1}^{n} \Bigg[- \beta_i |u_i - \bar{u}_i|^2 + \sum_{j=1}^{n} \frac{|a_{ij}|}{2} l_j \Big(|u_j - \bar{u}_j|^2 + |u_i - \bar{u}_i|^2 \Big) + \sum_{j=1}^{n} \frac{|b_{ij}|}{2} l_j \Big(|u_j - \bar{u}_j|^2 + |u_i - \bar{u}_i|^2 \Big) + \sum_{j=1}^{n} \tau \frac{|d_{ij}|}{2} l_j \Big(|u_j - \bar{u}_j|^2 + |u_i - \bar{u}_i|^2 \Big) \Bigg]. \tag{4.12}$$

We can write inequality (4.12) as

$$0 \leq \sum_{i=1}^{n} \Bigg[- \beta_i + \sum_{j=1}^{n} \bigg(\frac{|a_{ij}|}{2} + \frac{|b_{ij}|}{2} + \tau \frac{|d_{ij}|}{2} \bigg) l_j + \sum_{j=1}^{n} \bigg(\frac{|a_{ji}|}{2} + \frac{|b_{ji}|}{2} + \tau \frac{|d_{ji}|}{2} \bigg) l_i \Bigg] |u_i - \bar{u}_i|^2.$$

Using inequality (4.10), we must have $u_i = \bar{u}_i$ for all $i = 1, 2, ..., n$, which contradicts our supposition. Therefore, $F(u)$ is an injective mapping.

Now, we will show that $\|F(u)\| \to \infty$ as $\|u\| \to \infty$. Let $\hat{F}(u) = [\hat{F}_1(u), \hat{F}_2(u), ..., \hat{F}_n(u)]^T = F(u) - F(0)$. Since we have $g(0) = 0$, then we get

$$\hat{F}_i(u) = -\beta_i u_i + \sum_{j=1}^{n} a_{ij}\big(g_j(u_j) - g_j(0)\big) + \sum_{j=1}^{n} b_{ij}\big(g_j(u_j) - g_j(0)\big) + \sum_{j=1}^{n} \tau_j(t) d_{ij}\big(g_j(u_j) - g_j(0)\big), \ i = 1, 2, ..., n.$$

Further, we can obtain

$$\begin{aligned} u^T\hat{F}(u) =& \sum_{i=1}^{n}\Bigg[-\beta_i|u_i|^2 + \sum_{j=1}^{n} a_{ij}g_j(u_j)u_i + \sum_{j=1}^{n} b_{ij}g_j(u_j)u_i + \sum_{j=1}^{n}\tau_j(t)d_{ij}g_j(u_j)u_i\Bigg] \\ \le& \sum_{i=1}^{n}\Bigg[-\beta_i|u_i|^2 + \sum_{j=1}^{n}\frac{|a_{ij}|}{2}l_j\Big(|u_j|^2 + |u_i|^2\Big) + \sum_{j=1}^{n}\frac{|b_{ij}|}{2}l_j\Big(|u_j|^2 + |u_i|^2\Big) \\ &+ \sum_{j=1}^{n}\tau\frac{|d_{ij}|}{2}l_j\Big(|u_j|^2 + |u_i|^2\Big)\Bigg] \\ \le& \sum_{i=1}^{n}\Bigg[-\beta_i + \sum_{j=1}^{n}\left(\frac{|a_{ij}|}{2} + \frac{|b_{ij}|}{2} + \tau\frac{|d_{ij}|}{2}\right)l_j \\ &+ \sum_{j=1}^{n}\left(\frac{|a_{ji}|}{2} + \frac{|b_{ji}|}{2} + \tau\frac{|d_{ji}|}{2}\right)l_i\Bigg]|u_i|^2. \end{aligned}$$

$$\begin{aligned} u^T\hat{F}(u) \le& -\min_{1\le i\le n}\Bigg[\beta_i - \sum_{j=1}^{n}\left(\frac{|a_{ij}|}{2} - \frac{|b_{ij}|}{2} + \tau\frac{|d_{ij}|}{2}\right)l_j \\ &- \sum_{j=1}^{n}\left(\frac{|a_{ji}|}{2} + \frac{|b_{ji}|}{2} + \tau\frac{|d_{ji}|}{2}\right)l_i\Bigg]\|u\|^2. \end{aligned} \tag{4.13}$$

Taking norm $\|.\|$ both sides of inequality (4.13) and using the Cauchy-Schwartz inequality, we get

$$\|u\|\|\hat{F}(u)\| \geq \|u\hat{F}(u)\| \geq \min_{1\leq i\leq n}\left[\beta_i - \sum_{j=1}^{n}\left(\frac{|a_{ij}|}{2} - \frac{|b_{ij}|}{2} + \tau\frac{|d_{ij}|}{2}\right)l_j - \sum_{j=1}^{n}\left(\frac{|a_{ji}|}{2} + \frac{|b_{ji}|}{2} + \tau\frac{|d_{ji}|}{2}\right)l_i\right]\|u\|^2. \tag{4.14}$$

Further, inequality (5.12) could be written as

$$\|\hat{F}(u)\| \geq \min_{1\leq i\leq n}\left[\beta_i - \sum_{j=1}^{n}\left(\frac{|a_{ij}|}{2} - \frac{|b_{ij}|}{2} + \tau\frac{|d_{ij}|}{2}\right)l_j - \sum_{j=1}^{n}\left(\frac{|a_{ji}|}{2} + \frac{|b_{ji}|}{2} + \tau\frac{|d_{ji}|}{2}\right)l_i\right]\|u\|. \tag{4.15}$$

Let $\mathcal{M} = \min_{1\leq i\leq n}\left[\beta_i - \sum_{j=1}^{n}\left(\frac{|a_{ij}|}{2} - \frac{|b_{ij}|}{2} + \tau\frac{|d_{ij}|}{2}\right)l_j - \sum_{j=1}^{n}\left(\frac{|a_{ji}|}{2} + \frac{|b_{ji}|}{2} + \tau\frac{|d_{ji}|}{2}\right)l_i\right]$ and substitute it in inequality (4.15), then we get

$$\|\hat{F}(u)\| \geq \mathcal{M}\|u\|. \tag{4.16}$$

It is clear from inequality (4.16) that $\|\hat{F}(u)\| \to \infty$ as $\|u\| \to \infty$ which further implies that $\|F(u)\| \to \infty$ as $\|u\| \to \infty$. Therefore, according to Lemma 4.2.1, mapping $F(u)$ is a homeomorphism on $\mathbb{R}^n$ and has a unique equilibrium point. This completes the proof. □

Now suppose that (u^*, v^*) be a unique equilibrium point of system (4.5). Let $x(t) = u(t) - u^*$ and $y(t) = v(t) - v^*$, then system (4.5) with the impulsive control can be written as

$$\begin{cases} \frac{dx(t)}{dt} = & -\zeta x(t) + y(t), \\ \frac{dy(t)}{dt} = & -\Gamma y(t) + Cx(t) + Af(x(t)) + Bf(x(t-\sigma(t))) + D\int_{t-\tau(t)}^{t} f(x(s))ds, t \neq t_k, \\ x(t_k^+) = & P_k x(t_k^-), \\ y(t_k^+) = & Q_k y(t_k^-), \quad t = t_k, k \in \mathbb{N}, \end{cases} \tag{4.17}$$

where $\mathcal{S} = \{t_k : k \in \mathbb{N}\}$ is an increasing sequence of impulsive instants. The diagonal matrices $P_k = diag\{p_{1k}, p_{2k}, ..., p_{nk}\}$ and $Q_k = diag\{q_{1k}, q_{2k}, ..., q_{nk}\}$ are the strengths of the impulses at impulsive instant t_k which belong to the set $\mathfrak{D} = \{R_k : R_k$ is a diagonal matrix and $R_{k_1} \neq R_{k_2}$for $t_{k_1} \neq t_{k_2} \in \mathcal{S}\}$. The state variables $x(t)$ and $y(t)$ has a jump kind of discontinuity from left side of $t_k \in \mathcal{S}$, i.e., $x(t_k^+),\ x(t_k^-),\ y(t_k^+),\ y(t_k^-)$ exist and satisfy $x(t_k^+) = x(t_k) \neq x(t_k^-)$ and $y(t_k^+) = y(t_k) \neq y(t_k^-)$. The initial condition of system (4.17) is $x(s) = \psi(s),\ y(s) = \varphi(s) + \zeta\psi(s) \in \mathrm{PC}([-\bar{\tau}, 0], \mathbb{R}^n)$, and $f(x(t)) = g(x(t) + u^*) - g(u^*)$.

Remark 4.2.2. Basically, impulses are characterized into three categories on the basis of their absolute values. If $|q_{ik}| > 1$ or $|p_{ik}| > 1$, then impulses are called destabilizing impulses since they enlarge the absolute value of the state variable. If $|q_{ik}| < 1$ or $|p_{ik}| < 1$, then impulses are called stabilizing impulses since they reduce the absolute value of the state variable. If $|q_{ik}| = 1$ or $|p_{ik}| = 1$, the impulses are inactive impulses since they do not affect the absolute value of the state variable, negatively or positively.

Since the set $\mathcal{S} = \{t_k : k \in \mathbb{N}\}$ contains all the points at which stabilizing and destabilizing impulses are activated, so without loss of generality, we suppose $\check{t}_{ik}$

and $\hat{t}_{jk} \in \mathcal{S}$ are the activation points of stabilizing and destabilizing impulses, respectively, where i and j are the respective number of occurrence of stabilizing and destabilizing impulses. Despite the frequent occurrence of destabilizing impulses, which may lead to the unstable neural network, the system (4.17) can be stabilized if the stabilizing impulses prevail over the effects of destabilizing impulses. In other words, there should exist lower and upper bounds of the impulsive interval of destabilizing and stabilizing impulses, respectively, to show that destabilizing impulses do not occur frequently and the occurrence of stabilizing impulses should not be too low. The results obtained in [105] are based on $\inf\{\hat{t}_{jk}-\hat{t}_{j(k-1)}\}$ and $\sup\{\check{t}_{ik}-\check{t}_{i(k-1)}\}$, other than the infimum and supremum of the impulsive interval, we have used the concept of average impulsive interval of destabilizing and stabilizing impulses. The following definitions are necessary to show the main results.

Definition 4.2.2. The average impulsive interval of destabilizing impulses is equal to $\hat{T}_a$ not less than $\inf\{\hat{t}_{jk} - \hat{t}_{j(k-1)}\}$, and the average impulsive interval of stabilizing impulses is equal to $\check{T}_a$ not more than $\sup\{\check{t}_{ik}-\check{t}_{i(k-1)}\}$ if there exist positive constants N_0, $\hat{T}_a$ and $\check{T}_a$ such that

$$\hat{N}_{\mathcal{S}}(t,t_0) \leq \frac{t-t_0}{\hat{T}_a} + N_0 \qquad \textit{and} \qquad \check{N}_{\mathcal{S}}(t,t_0) \geq \frac{t-t_0}{\check{T}_a} - N_0, \tag{4.18}$$

where $\hat{N}_{\mathcal{S}}(t,t_0)$ and $\check{N}_{\mathcal{S}}(t,t_0)$ are the number of destabilizing and stabilizing impulses occurred in the interval (t_0,t). For the sake of simplicity, we will write just $\hat{N}_{\mathcal{S}}$ and $\check{N}_{\mathcal{S}}$ instead of $\hat{N}_{\mathcal{S}}(t,t_0)$ and $\check{N}_{\mathcal{S}}(t,t_0)$, respectively.

Lemma 4.2.3. [106] Let $\xi_1 \in \mathbb{R},\ \xi_2 \geq 0,\ \delta > 1,\ b_k \geq \frac{1}{\delta}$ are real constants and $0 \leq W(t) \in$PC$(\mathbb{R},\ \mathbb{R}_+)$ such that

$$\begin{cases} D^+(W(t)) \leq \xi_1 W(t) + \xi_2 \sup_{t-\bar{\tau}\leq s\leq t} W(s), \quad t \neq t_k, k \in \mathbb{N}, \\ W(t_k^+) \leq b_k W(t_k^-), \quad t = t_k, \\ W(s) = H(s), s \in [-\bar{\tau},\ 0]. \end{cases} \tag{4.19}$$

Further, suppose $\xi_1 + \xi_2\delta - \frac{\ln\delta}{\varsigma} < 0$, where $\varsigma = \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\} < \infty$ and $0 < \lambda < \frac{\ln\delta}{\varsigma} - \xi_1 - \xi_2\delta e^{\lambda\bar{\tau}}$. If there exist $\eta \geq 0$ and $\mathfrak{M} \geq \delta$ such that $\delta^{k+1} \prod_{t_0<t_k<t} b_k \leq \mathfrak{M} e^{\eta(t_k - t_0)}$, $\forall k \in \mathbb{N}$, then $W(t) \in$PC$(\mathbb{R},\ \mathbb{R}_+)$ is estimated as

$$W(t) \leq \mathfrak{M} \sup_{t_0-\bar{\tau}\leq s\leq t_0} W(s) e^{-(\lambda-\eta)(t-t_0)},\ \forall t \geq t_0. \tag{4.20}$$

Remark 4.2.3. The inequality $\xi_1 + \xi_2\delta - \frac{\ln\delta}{\varsigma} < 0$ implies that there exists $\lambda \in \mathbb{R}$ such that $0 < \lambda < \frac{\ln\delta}{\varsigma} - \xi_1 - \xi_2\delta e^{\lambda\bar{\tau}}$ holds. For this, assume $\xi_1 + \xi_2\delta - \frac{\ln\delta}{\varsigma} < 0$ and define $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) = \frac{\ln\delta}{\varsigma} - \xi_1 - \xi_2\delta e^{x\bar{\tau}} - x, \quad \phi(0) = \frac{\ln\delta}{\varsigma} - \xi_1 - \xi_2\delta > 0,$$

then there exists $\epsilon > 0$ such that $\phi(x) > 0$ for all $x \in (-\epsilon,\ \epsilon)$. One needs to just pick $\lambda \in (0,\ \epsilon)$ to verify $0 < \lambda < \frac{\ln\delta}{\varsigma} - \xi_1 - \xi_2\delta e^{\lambda\bar{\tau}}$.

Definition 4.2.3. The solutions of impulsive system (4.17) converge globally exponentially to a zero equilibrium point if there exist $M > 1$ and $\lambda > 0$ such that

$$\|e(t)\|_q \leq M \sup_{-\bar{\tau}\leq s\leq 0} \|e(s)\|_q e^{-\lambda t}, \forall t \geq 0, \tag{4.21}$$

where $q = 1, 2, \infty$; $e(t) = [x(t),\ y(t)]^T$ and λ is the rate of convergence.

4.3 Global exponential stability criteria of inertial BAM neural network

In this section, in order to ensure the exponential stability of the equilibrium point of system (4.17), we shall derive the sufficient criteria in the form of matrix measure.

Theorem 4.1. *Under Assumption 4.1, the trajectories of system (4.17) are said to converge globally exponentially at the convergence rate* $\bar{r}_1 > 0$ *to a unique equilibrium point if there exist* $\delta > 1$, $\zeta = diag\{\zeta_1, \zeta_2, ..., \zeta_n\}$, *and the matrix measure* $\mu_q(.)$ $(q = 1, 2, \infty)$ *such that*

$$\mu_q(\mathcal{H}) + l_q\|A\|_q + (l_q\|B\|_q + \tau l_q\|D\|_q)\delta - \frac{\ln\delta}{\varsigma} < 0, \tag{4.22}$$

where $\mathcal{H} = \begin{pmatrix} -\zeta & E \\ C & -\Gamma \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$, $l_q = \max_{1\leq j\leq n}\{l_j\}$, $\varsigma = \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\} < \infty$, *and the rate of convergence* $\bar{r}_1$ *will be different for different types of impulses which can be described in the following cases:*

Case 1: For $\frac{1}{\delta} \leq \eta_k < 1, \forall k \in \mathbb{N}$, *the rate of convergence will be* $\bar{r}_1 = \lambda_1 - \bar{\alpha}_1$ *such that*

$$\|e(t)\|_q \leq \mathcal{K}_1 \sup_{-\bar{\tau}\leq s\leq 0} \|e(s)\|_q e^{-(\lambda_1-\bar{\alpha}_1)t}, \quad \forall t \geq 0, \tag{4.23}$$

where $\eta_k = \left\| \begin{pmatrix} P_k & 0 \\ 0 & Q_k \end{pmatrix} \right\|_q$, $\bar{\eta} = \max\{\eta_1, \eta_2, ..., \eta_{\check{N}_S}\}$,$\mathcal{K}_1 = \delta^{\check{N}_S+1}\bar{\eta}^{-N_0} \geq \delta$, $\bar{\alpha}_1 = \frac{\ln\bar{\eta}}{\check{T}_a}$, *and* $\lambda_1 \in \left(0, \frac{\ln\delta}{\varsigma} - \mu_q(\mathcal{H}) - l_q\|A\|_q - (l_q\|B\|_q + \tau l_q\|D\|_q)\delta e^{\lambda_1\bar{\tau}}\right)$.

Case 2: For $\eta_k \geq 1,\ \forall k \in \mathbb{N}$, the rate of convergence will be $\bar{r}_1 = \lambda_1 - \tilde{\alpha}_2$ such that

$$\|e(t)\|_q \leq \mathcal{K}_2 \sup_{-\bar{\tau} \leq s \leq 0} \|e(s)\|_q e^{-(\lambda_1 - \tilde{\alpha}_2)t}, \quad \forall t \geq 0, \tag{4.24}$$

where $\tilde{\eta} = \max\{\eta_1, \eta_2, ..., \eta_{\hat{N}_\mathcal{S}}\}$, $\mathcal{K}_2 = \delta^{\hat{N}_\mathcal{S}+1}\tilde{\eta}^{N_0} \geq \delta$, $\tilde{\alpha}_2 = \frac{\ln \tilde{\eta}}{\hat{T}_a}$, and λ_1 be same as given in Case 1.

Proof. Define $e(t) = [x(t), y(t)]^T$. When $t \neq t_k$, the right-hand Dini derivative of $\|e(t)\|_q$ with respect to time along system (4.17) is as

$$\begin{aligned} D^+(\|e(t)\|_q) =& \overline{\lim_{h \to 0^+}} \frac{\|e(t+h)\|_q - \|e(t)\|_q}{h} \\ =& \overline{\lim_{h \to 0^+}} \frac{\|e(t) + h\dot{e}(t) + O(h^2)\|_q - \|e(t)\|_q}{h}, \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} \|e(t) + h\dot{e}(t) + O(h^2)\|_q =& \left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + h \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} + O(h^2) \right\|_q \\ \leq& \left\| e(t) + h \begin{pmatrix} -\zeta & E \\ C & -\Gamma \end{pmatrix} e(t) \right\|_q + h\|A\|_q \|f(x(t))\|_q \\ &+ h\|B\|_q \|f(x(t-\sigma(t)))\|_q + h\|D\|_q \int_{t-\tau(t)}^{t} \|f(x(s))\|_q ds \\ &+ O(h^2). \end{aligned} \tag{4.26}$$

From Assumption 4.1, we have

$$\begin{aligned} \|f(x(t))\|_q \leq l_q \|x(t)\|_q \leq l_q \|e(t)\|_q, \|f(x(t-\sigma(t)))\|_q \leq& l_q \|x(t-\sigma(t))\|_q \\ \leq& l_q \|e(t-\sigma(t))\|_q. \end{aligned} \tag{4.27}$$

Using (4.27) in the equation (4.26), we get

$$\|e(t) + h\dot{e}(t) + O(h^2)\|_q \leq \|E + h\mathcal{H}\|_q \|e(t)\|_q + hl_q\|A\|_q\|e(t)\|_q + hl_q\|B\|_q\|e(t-\sigma(t))\|_q + hl_q\|D\|_q \int_{t-\tau(t)}^{t} \|e(s)\|_q ds + O(h^2). \tag{4.28}$$

Substituting inequality (4.28) in (4.25), we get

$$\begin{aligned} D^+(\|e(t)\|_q) \leq & \overline{\lim_{h\to 0^+}} \frac{\|E + h\mathcal{H}\|_q - 1}{h} \|e(t)\|_q + l_q\|A\|_q\|e(t)\|_q + l_q\|B\|_q\|e(t-\sigma(t))\|_q \\ & + l_q\|D\|_q \int_{t-\tau(t)}^{t} \|e(s)\|_q ds \\ \leq & \big(\mu_q(\mathcal{H}) + l_q\|A\|_q\big)\|e(t)\|_q + l_q\|B\|_q \sup_{t-\sigma\leq s\leq t} \|e(s)\|_q \\ & + l_q\|D\|_q\tau(t) \sup_{t-\tau\leq s\leq t} \|e(s)\|_q \\ \leq & \big(\mu_q(\mathcal{H}) + l_q\|A\|_q\big)\|e(t)\|_q + \big(l_q\|B\|_q + l_q\|D\|_q\tau\big) \sup_{t-\bar{\tau}\leq s\leq t} \|e(s)\|_q. \end{aligned} \tag{4.29}$$

When $t = t_k,\ k \in \mathbb{N}$,

$$\begin{aligned} \|e(t_k)\|_q = & \left\| \begin{pmatrix} x(t_k) \\ y(t_k) \end{pmatrix} \right\|_q \\ = & \left\| \begin{pmatrix} P_k & 0 \\ 0 & Q_k \end{pmatrix} \begin{pmatrix} x(t_k^-) \\ y(t_k^-) \end{pmatrix} \right\|_q \\ \leq & \left\| \begin{pmatrix} P_k & 0 \\ 0 & Q_k \end{pmatrix} \right\|_q \left\| \begin{pmatrix} x(t_k^-) \\ y(t_k^-) \end{pmatrix} \right\|_q \\ \leq & \eta_k \|e(t_k^-)\|_q. \end{aligned} \tag{4.30}$$

Assembling inequalities (4.29) and (4.30), we get

$$\begin{cases} D^+(\|e(t)\|_q) \le \big(\mu_q(\mathcal{H}) + l_q\|A\|_q\big)\|e(t)\|_q + \big(l_q\|B\|_q + l_q\|D\|_q\tau\big) \quad \sup_{t-\bar{\tau}\le s\le t}\|e(s)\|_q, \\ \hfill t \ne t_k, k \in \mathbb{N}, \\ \|e(t_k)\|_q \le \eta_k\|e(t_k^-)\|_q,\ t = t_k, \\ \|e(s)\|_q = \left\| \begin{pmatrix} \psi(s) \\ \varphi(s) + \zeta\psi(s) \end{pmatrix} \right\|_q, \ \forall s \in [\bar{\tau},\ 0]. \end{cases} \tag{4.31}$$

From inequality (4.22), we have $\mu_q(\mathcal{H}) + l_q\|A\|_q + (l_q\|B\|_q + \tau l_q\|D\|_q)\delta - \frac{\ln\delta}{\varsigma} < 0$. Using Lemma 4.2.3, we can estimate any solution $\|e(t)\|_q \in$PC$(\mathbb{R},\ \mathbb{R}_+)$ of system (4.31) under different types of time-varying impulses as

Case 1: If $\frac{1}{\delta} \le \eta_k < 1, \forall k \in \mathbb{N}$, then from Definition 4.2.2, we have

$$\begin{aligned} \delta^{\check{N}_{\mathcal{S}}+1}\prod_{t_0<t_k<t}\eta_k \le & \delta^{\check{N}_{\mathcal{S}}+1}\bar{\eta}^{\check{N}_{\mathcal{S}}} \\ \le & \delta^{\check{N}_{\mathcal{S}}+1}\bar{\eta}^{\frac{t-t_0}{\check{T}_a}-N_0} \\ = & \delta^{\check{N}_{\mathcal{S}}+1}\bar{\eta}^{-N_0}e^{\frac{\ln\bar{\eta}}{\check{T}_a}(t-t_0)}. \end{aligned}$$

By Lemma 4.2.3, we obtain

$$\|e(t)\|_q \le \mathcal{K}_1 \sup_{-\bar{\tau}\le s\le 0}\|e(s)\|_q e^{-(\lambda_1-\bar{\alpha}_1)t}, \quad \forall t \ge 0. \tag{4.32}$$

It can be easily observed from inequality (4.32) that the trajectories of system (4.17) converge globally exponentially to a unique equilibrium point at the convergence rate $\bar{r}_1 = \lambda_1 - \bar{\alpha}_1$ under the effects of stabilizing time-varying impulses.

Case 2: If $\eta_k \geq 1, \ \forall k \in \mathbb{N}$, then from Definition 4.2.2, we have

$$\begin{aligned}\delta^{\hat{N}_{\mathcal{S}}+1} \prod_{t_0<t_k<t} \eta_k \leq & \delta^{\hat{N}_{\mathcal{S}}+1} \tilde{\eta}^{\hat{N}_{\mathcal{S}}} \\ \leq & \delta^{\hat{N}_{\mathcal{S}}+1} \tilde{\eta}^{\frac{t-t_0}{\hat{T}_a}+N_0} \\ = & \delta^{\hat{N}_{\mathcal{S}}+1} \tilde{\eta}^{N_0} e^{\frac{\ln \tilde{\eta}}{\hat{T}_a}(t-t_0)}.\end{aligned}$$

Using Lemma 4.2.3, we get

$$\|e(t)\|_q \leq \mathcal{K}_2 \sup_{-\bar{\tau} \leq s \leq 0} \|e(s)\|_q e^{-(\lambda_1 - \tilde{\alpha}_2)t}, \quad \forall t \geq 0. \tag{4.33}$$

It is concluded from inequality (4.33) that every trajectory of system (4.17) converges globally exponentially to a unique equilibrium point with the convergence rate $\bar{r}_1 = \lambda_1 - \tilde{\alpha}_2$ under the effects of destabilizing time-varying impulses. Hence, the proof is completed. □

Remark 4.3.1. Note that the result in Theorem 4.1 does not contain the information about each $l_j, \ j = 1, 2, ...$. Therefore, we will proceed to derive more precise stability criteria which utilize the value of each l_j. For this, we shall introduce the following lemma in support of the next theorem.

Lemma 4.3.1. [107] Let Assumption 4.2 holds, and $\mu_q(.)$ be the matrix measure corresponding to the matrix norm $\|.\|_q$ induced on $\mathbb{R}^{n\times n}$, then $\mu_q(\mathcal{W}F(x(t))) \leq \mu_q(\bar{\mathcal{W}}L)$, where $F(x(t))$ =diag$\Big\{\frac{f_1(x_1(t))}{x_1(t)}, \frac{f_2(x_2(t))}{x_2(t)}, ..., \frac{f_n(x_n(t))}{x_n(t)}\Big\}$, L =diag$\{l_1, l_2, ..., l_n\}$, and

$$\bar{\mathcal{W}} = \bar{w}_{ij} = \begin{cases} \max(0, w_{ii}), & i = j, \\ w_{ij}, \quad i \neq j, \end{cases}$$

for $q = 1, \infty$.

Theorem 4.2. *Under Assumption 4.2, trajectories of system (4.17) are said to converge globally exponentially at the convergence rate $\bar{r}_2 > 0$ to the unique equilibrium point if there exist $\delta > 1,\ \zeta = diag\{\zeta_1, \zeta_2, ..., \zeta_n\}$, and the matrix measure $\mu_q(.)$ $(q = 1, \infty)$ such that*

$$\mu_q(\mathcal{H}) + \mu_q(\hat{A}\hat{L}) + (l_q\|B\|_q + \tau l_q\|D\|_q)\delta - \frac{\ln\delta}{\varsigma} < 0, \tag{4.34}$$

where $\hat{A} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$, $\hat{L} = diag\{l_1, l_2, ..., l_n, 1, 1, 1, ..., 1\}$, *and the convergence rate $\bar{r}_2$ will be different for different types of impulses which can be described as*

Case 1: For $\frac{1}{\delta} \leq \eta_k < 1, \forall k \in \mathbb{N}$, the rate of convergence will be $\bar{r}_2 = \lambda_2 - \bar{\alpha}_1$ such that

$$\|e(t)\|_q \leq \mathcal{K}_1 \sup_{-\bar{\tau}\leq s\leq 0} \|e(s)\|_q e^{-(\lambda_2-\bar{\alpha}_1)t}, \quad \forall t \geq 0, \tag{4.35}$$

where $\lambda_2 \in \left(0,\ \frac{\ln\delta}{\varsigma} - \mu_q(\mathcal{H}) - \mu_q(\hat{A}\hat{L}) - (l_q\|B\|_q + \tau l_q\|D\|_q)\delta e^{\lambda_2\bar{\tau}}\right)$.

Case 2:For $\eta_k \geq 1,\ \forall k \in \mathbb{N}$, the rate of convergence will be $\bar{r}_2 = \lambda_2 - \tilde{\alpha}_2$ such that

$$\|e(t)\|_q \leq \mathcal{K}_2 \sup_{-\bar{\tau}\leq s\leq 0} \|e(s)\|_q e^{-(\lambda_2-\tilde{\alpha}_2)t}, \quad \forall t \geq 0. \tag{4.36}$$

Proof. When $t \neq t_k,\ k \in \mathbb{N}$, similar to the proof of Theorem 4.1, we have

$$\begin{aligned}
D^+(\|e(t)\|_q) =& \overline{\lim_{h\to 0^+}} \frac{\|e(t+h)\|_q - \|e(t)\|_q}{h} \\
=& \overline{\lim_{h\to 0^+}} \frac{\|e(t) + h\dot{e}(t) + O(h^2)\|_q - \|e(t)\|_q}{h} \\
\leq& \overline{\lim_{h\to 0^+}} \frac{1}{h} \Bigg[\left\| e(t) + h \begin{pmatrix} -\zeta & E \\ C & -\Gamma \end{pmatrix} e(t) + h \begin{pmatrix} 0 \\ Af(x(t)) \end{pmatrix} \right\|_q \\
&+ h\|B\|_q \|f(x(t-\sigma(t)))\|_q + h\|D\|_q \int_{t-\tau(t)}^{t} \|f(x(s))\|_q ds - \|e(t)\|_q \Bigg].
\end{aligned} \tag{4.37}$$

Since $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$, $y(t) = [y_1(t), y_2(t), ..., y_n(t)]^T$, $f(x(t)) = [f_1(t), f_2(t), ..., f_n(t)]^T$, then we can define

$$F_1(x(t)) = diag\Big\{ \frac{f_1(x_1(t))}{x_1(t)}, \frac{f_2(x_2(t))}{x_2(t)}, ..., \frac{f_n(x_n(t))}{x_n(t)} \Big\}, \quad F_2(y(t)) = diag\{1, 1, 1, ..., 1\}.$$

So $f(x(t))$ and $y(t)$ could be written as

$$f(x(t)) = F_1(x(t))x(t), \quad y(t) = F_2(y(t))y(t).$$

Now let

$$F(e(t)) = \begin{pmatrix} F_1(x(t)) & 0 \\ 0 & F_2(y(t)) \end{pmatrix},$$

then

$$\begin{pmatrix} 0 \\ Af(x(t)) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} f(x(t)) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} F_1(x(t)) & 0 \\ 0 & F_2(y(t)) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$
$$= \hat{A}F(e(t))e(t). \tag{4.38}$$

Using (4.38) in inequality (4.37), we obtain

$$\begin{aligned} D^+(\|e(t)\|_q) \leq & \overline{\lim_{h\to 0^+}} \frac{1}{h} \left[\left\| e(t) + h \begin{pmatrix} -\zeta & E \\ C & -\Gamma \end{pmatrix} e(t) + h\hat{A}F(e(t))e(t) \right\|_q \right. \\ & \left. + h\|B\|_q \|f(x(t-\sigma(t)))\|_q + h\|D\|_q \int_{t-\tau(t)}^{t} \|f(x(s))\|_q ds - \|e(t)\|_q \right] \\ \leq & \overline{\lim_{h\to 0^+}} \frac{\|E + h\big(\mathcal{H} + \hat{A}F(e(t))\big)\|_q - 1}{h} \|e(t)\|_q + l_q\|B\|_q\|e(t-\sigma(t))\|_q \\ & + l_q\|D\|_q \int_{t-\tau(t)}^{t} \|e(s)\|_q ds \\ \leq & \mu_q\big(\mathcal{H} + \hat{A}F(e(t))\big)\|e(t)\|_q + l_q\|B\|_q \sup_{t-\sigma\leq s\leq t} \|e(s)\|_q \\ & + l_q\|D\|_q\tau(t) \sup_{t-\tau\leq s\leq t} \|e(s)\|_q. \end{aligned} \tag{4.39}$$

From (1.3.3) and Lemmas 4.3.1, we have

$$\begin{aligned} \mu_q\big(\mathcal{H} + \hat{A}F(e(t))\big) \leq & \mu_q(\mathcal{H}) + \mu_q(\hat{A}F(e(t))) \\ \leq & \mu_q(\mathcal{H}) + \mu_q(\hat{A}\hat{L}), \end{aligned} \tag{4.40}$$

where $\hat{L}$ =diag$\{l_1, l_2, ..., l_n, 1, 1, 1, ..., 1\}$. Putting the inequality (4.40) in (4.39), we get

$$D^+(\|e(t)\|_q) \leq \big(\mu_q(\mathcal{H}) + \mu_q(\hat{A}\hat{L})\big)\|e(t)\|_q + \big(l_q\|B\|_q + l_q\|D\|_q\tau\big) \sup_{t-\bar{\tau}\leq s\leq t} \|e(s)\|_q. \tag{4.41}$$

When $t = t_k$, we get the inequality $\|e(t_k)\|_q \leq \eta_k\|e(t_k^-)\|_q$. From the inequality (4.34), we have $\mu_q(\mathcal{H}) + \mu_q(\hat{A}\hat{L}) + (l_q\|B\|_q + \tau l_q\|D\|_q)\delta - \frac{\ln\delta}{\varsigma} < 0$. Now, following the proof as given in Theorem 4.1, we get

Case 1: For $\frac{1}{\delta} \leq \eta_k < 1, \forall k \in \mathbb{N}$, we have

$$\|e(t)\|_q \leq \mathcal{K}_1 \sup_{-\bar{\tau}\leq s\leq 0} \|e(s)\|_q e^{-(\lambda_2-\bar{\alpha}_1)t}, \quad \forall t \geq 0, \tag{4.42}$$

and

Case 2: For $\eta_k \geq 1, \ \forall k \in \mathbb{N}$, we have

$$\|e(t)\|_q \leq \mathcal{K}_2 \sup_{-\bar{\tau}\leq s\leq 0} \|e(s)\|_q e^{-(\lambda_2-\tilde{\alpha}_2)t}, \quad \forall t \geq 0. \tag{4.43}$$

It is clear from inequalities (4.42) and (4.43) that the trajectories of system (4.17) are converging globally exponentially to the unique equilibrium point with the convergence rates $\bar{r}_1 = \lambda_2 - \bar{\alpha}_1$ and $\bar{r}_2 = \lambda_2 - \tilde{\alpha}_2$ depending on $\frac{1}{\delta} \leq \eta_k < 1$ and $\eta_k > 1$, respectively. Hence, the proof is completed. □

Remark 4.3.2. According to the definition of matrix measure, the value of a matrix measure $\mu_q(\mathcal{W})$ $(q = 1, 2, \infty)$ can be negative, positive, or zero, whereas a matrix norm can only have a non-negative value. Thus, using matrix measure one can better utilize the elements of connection weights matrix.

4.4 Numerical simulations and discussions

In this section, two examples are given to illustrate the theoretical results obtained in the previous section.

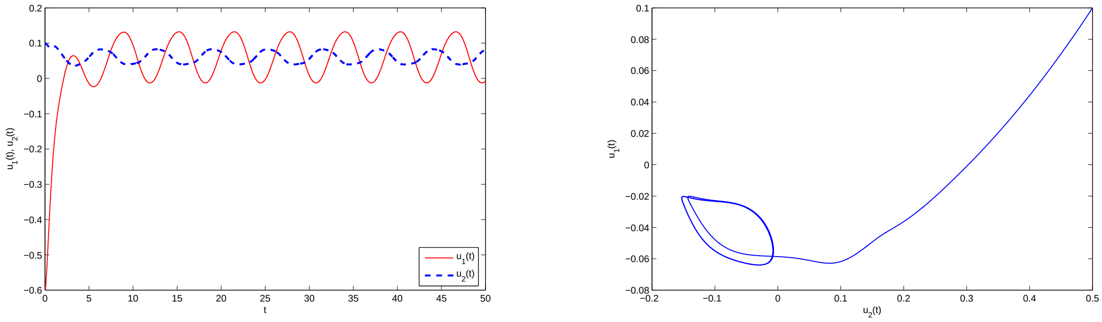
Example 4.4.1. Let us consider the following state equations of an inertial BAM neural network with mixed time-varying delays for $n = 2$ as

$$\begin{cases} \frac{d^2u_1(t)}{dt^2} = & -7\frac{du_1(t)}{dt} - 10u_1(t) + 2g_1(u_1(t)) - g_2(u_2(t)) + g_1(u_1(t-\sigma_1(t))) \\ & -g_2(u_2(t-\sigma_2(t))) + \int_{t-\tau_1(t)}^{t} g_1(u_1(s))ds + \int_{t-\tau_2(t)}^{t} g_2(u_2(s))ds + 0.8\sin t, \\ \frac{d^2u_2(t)}{dt^2} = & -8\frac{du_2(t)}{dt} - 15u_2(t) - g_1(u_1(t)) - 3g_2(u_2(t)) - g_1(u_1(t-\sigma_1(t))) \\ & -g_2(u_2(t-\sigma_2(t))) + 2\int_{t-\tau_1(t)}^{t} g_1(u_1(s))ds - \int_{t-\tau_2(t)}^{t} g_2(u_2(s))ds + 0.4\cos t, \end{cases} \tag{4.44}$$

where $g_i(u_i(t)) = \tanh(u_i(t)),\ \sigma_i(t) = 0.5\sin^2 t + 0.5,\ \tau_i(t) = \cos^2 t,\ i = 1, 2$. Setting $\zeta_1 = 3,\ \zeta_2 = 4$, the system of equations (4.44) can be written in the form of (4.17) with the corresponding matrices obtained as

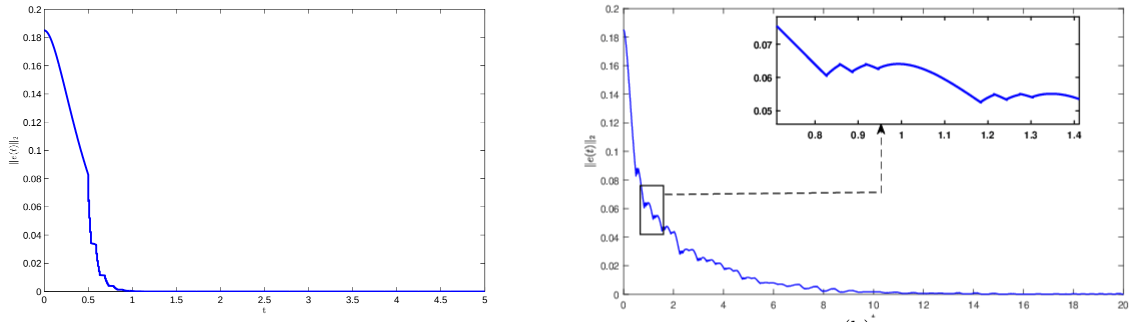
$$\zeta = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix},\ \Gamma = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix},\ C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},\ A = \begin{pmatrix} 2 & -1 \\ -1 & -3 \end{pmatrix},$$
$$B = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix},\ D = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

By simple calculations for $q = 2$, we get $l_q = 1,\ \sigma = 1,\ \tau = 1,\ \bar{\tau} = 1,\ \mu_2(\mathcal{H}) =$



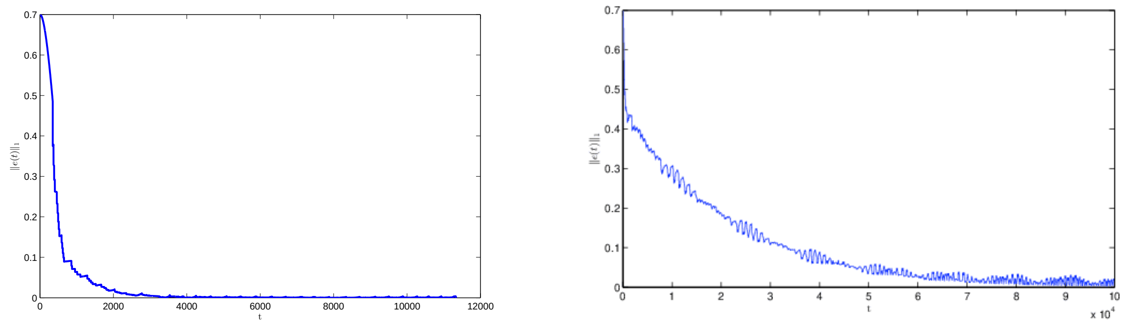
(a) (b)

FIGURE 4.1: (a) and (b) demonstrate time evolution and phase portrait of states' trajectories for the system (4.44), respectively



(a) (b)

FIGURE 4.2: (a) and (b) demonstrate time evolution of $\|e(t)\|_2$ for stabilizing and destabilizing impulses with $\hat{T}_a = 0.071$, respectively.



(a) (b)

FIGURE 4.3: (a) demonstrate time evolution of $\|e(t)\|_1$ for stabilizing with $\hat{T}_a = 0.03$. (b) demonstrate time evolution of $\|e(t)\|_1$ for destabilizing impulses with $\hat{T}_a = 0.04$.

-1.9188, $\|A\|_2 = 3.1926$, $\|B\|_2 = 1.4142$, and $\|D\|_2 = 2.3028$. The inequalities

$$-\beta_1 + \sum_{j=1}^{2}\left(\frac{|a_{1j}|}{2} + \frac{|b_{1j}|}{2} + \tau\frac{|d_{1j}|}{2}\right)l_j + \sum_{j=1}^{2}\left(\frac{|a_{j1}|}{2} + \frac{|b_{j1}|}{2} + \tau\frac{|d_{j1}|}{2}\right)l_1 = -2.5 < 0,$$
$$-\beta_2 + \sum_{j=1}^{2}\left(\frac{|a_{2j}|}{2} + \frac{|b_{2j}|}{2} + \tau\frac{|d_{2j}|}{2}\right)l_j + \sum_{j=1}^{2}\left(\frac{|a_{j2}|}{2} + \frac{|b_{j2}|}{2} + \tau\frac{|d_{j2}|}{2}\right)l_2 = -6.5 < 0,$$

hold for $i = 1, 2$. It is obvious from Lemma 4.2.2 that system (4.44) must have a unique equilibrium point. The phase trajectory and the time evolution of system (4.44) with the initial condition (0.5, 0, 0.1, 0) are depicted in Fig. (4.1(b)) and Fig. (4.1(a)), respectively. If $\delta = 2$, then the maximum impulsive interval of the impulsive sequence $\mathcal{S} = \{t_k : k \in \mathbb{N}\}$ is $\varsigma = 0.0714$ for which the inequality $\mu_2(\mathcal{H}) + l_q\|A\|_2 + (l_q\|B\|_2 + \tau l_q\|D\|_2)\delta - \frac{\ln\delta}{\varsigma} = -1.0001 < 0$ holds; and we choose $\lambda_1 = 0.1016 \in (0, 0.2151)$.

The notations $\check{t}_{ik}$ and $\hat{t}_{jk}$ for $i = 1, 2, ..., \check{N}_{\mathcal{S}}, j = 1, 2, ..., \hat{N}_{\mathcal{S}}, k \in \mathbb{N}$ are the activation times of the stabilizing and destabilizing impulses, respectively. Suppose both the stabilizing and destabilizing impulses are time-invariant, i.e., $P_{k_1} = P_{k_2}, Q_{k_1} = Q_{k_2}$, $\check{t}_{ik} = \check{t}_k$, $\hat{t}_{jk} = \hat{t}_k$ for all $i = 1, 2, ..., \check{N}_{\mathcal{S}}, j = 1, 2, ..., \hat{N}_{\mathcal{S}}$ and $k_1, k_2 \in \mathbb{N}$.

Case 1: Let $P_k = \begin{pmatrix} -0.9 & 0 \\ 0 & 0.5 \end{pmatrix}$, and $Q_k = \begin{pmatrix} -0.2 & 0 \\ 0 & 0.3 \end{pmatrix}$, then $0.5 < \eta_k = 0.9 < 1, \forall k \in \mathbb{N}$ implies $\bar{\eta} = 0.9$. Suppose the average impulsive interval of the stabilizing impulsive sequence $\check{T}_a = 0.02$, and the positive constant $N_0 = 1$, then we have $\bar{\alpha}_1 = -5.2680$. Thus, from Theorem 4.1, the trajectory of system (4.44) converges globally exponentially with the convergence rate $\bar{r}_1 = 0.1016 + 5.2680 = 5.3696$ under the effects of stabilizing impulses, which is shown in Fig. (4.2(a)).

Case 2: Let $P_k = \begin{pmatrix} -1.002 & 0 \\ 0 & 1.003 \end{pmatrix}$, $Q_k = \begin{pmatrix} -1.004 & 0 \\ 0 & 1.007 \end{pmatrix}$, the average impulsive interval of the destabilizing impulses $\hat{T}_a = 0.071$, and positive constant

$N_0 = 5$, then we can obtain $\eta_k = 1.007 > 1$, $\forall k \in \mathbb{N}$, $\tilde{\alpha}_2 = 0.0982$, and $\tilde{\eta} = 1.007$. Thus, from Theorem 4.1, it is clear that even for destabilizing impulses, the trajectory of system (4.44) is converging globally exponentially at the convergence rate $\bar{r}_1 = 0.1016 - 0.0982 = 0.0034$, which can be seen in Fig. (4.2(b)). Hence, Theorem 4.1 is verified.

Example 4.4.2. Consider system (4.44) to verify the results of Theorem 4.2. By simple calculations for $q = 1$, we obtain $\mu_1(\mathcal{H}) = -1$, $\mu_1(\hat{A}\hat{L}) = 4$, $\|B\|_1 = 2$, and $\|D\|_1 = 3$. Let $\delta = 3$, then the maximum impulsive interval of the impulsive sequence $\mathcal{S} = \{t_k : k \in \mathbb{N}\}$ is $\varsigma = 0.05$ for which the inequality $\mu_1(\mathcal{H}) + \mu_1(\hat{A}\hat{L}) + (l_q\|B\|_1 + \tau l_q\|D\|_1)\delta - \frac{\ln\delta}{\varsigma} = -3.978 < 0$ holds; and we choose $\lambda_2 = 0.2177 \in (0, 0.3240)$.

Case 1: Consider the same impulsive strength as in the Case 1 of Example 1, i.e., $\frac{1}{3} < \bar{\eta} = 0.9 < 1$ and let the average impulsive interval of stabilizing impulsive sequence is $\check{T}_a = 0.03$,and positive constant $N_0 = 5$, then we have $\bar{\alpha}_1 = -3.5120$. Hence, the rate of convergence at which the trajectory of system (4.44) is converging globally exponentially, is $\bar{r}_2 = 0.2177 + 3.5120 = 3.7297$. The convergence of trajectory of system (4.17) is shown in Fig. (4.3(a)).

Case 2: For the same impulsive strength as in the Case 2 of Theorem 4.1, we have $\tilde{\eta} = 1.007 > 1$. Let the average impulsive interval of the destabilizing impulsive sequence is $\hat{T}_a = 0.04$, and the positive constant $N_0 = 5$, then we get $\tilde{\alpha}_2 = 0.1744$. Hence, under the effects of destabilizing impulses, system (4.44) is getting stabilized with the convergence rate $\bar{r}_2 = 0.2177 - 0.1744 = 0.0433$. The time-evolution of the system's trajectory is depicted in Fig. (4.3(b)). Thus, the obtained results of Theorem 4.2 are verified.

4.5 Conclusion

This chapter has investigated the effects of time-varying stabilizing and destabilizing impulses on exponential stability of inertial BAM neural networks with mixed time-varying delays. The original system has been transformed into the system of first order differential equations by implementing a suitable variable transformation. Using the concept of homeomorphism, a distributed delay-dependent sufficient criterion has been found under which the system acquires a unique equilibrium point. The matrix measure technique together with the extended impulsive differential inequality have been successfully applied to derive sufficient criteria for ensuring the global exponential stability of inertial BAM neural networks with mixed time-varying delays for two classes of activation functions. Meanwhile, using the concept of the average impulsive interval, the exponential convergence rates of the system's trajectories for each case of stabilizing and destabilizing variable impulses have been investigated. Two numerical examples are given to validate the effectiveness of the theoretical results.

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