Chapter 3

Projective synchronization of delayed neural networks with mismatched parameters and impulsive effects

3.1 Introduction

This chapter is concerned about the effects of impulses on projective synchronization of neural networks. We have considered two non-identical Hopfield neural networks with discrete and distributed time-varying delays as master and response systems. In order to study the effects of extended range of impulses, the impulsive controller has been designed based on the controller introduced in Tang et al.[87]. The controller is a symbiosis of continuous (linear feedback term) and discontinuous (impulsive term) terms. The benefits of combining the two types of controller have been discussed elaborately with a numerical verification. If impulses affect projective synchronization negatively then the linear feedback term counteracts the negative effects and compels the response system to get projectively synchronized with the master system. If impulse affect positively then the impulsive term is sufficient to achieve the projective synchronization between the systems' network. Due to the existence of parameter mismatch and projective factor between the states of the networks, a weak projective synchronization is achieved under different types of impulsive effects with a small synchronization error bound.

The optimal synchronization error bound for different ranges of impulsive strength has been discussed by using fundamental calculus. We finished this chapter by giving an numerical example to verify the obtained results.

3.2 Problem formulation and preliminaries

The state equation of a neural network with mixed time-varying delays is written as follows:

$$\begin{aligned}\dot{u}_i(t) =& -\bar{a}_i u_i(t) + \sum_{j=1}^{n} \bar{b}_{ij} f_j(u_j(t)) + \sum_{j=1}^{n} \bar{c}_{ij} g_j(u_j(t-\sigma_1(t))) \sum_{j=1}^{n} \bar{d}_{ij} \int_{t-\sigma_2(t)}^{t} h_j(u_j(s))ds \\ &+ I_i, \\ u_i(t) =& \phi_i(t) \in C([-\sigma, 0], \mathbb{R}),\ i = 1, 2, 3, ...n, \end{aligned} \tag{3.1}$$

which can be re-written in a compact form as

$$\dot{u}(t) = -\bar{A}u(t) + \bar{B}f(u(t)) + \bar{C}g(u(t-\sigma_1(t))) + \bar{D}\int_{t-\sigma_2(t)}^{t} h(u(s))ds + I, \tag{3.2}$$

where $u(t) = [u_1(t), u_2(t), ..., u_n(t)]^T \in \mathbb{R}^n$ is the state vector of the neurons;$f(u(t)) = [f_1(u_1(t)), f_2(u_2(t)), ...f_n(u_n(t))]^T \in \mathbb{R}^n$,$g(u(t-\sigma_1(t))) = [g_1(u_1(t-\sigma_1(t))), g_2(u_2(t-\sigma_1(t))), ..., g_n(u_n(t-\sigma_1(t)))]^T \in \mathbb{R}^n$ and $h(u(t)) = [h_1(u_1(t)), h_2(u_2(t)), ..., h_n(u_n(t))]^T \in \mathbb{R}^n$ are the activation functions of the neurons at time t, time-varying discrete delay $t-\sigma_1(t)$ and distributed delay $t-\sigma_2(t)$, respectively, such that $f(0) = 0, g(0) = 0$ and $h(0) = 0$; $\bar{A} = diag(\bar{a}_1, \bar{a}_2, ..., \bar{a}_n) > 0$; $\bar{B} = (\bar{b}_{ij})_{n\times n}$ is the connection weights matrix at time t; $\bar{C} = (\bar{c}_{ij})_{n\times n}$ and $\bar{D} = (\bar{d}_{ij})_{n\times n}$ are the connection weights matrices at $t-\sigma_1(t)$ and $t-\sigma_2(t)$, respectively; $0 < \sigma_1(t) \le \sigma_1$ and $0 \le \sigma_2(t) \le \sigma_2$ are discrete and distributed time-varying delays, where $\sigma = max\{\sigma_1, \sigma_2\}$; $\phi_i(t) \in C([-\sigma, 0], \mathbb{R})$

and I_i's are the initial condition and the external input vector for $i = 1, 2, ..., n$, respectively.

Let us consider another state equation of a neural network given as follows:

$$\begin{aligned}\dot{v}_i(t) =& -\tilde{a}_i v_i(t) + \sum_{j=1}^{n} \tilde{b}_{ij} f_j(v_j(t)) \sum_{j=1}^{n} \tilde{c}_{ij} g_j(v_j(t-\sigma_1(t))) \sum_{j=1}^{n} \tilde{d}_{ij} \int_{t-\sigma_2(t)}^{t} h_j(v_j(s))ds \\ & + U_i(t) + J_i, \\ v_i(t) =& \varphi_i(t) \in C([-\sigma, 0], \mathbb{R}),\ \ i = 1, 2, 3, ...n, \end{aligned} \tag{3.3}$$

whose compact form is given by

$$\dot{v}(t) = -\tilde{A}v(t) + \tilde{B}f(v(t)) + \tilde{C}g(v(t-\sigma_1(t))) + \tilde{D}\int_{t-\sigma_2(t)}^{t} h(v(s))ds + U(t) + J. \tag{3.4}$$

Equation (3.3) or (3.4) is assumed as a response system and equation (3.1) or (3.2) as a master system, where $v(t) = [v_1(t), v_2(t), ..., v_n(t)]^T \in \mathbb{R}^n$ is the state vector and $\tilde{A} = diag(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) > 0$; $\tilde{B} = (\tilde{b_{ij}})_{n\times n}$, $\tilde{C} = (\tilde{c}_{ij})_{n\times n}$ and $\tilde{D} = (\tilde{d}_{ij})_{n\times n}$ are the connection weights matrices; the initial condition of equation (3.4) is $\varphi(t) = [\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)]^T \in C([-\sigma, 0], \mathbb{R}^n)$; $U(t)$ be the impulsive control function that will be designed later.

Our interest is to investigate the effects of impulses belonging in the large range on the projective synchronization between two different neural networks, where systems' parameters are mismatched, i.e., $\tilde{A} \neq \bar{A}$, $\tilde{B} \neq \bar{B}$, $\tilde{C} \neq \bar{C}$, and $\tilde{D} \neq \bar{D}$. Now, define the error system as $e(t) = [e_1(t), e_2(t), ..., e_n(t)]^T$ such that $e(t) = v(t) - \alpha u(t)$, where α is considered as a projective factor. Subtracting the equation (3.2) from the equation (3.4), the derivative of error system can be written as follows:

$$\begin{aligned}\dot{e}(t) =& -\tilde{A}e(t) + \tilde{B}\hat{f}(e(t)) + \tilde{C}\hat{g}(e(t-\sigma_1(t))) \\ &+ \tilde{D}\int_{t-\sigma_2(t)}^{t}\hat{h}(e(s))ds + H(u(t),\alpha,\sigma_1(t),\sigma_2(t)) + U(t),\end{aligned} \tag{3.5}$$

where $\hat{f}(e(t)) = f(v(t)) - f(\alpha u(t)), \hat{g}(e(t-\sigma_1(t))) = g(v(t-\sigma_1(t))) - g(\alpha u(t-\sigma_1(t))), \hat{h}(e(s)) = h(v(s)) - h(\alpha u(s))$, and

$$\begin{aligned}H(u(t),\alpha,\sigma_1(t),\sigma_2(t)) =& \alpha(-\tilde{A}+\bar{A})u(t) + \tilde{B}f(\alpha u(t)) - \alpha\bar{B}f(u(t)) \\ &+ \tilde{C}g(\alpha u(t-\sigma_1(t))) - \alpha\bar{C}g(u(t-\sigma_1(t))) \\ &+ \int_{t-\sigma_2(t)}^{t}\big(\tilde{D}h(\alpha u(s)) - \alpha\bar{D}h(u(s))\big)ds + (J-\alpha I).\end{aligned} \tag{3.6}$$

Due to the presence of parameter mismatches and a projective factor $\alpha \in \mathbb{R}$, it is clear that the equilibrium point of equation (3.5) is non-zero i.e., $e \neq 0$. So, it is impossible to achieve the complete projective synchronization between the systems (3.4) and (3.2). However, the weak projective synchronization can be achieved up to a small error bound $\bar{e}$ by designing an impulsive controller as

$$U(t) = -\Gamma e(t) + \sum_{k=1}^{\infty}\tau e(t)\delta(t-t_k), \tag{3.7}$$

where $\Gamma = diag(\gamma_1, \gamma_2, ..., \gamma_n) \geq 0$ is the feedback gain and $\tau \neq -1$ is the strength of impulse at a different impulsive instant that will be discussed later. The sequence of impulsive instant $\xi = \{t_1, t_2, ..., t_k\}$ is increasing sequence i.e., $t_{k-1} < t_k$ and $\lim_{k\to\infty} t_k = \infty$. $\delta(.)$ is the Dirac impulsive function.

Substituting the controller (3.7) in the equation (3.4), the error system (3.5) could be rewritten as

$$
\begin{aligned}
\dot{e}(t) =& -(\tilde{A}+\Gamma)e(t)+\tilde{B}\hat{f}(e(t))+\tilde{C}\hat{g}(e(t-\sigma_1(t)))\\
&+\tilde{D}\int_{t-\sigma_2(t)}^{t}\hat{h}(e(s))ds+H(u(t),\alpha,\sigma_1(t),\sigma_2(t)),\quad t\neq t_k,\\
\Delta e(t_k) =&\ e(t_k^+)-e(t_k^-)=\tau e(t_k^-),\qquad t=t_k,\quad k=1,2,3,... \qquad (3.8)
\end{aligned}
$$

The state vector $e(t)$ is right-hand continuous at $t = t_k,\ \ k = 1, 2, ...$, i.e., $\lim_{h\to 0^+} e(t_k+h) = e(t_k^+) = e(t_k)$. Therefore, the solution of (3.8) has jump kind of discontinuity at left side of $t = t_k$. Furthermore, it is assumed that the equation (3.8) satisfies the initial condition $e_i(t) = (\varphi_i(t) - \alpha\phi_i(t)) \in C([-\sigma, 0], \mathbb{R})$.

Definition 3.2.1. The systems (3.2) and (3.4) are said to be weak projective synchronized with an error bound $\bar{e}$ if the error neural network (3.8) converges exponentially into a compact domain $\bar{\Delta} = \{e(t) \in \mathbb{R}^n | \|e(t)\|_q \leq \bar{e}\}$ containing the origin, as $t \to \infty$, where $\|.\|_q$ is defined in the subsection (1.3.3).

Definition 3.2.2. [88] Suppose $N_\xi(s, t)$ is the number of activation time of impulse at t_k of the impulsive sequence $\xi = \{t_1, t_2, t_3, ...\}$ in the time interval (s, t). Then $N_\xi(s, t)$ can be estimated by its upper and lower bounds if there exist positive real number T_a and positive integer N_0 such that

$$
\frac{t-s}{T_a} - N_0 \leq N_\xi(s,t) \leq \frac{t-s}{T_a} + N_0, \quad \forall\, t \geq s > 0, \qquad (3.9)
$$

where the average impulsive interval of the impulsive sequence ξ is equal to T_a.

To prove the main results of this chapter we have the following assumptions and lemma.

Assumption 3.1. The activation functions $f(.), g(.)$ and $h(.)$ satisfy the Lipschitz condition i.e., for any $u_1, u_2 \in \mathbb{R}^n$ the following conditions hold

$$\|f(u_1) - f(u_2)\|_q \leq L_f^q \|u_1 - u_2\|_q, \tag{3.10}$$

$$\|g(u_1) - g(u_2)\|_q \leq L_g^q \|u_1 - u_2\|_q, \tag{3.11}$$

$$\|h(u_1) - h(u_2)\|_q \leq L_h^q \|u_1 - u_2\|_q, \tag{3.12}$$

where $L_f^q > 0$, $L_g^q > 0$ and $L_h^q > 0$ are Lipchitz constants.

Assumption 3.2. It is assumed that the solution of equation (3.2) is bounded for any initial function $u(t) \in C([-\sigma, 0], \mathbb{R}^n)$, i.e., there exist positive constant $M \in \mathbb{R}_+$ and time instant t_0 such that $\|u(t)\|_q \leq M$ for all $t \geq t_0$.

Lemma 3.2.1. [89] If there exists a positive constant c such that

$$\begin{cases} D^+(u(t)) \leq G(t, u(t), u(t - \sigma_1(t))) + c\int_{t-\sigma_2(t)}^{t} u(s)ds, \ t \neq t_k \\ u(t_k) \leq I_k(u(t_k^-)), \quad k \in \mathbb{N} \end{cases} \tag{3.13}$$

and

$$\begin{cases} D^+(v(t)) > G(t, v(t), v(t - \sigma_1(t))) + c\int_{t-\sigma_2(t)}^{t} v(s)ds, \ t \neq t_k \\ v(t_k) \geq I_k(v(t_k^-)), \quad k \in \mathbb{N}, \end{cases} \tag{3.14}$$

where $G(t,\ u,\ \bar{u}_1) : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is non-decreasing functional in $\bar{u}_1$ for any fixed (t, u) and $I_k(u) : \mathbb{R} \to \mathbb{R}$ is non-decreasing functional in u. Then $u(t) \leq v(t),\ \forall t \in [-\sigma, 0]$ implies that $u(t) \leq v(t)$ for $t \geq 0$.

Before getting the main results, we would like to show that $H(u(t), \alpha, \sigma_1(t), \sigma_2(t))$ is bounded. By applying the q-norm in both sides of (3.6), we obtain

$$\begin{aligned}\|H(u(t), \alpha, \sigma_1(t), \sigma_2(t))\|_q \leq & \|\alpha(\tilde{A} + \bar{A})u(t)\|_q + \|\tilde{B}f(\alpha u(t)) - \alpha\bar{B}f(u(t))\|_q \\ & + \|\tilde{C}g(\alpha u(t - \sigma_1(t))) - \alpha\bar{C}g(u(t - \sigma_2(t)))\|_q \\ & + \left\| \int_{t-\sigma_2(t)}^{t} \big(\tilde{D}h(\alpha u(s)) - \alpha\bar{D}h(u(s))\big) ds \right\|_q + \|J - \alpha I\|_q.\end{aligned} \tag{3.15}$$

Using Assumptions 3.1 and 3.2, we get the followings:

$$\begin{aligned}\|H(u(t), \alpha, \sigma_1(t), \sigma_2(t))\|_q \leq & (\|\tilde{A}\|_q + \|\bar{A}\|_q)|\alpha|M + (\|\bar{B}\|_q + \|\tilde{B}\|_q)|\alpha|L_f M \\ & + (\|\bar{C}\|_q + \|\tilde{C}\|_q)|\alpha|L_g M \\ & + (\|\tilde{D}\|_q + \|\bar{D}\|_q)|\alpha|\sigma_2 L_h M.\end{aligned} \tag{3.16}$$

From (3.16) we can conclude that $\|H(u(t), \alpha, \sigma_1(t), \sigma_2(t))\|_q$ is bounded for $t \geq -\sigma$. That is, there exists a positive constant Ξ such that $\Xi = \sup_{t\geq 0} \|H(u(t), \alpha, \sigma_1(t), \sigma_2(t))\|_q$, where $\Xi < \infty$.

3.3 Main Results

In this section, we will derive sufficient criteria to achieve the weak projective synchronization between the parameter mismatched systems (3.2) and (3.4) under the impulsive controller (3.7) for two different ranges of impulsive effects.

Theorem 3.1. *Consider the impulsive differential equation (3.8) with mixed time-varying delays having Assumptions 3.1 and 3.2. Suppose the average impulsive interval of the impulsive sequence $\xi = \{t_1, t_2, t_3, ...\}$, defined in Definition 3.2.2, is equal* *to T_a. If there exist non-singular matrix Q and matrix measure $\mu_q(.)$ $(q = 1, 2, \infty)$ such that*

Case I: For $\tau \in (-2, 0]$ and $\tau \neq -1$, if the condition

$$\zeta_1 + \frac{\ln \mu}{T_a} + \mu^{-N_0}\zeta_2 + \mu^{-N_0}\zeta_3\sigma_2 < 0 \tag{3.17}$$

is satisfied, where

$$\zeta_1 = \mu_q(Q(-\tilde{A} - \Gamma)Q^{-1}) + L_f\|Q\tilde{B}\|_q\|Q^{-1}\|_q, \tag{3.18}$$

$$\zeta_2 = L_g\|Q\tilde{C}\|_q\|Q^{-1}\|_q, \tag{3.19}$$

$$\zeta_3 = L_h\|Q\tilde{D}\|_q\|Q^{-1}\|_q, \tag{3.20}$$

then we get a small compact set $\bar{\Delta}$ containing the origin into which the solution of the equation (3.8) converges exponentially with the convergence rate $\lambda > 0$, where

$$\bar{\Delta} = \left\{ e(t) \in \mathbb{R}^n : \|e(t)\|_q \leq \frac{\|Q^{-1}\|_q\|Q\|_q\Xi}{-\left(\zeta_1 + \frac{\ln \mu}{T_a}\right)\mu^{N_0} - \zeta_2 - \zeta_3\sigma_2} \right\}, \tag{3.21}$$

$\mu = |1 + \tau|$ and λ is the unique solution of the equation $\lambda + \zeta_1 + \frac{\ln \mu}{T_a} + \mu^{-N_0}\zeta_2 e^{\lambda\sigma_1} + \mu^{-N_0}\zeta_3\frac{e^{\lambda\sigma_2}-1}{\lambda} = 0$. It is clear from (3.21) that the weak projective synchronization between the master system (3.1) and the response system (3.4) is achieved with a small error bound $\bar{e}$ under the impulsive controller (3.7).

Case II: For the impulsive effect $\tau \in (-\infty, -2]$ or $\tau > 0$, if the following inequality

$$\zeta_1 + \frac{\ln \mu}{T_a} + \mu^{N_0}\zeta_2 + \mu^{N_0}\zeta_3\sigma_2 < 0 \tag{3.22}$$

holds, then the solution of the equation (3.8) converges exponentially into a small compact set $\tilde{\Delta}$ containing the origin with convergence rate $\lambda' > 0$, where

$$\tilde{\Delta} = \left\{ e(t) \in \mathbb{R}^n : \|e(t)\|_q \leq \frac{\|Q^{-1}\|_q \|Q\|_q \Xi}{-\left(\zeta_1 + \frac{\ln \mu}{T_a}\right) - \mu^{N_0}\zeta_2 - \mu^{N_0}\zeta_3\sigma_2} \right\}. \tag{3.23}$$

The constants $\zeta_1, \zeta_2, \zeta_3$ and μ are same as in Case I.The convergence rate λ' is a unique solution of the equation $\lambda' + \zeta_1 + \frac{\ln \mu}{T_a} + \mu^{N_0}\zeta_2 e^{\lambda' \sigma_1} + \mu^{N_0}\zeta_3 \frac{e^{\lambda' \sigma_2 - 1}}{\lambda'} = 0$. It is clear from (3.23) that the weak projective synchronization between the master system (3.2) and the response system (3.4) is achieved with a small error bound $\bar{e}$ under the impulsive controller given in equation (3.7).

Proof. Let us construct the Lyapunov function in the form of $q-$norm as

$$V(e(t)) = \|Qe(t)\|_q. \tag{3.24}$$

For $t \neq t_k$,

$$D^+(V(e(t))) = \overline{\lim_{\epsilon \to 0^+}} \frac{\|Qe(t+\epsilon)\|_q - \|Qe(t)\|_q}{\epsilon},$$

where $D^+(V(e(t)))$ denotes the right-upper Dini derivative of equation(3.24) with respect to t along the solution of equation (3.8). Now, using Taylor's theorem, we get

$$
\begin{aligned}
D^{+}(V(e(t))) =& \overline{\lim_{\epsilon \to 0^{+}}} \frac{\|Qe(t) + \epsilon Q\dot{e}(t) + o(\epsilon)\|_q - \|Qe(t)\|_q}{\epsilon} \\
=& \overline{\lim_{\epsilon \to 0^{+}}} \frac{1}{\epsilon} \Bigg\{ \Bigg\| Qe(t) + \epsilon Q \Bigg[-(\tilde{A} + \Gamma)e(t) + \tilde{B}\hat{f}(e(t)) + \tilde{C}\hat{g}(e(t - \sigma_1(t))) \\
& + \tilde{D} \int_{t-\sigma_2(t)}^{t} \hat{h}(e(s))ds + H(u(t), \alpha, \sigma_1(t), \sigma_2(t)) \Bigg] + o(\epsilon) \Bigg\|_q - \|Qe(t)\|_q \Bigg\}. \\
\leq & \overline{\lim_{\epsilon \to 0^{+}}} \frac{1}{\epsilon} \Big[\|Qe(t) + \epsilon Q(-\tilde{A} - \Gamma)e(t)\|_q - \|Qe(t)\|_q \Big] + \|Q\tilde{B}\hat{f}(e(t))\|_q \\
& + \|Q\tilde{C}\hat{g}(e(t - \sigma_1(t)))\|_q + \|Q\tilde{D} \int_{t-\sigma_2(t)}^{t} \hat{h}(e(s))ds\|_q \\
& + \|QH(u(t), \alpha, \sigma_1(t), \sigma_2(t))\|_q.
\end{aligned} \tag{3.25}
$$

Now, we will use the Assumptions 3.1, 3.2 and the result obtained in (3.16) to rewrite the inequality (3.25) as

$$
\begin{aligned}
D^{+}(V(e(t))) \leq & \overline{\lim_{\epsilon \to 0^{+}}} \frac{1}{\epsilon} \big[\|I + \epsilon Q(-\tilde{A} - \Gamma)Q^{-1}\|_q - 1 \big] \|Qe(t)\|_q \\
& + L_f \|Q\tilde{B}\|_q \|Q^{-1}\|_q \|Qe(t)\|_q + L_g \|Q\tilde{C}\|_q \|Q^{-1}\|_q \|Qe(t - \sigma_1(t))\|_q \\
& + L_h \|Q\tilde{D}\|_q \|Q^{-1}\|_q \int_{t-\sigma_2(t)}^{t} \|Qe(s)\|_q ds + \|Q\|_q \Xi.
\end{aligned}
$$

Using Definition 1.3.4 and the equation (3.24), we get

$$
\begin{aligned}
D^{+}(V(e(t))) \leq & \left(\mu_q(Q(-\tilde{A} - \Gamma)Q^{-1}) + L_f \|Q\tilde{B}\|_q \|Q^{-1}\|_q \right) V(e(t)) \\
& + L_g \|Q\tilde{C}\|_q \|Q^{-1}\|_q V(e(t - \sigma_1(t))) + L_h \|Q\tilde{D}\|_q \|Q^{-1}\|_q \int_{t-\sigma_2(t)}^{t} V(e(s))ds \\
& + \|Q\|_q \Xi.
\end{aligned} \tag{3.26}
$$

From the equations (3.18), (3.19) and (3.20), we have $\zeta_1 = \mu_q(Q(-\tilde{A} - \Gamma)Q^{-1}) + L_f \|Q\tilde{B}\|_q \|Q^{-1}\|_q$, $\zeta_2 = L_g \|Q\tilde{C}\|_q \|Q^{-1}\|_q$, and $\zeta_3 = L_h \|Q\tilde{D}\|_q \|Q^{-1}\|_q$. Substituting

ζ_1, ζ_2 and ζ_3 in inequality (3.26), we get

$$D^+(V(e(t))) \leq \zeta_1 V(e(t)) + \zeta_2 V(e(t-\sigma_1(t))) + \zeta_3 \int_{t-\sigma_2(t)}^{t} V(e(s))ds + \|Q\|_q \Xi, \ t \neq t_k. \tag{3.27}$$

On the basis of (3.8), the Lyapunov function for discrete time $t = t_k, k = 1, 2, 3, ...$ can be written as

$$\begin{aligned} V(e(t_k^+)) &= \|Qe(t_k^+)\|_q = \|Q(1+\tau)e(t_k^-)\|_q \\ V(e(t_k)) &= |1+\tau| \|Qe(t_k^-)\|_q \\ V(e(t_k)) &= \mu \|Qe(t_k^-)\|_q. \end{aligned} \tag{3.28}$$

Based on the comparison principle; using (3.27) and (3.28), we have constructed the following impulsive differential equation with mixed time-varying delays as

$$\begin{cases} \dot{z}(t) = \zeta_1 z(t) + \zeta_2 z(t-\sigma_1(t)) + \zeta_3 \int_{t-\sigma_2(t)}^{t} z(s)ds + \|Q\|_q \Xi + \epsilon, \\ z(t_k^+) = \mu z(t_k^-), \quad t = t_k, \quad k = 1, 2, ..., \\ z(t) = \|Q\|_q \|\varphi(t) - \alpha\phi(t)\|_q, \qquad \forall t \in [-\sigma, 0], \end{cases} \tag{3.29}$$

where $z(t)$ is a unique solution of (3.29) for any $\epsilon > 0$. Using Lemma 3.2.1, we conclude that $V(t) \leq z(t), \ \forall t > 0$. Based on the formula for variation of parameters [90], $z(t)$ can be written as

$$z(t) = W(t,0)z(0) + \int_0^t W(t,s)\left[\zeta_2 z(s-\sigma_1(s)) + \zeta_3 \int_{s-\sigma_2(s)}^{s} z(r)dr + \|Q\|_q \Xi + \epsilon\right]ds, \tag{3.30}$$

where $W(t,s) = e^{\zeta_1(t-s)} \prod\limits_{s \le t_k \le t} \mu$ is the Cauchy matrix of the following linear impulsive system:

$$\begin{cases} \dot{z}(t) = \zeta_1 z(t), & t \neq t_k, \\ z(t_k^+) = \mu z(t_k^-), & t = t_k, k = 1, 2, ... \end{cases}$$

Calculation of the Cauchy matrix $W(t,s)$ will be done in two cases including different intervals of impulsive effect.

Case I: If we consider $-2 < \tau \le 0$ except $\tau = -1$, that is $0 < \mu \le 1$, then we have $N_\xi(s,t) \ge \frac{t-s}{T_a} - N_0$ from Definition 3.2.2. The Cauchy matrix can be estimated as

$$\begin{aligned} W(t,s) &= e^{\zeta_1(t-s)} \prod_{s \le t_k \le t} \mu \\ &\le e^{\zeta_1(t-s)} \mu^{N_\xi(s,t)} \\ &\le e^{\zeta_1(t-s)} \mu^{\left(\frac{t-s}{T_a} - N_0\right)} \\ &\le \mu^{-N_0} e^{\left(\zeta_1 + \frac{\ln \mu}{T_a}\right)(t-s)}. \end{aligned} \tag{3.31}$$

Substituting the inequality (3.31) in the integral equation (3.30), we get

$$\begin{aligned} z(t) \le & \mu^{-N_0} \|Q\|_q \|\varphi(0) - \alpha\phi(0)\|_q e^{\left(\zeta_1 + \frac{\ln \mu}{T_a}\right)t} \\ & + \int_0^t \mu^{-N_0} e^{\left(\zeta_1 + \frac{\ln \mu}{T_a}\right)(t-s)} \left[\zeta_2 z(s - \sigma_1(s)) + \zeta_3 \int_{s-\sigma_2(s)}^{s} z(r)dr + \|Q\|_q \Xi + \epsilon \right] ds, t \ge 0. \end{aligned}$$

Suppose $\eta = \mu^{-N_0} \|Q\|_q \sup_{-\sigma \le t \le 0} \|\varphi(t) - \alpha\phi(t)\|_q$, then we have

$$\begin{aligned} z(t) \le & \eta e^{\left(\zeta_1 + \frac{\ln \mu}{T_a}\right)t} + \int_0^t e^{\left(\zeta_1 + \frac{\ln \mu}{T_a}\right)(t-s)} \Big[\mu^{-N_0} \zeta_2 z(s - \sigma_1(s)) \\ & + \mu^{-N_0} \zeta_3 \int_{s-\sigma_2(s)}^{s} z(r)dr + \mu^{-N_0} \|Q\|_q \Xi + \mu^{-N_0} \epsilon \Big] ds. \end{aligned} \tag{3.32}$$

Let us define a continuous function $\psi(\lambda)=\lambda+\zeta_1+\frac{\ln\mu}{T_a}+\mu^{-N_0}\zeta_2e^{\lambda\sigma_1}+\mu^{-N_0}\zeta_3\frac{e^{\lambda\sigma_2}-1}{\lambda}$. Since $\psi(0^+)=\zeta_1+\frac{\ln\mu}{T_a}+\mu^{-N_0}\zeta_2+\mu^{-N_0}\zeta_3\lim_{\lambda\to0^+}\frac{e^{\lambda\sigma_2}-1}{\lambda}=\zeta_1+\frac{\ln\mu}{T_a}+\mu^{-N_0}\zeta_2+\mu^{-N_0}\zeta_3\sigma_2<0$ (from (3.17)), $\psi(+\infty)>0$ and $\psi'(\lambda)=1+\sigma_1\mu^{-N_0}\zeta_2e^{\lambda\sigma_1}+\mu^{-N_0}\zeta_3\frac{(\lambda\sigma_2-1)e^{\lambda\sigma_2}+1}{\lambda^2}>0,\ \ \forall\lambda>0$. Therefore, from the basic theory of calculus, we can say that $\psi(\lambda)$ must possess a unique root $\lambda>0$.

Since $\lambda>0,\ \epsilon>0,\ -(\zeta_1+\frac{\ln\mu}{T_a}+\mu^{-N_0}\zeta_2+\mu^{-N_0}\zeta_3\sigma_2)>0$ and $0<\mu\le1$, then we have

$$\begin{aligned}
z(t)&=\|Q\|_q\|\varphi(t)-\alpha\phi(t)\|_q\\
&\le\mu^{-N_0}\|Q\|_q\sup_{-\sigma\le t\le0}\|\varphi(t)-\alpha\phi(t)\|_q\\
&<\eta e^{-\lambda t}+\frac{\epsilon+\|Q\|_q\Xi}{-\mu^{N_0}\big(\zeta_1+\frac{\ln\mu}{T_a}\big)-\zeta_2-\zeta_3\sigma_2},-\sigma\le t\le0.
\end{aligned}\tag{3.33}$$

Next, we will proceed to show that the above inequality (3.33) holds for all $t>0$. To show this, we will use the mathematical method, viz., proof by contradiction. Let us assume the following:

$$z(t)<\eta e^{-\lambda t}+\frac{\epsilon+\|Q\|_q\Xi}{-\mu^{N_0}\big(\zeta_1+\frac{\ln\mu}{T_a}\big)-\zeta_2-\zeta_3\sigma_2},\ \forall t>0.\tag{3.34}$$

If the inequality (3.34) does not hold, then there exists $t^*>0$ such that

$$z(t^*)\ge\eta e^{-\lambda t^*}+\frac{\epsilon+\|Q\|_q\Xi}{-\mu^{N_0}\big(\zeta_1+\frac{\ln\mu}{T_a}\big)-\zeta_2-\zeta_3\sigma_2},\tag{3.35}$$

and

$$z(t)<\eta e^{-\lambda t}+\frac{\epsilon+\|Q\|_q\Xi}{-\mu^{N_0}\big(\zeta_1+\frac{\ln\mu}{T_a}\big)-\zeta_2-\zeta_3\sigma_2},\ t<t^*.\tag{3.36}$$

For the sake of simplicity, suppose $\Omega = -\mu^{N_0}\big(\zeta_1 + \frac{\ln\mu}{T_a}\big) - \zeta_2 - \zeta_3\sigma_2$. Then from the inequalities (3.32) and (3.36), we have

$$\begin{aligned}
z(t^*) <& \eta e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)t^*} + \int_0^{t^*} e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)(t^*-s)}\Bigg\{\mu^{-N_0}\zeta_2\Bigg(\eta e^{-\lambda(s-\sigma_1(s))} \\
&+ \frac{\epsilon + \|Q\|_q\Xi}{\Omega}\Bigg) + \mu^{-N_0}\zeta_3\int_{s-\sigma_2(s)}^{s}\Bigg(\eta e^{-\lambda r} + \frac{\epsilon + \|Q\|_q\Xi}{\Omega}\Bigg)dr \\
&+ \mu^{-N_0}(\|Q\|_q\Xi + \epsilon)\Bigg\}ds \\
<& \eta e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)t^*} + \mu^{-N_0}\zeta_2\eta e^{\lambda\sigma_1} e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)t^*} \times \int_0^{t*} e^{-\left(\lambda+\zeta_1+\frac{\ln\mu}{T_a}\right)s}ds \\
&+ \frac{\zeta_2\mu^{-N_0}(\epsilon + \|Q\|_q\Xi)}{\Omega} e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)t^*}\int_0^{t^*} e^{-\left(\zeta_1+\frac{\ln\mu}{T_a}\right)s}ds \\
&+ \mu^{-N_0}\zeta_3\eta\left(\frac{e^{\lambda\sigma_2}-1}{\lambda}\right)e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)t^*}\int_0^{t^*} e^{-\left(\lambda+\zeta_1+\frac{\ln\mu}{T_a}\right)s}ds \\
&+ \frac{\zeta_3\mu^{-N_0}\sigma_2(\epsilon + \|Q\|_q\Xi)}{\Omega} e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)t^*}\int_0^{t^*} e^{-\left(\zeta_1+\frac{\ln\mu}{T_a}\right)s}ds \\
&+ \mu^{-N_0}(\|Q\|_q\Xi + \epsilon)e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)t^*}\int_0^{t^*} e^{-\left(\zeta_1+\frac{\ln\mu}{T_a}\right)s}ds \\
&< \eta e^{-\lambda t^*} + \left(\frac{\epsilon + \|Q\|_q\Xi}{\Omega}\right)\left(1 - e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)t^*}\right) \\
&< \eta e^{-\lambda t^*} + \left(\frac{\epsilon + \|Q\|_q\Xi}{\Omega}\right).
\end{aligned} \tag{3.37}$$

It is obvious from the inequality (3.37) that it contradicts the assumption (3.35). Thus, the inequality (3.37) is true for all $t > 0$. Let $\epsilon \to 0$, then we have

$$V(e(t)) \leq z(t) < \eta e^{-\lambda t} + \left(\frac{\|Q\|_q\Xi}{-\mu^{N_0}\big(\zeta_1 + \frac{\ln\mu}{T_a}\big) - \zeta_2 - \zeta_3\sigma_2}\right). \tag{3.38}$$

From (3.24), we have

$$\|Q^{-1}\|_q V(e(t)) = \|Q^{-1}\|_q\|Qe(t)\|_q \geq \|e(t)\|_q.$$

The inequality (3.38) can be written as

$$\|e(t)\|_q \leq \|Q^{-1}\|_q V(e(t)) \leq \|Q^{-1}\|_q \eta e^{-\lambda t} + \frac{\|Q^{-1}\|_q \|Q\|_q \Xi}{-\mu^{N_0}\big(\zeta_1 + \frac{\ln\mu}{T_a}\big) - \zeta_2 - \zeta_3\sigma_2}. \tag{3.39}$$

It can be easily observed from the inequality (3.39) that the error system (3.8) is converging exponentially with the rate of convergence $\lambda > 0$, as time approaches infinity, into the compact set $\bar{\Delta}$ containing the origin, where

$$\bar{\Delta} = \left\{ e(t) \in \mathbb{R}^n : \|e(t)\|_q \leq \bar{e} = \frac{\|Q^{-1}\|_q \|Q\|_q \Xi}{-\mu^{N_0}\big(\zeta_1 + \frac{\ln\mu}{T_a}\big) - \zeta_2 - \zeta_3\sigma_2} \right\}, \tag{3.40}$$

and λ is the unique solution of continuous function $\psi(\lambda) = \lambda + \zeta_1 + \frac{\ln\mu}{T_a} + \mu^{-N_0}\zeta_2 e^{\lambda\sigma_1} + \mu^{-N_0}\zeta_3 \frac{e^{\lambda\sigma_2}-1}{\lambda} = 0$. That is, the weak projective synchronization between the systems (3.2) and (3.4) is achieved up to an error bound $\bar{e}$. This completes the first case.

Case II: If $\tau \leq -2$ or $\tau > 0$, then as in the Case I we have $N_\xi(t,s) \leq \frac{t-s}{T_a} + N_0$ from Definition 3.2.2 and $\mu > 1$. The Cauchy matrix must be written as

$$\begin{aligned} W(t,s) &= e^{\zeta_1(t-s)} \prod_{s \leq t_k \leq t} \mu \\ &\leq e^{\zeta_1(t-s)} \mu^{\left(\frac{t-s}{T_a} + N_0\right)} \\ &\leq \mu^{N_0} e^{\left(\zeta_1 + \frac{\ln\mu}{T_a}\right)(t-s)}. \end{aligned} \tag{3.41}$$

From the inequality of Cauchy matrix (3.41) and the integral equation (3.32), we have

$$\begin{aligned} z(t) \leq & \mu^{N_0} \|Q\|_q \|\varphi(0) - \alpha\phi(0)\|_q e^{\left(\zeta_1 + \frac{\ln\mu}{T_a}\right)t} + \int_0^t \mu^{N_0} e^{\left(\zeta_1 + \frac{\ln\mu}{T_a}\right)(t-s)} \Big[\zeta_2 z(s - \sigma_1(s)) \\ & + \zeta_3 \int_{s-\sigma_2(s)}^{s} z(r)dr + \|Q\|_q \Xi + \epsilon \Big] ds. \end{aligned} \tag{3.42}$$

Suppose $\bar{\eta} = \mu^{N_0}\|Q\|_q \sup_{-\sigma\le t\le 0}\|\varphi(t)-\alpha\phi(t)\|_q$, then we have

$$\begin{aligned} z(t) \le & \bar{\eta} e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)t} + \int_0^t e^{\left(\zeta_1+\frac{\ln\mu}{T_a}\right)(t-s)}\Big[\mu^{N_0}\zeta_2 z(s-\sigma_1(s)) \\ & + \mu^{N_0}\zeta_3\int_{s-\sigma_2(s)}^{s} z(r)dr + \mu^{N_0}\|Q\|_q\Xi + \mu^{N_0}\epsilon\Big]ds, \end{aligned} \tag{3.43}$$

for all $t > 0$. Approaching as in Case I, the following inequality holds for all $t > 0$

$$z(t) < \bar{\eta}e^{-\lambda' t} + \frac{\|Q\|_q\Xi}{-\mu^{-N_0}\left(\zeta_1+\frac{\ln\mu}{T_a}\right)-\zeta_2-\zeta_3\sigma_2}. \tag{3.44}$$

From the Lemma 3.2.1 and the equation (3.24), we have

$$\begin{aligned} \|e(t)\|_q \le & \|Q^{-1}\|_q V(e(t)) \le \|Q^{-1}\|_q z(t) \\ & < \|Q^{-1}\|_q\bar{\eta}e^{-\lambda' t} + \frac{\|Q^{-1}\|_q\|Q\|_q\Xi}{-\mu^{-N_0}\left(\zeta_1+\frac{\ln\mu}{T_a}\right)-\zeta_2-\zeta_3\sigma_2}. \end{aligned} \tag{3.45}$$

We conclude from inequality (3.45) that the error system (3.8) is converging exponentially with the convergence rate $\lambda' > 0$ into the compact set $\tilde{\Delta}$ containing the origin, where

$$\tilde{\Delta} = \left\{ e(t)\in\mathbb{R}^n : \|e(t)\|_q \le \bar{e} = \frac{\|Q^{-1}\|_q\|Q\|_q\Xi}{-\mu^{-N_0}\left(\zeta_1+\frac{\ln\mu}{T_a}\right)-\zeta_2-\zeta_3\sigma_2}\right\}$$

and λ' is the unique positive root of the continuous function $\psi(\lambda') = \lambda' + \zeta_1 + \frac{\ln\mu}{T_a} + \mu^{N_0}\zeta_2 e^{\lambda'\sigma_1} + \mu^{N_0}\zeta_3\frac{e^{\lambda'\sigma_2}-1}{\lambda'}$. That is , the weak projective synchronization between the systems (3.2) and (3.4) is achieved up to an error bound $\bar{e}$ under the impulsive controller (3.7). Hence, proof of the second case is completed. □

Remark 3.3.1. It is obvious from $e(t) = v(t) - \alpha u(t)$ that the problem in this chapter can also be extended to quasi synchronization and quasi anti-synchronization depending on the values of α. If we consider $\alpha = 1$, then the problem becomes quasi synchronization whereas for $\alpha = -1$ it becomes quasi anti-synchronization.

Remark 3.3.2. If we assume $\bar{A} = \tilde{A}$, $\bar{B} = \tilde{B}$, $\bar{C} = \tilde{C}$, $\bar{D} = \tilde{D}$, $\alpha = 1$ and the external input vectors $I = J$ in (3.6), then $H(u(t), \alpha, \sigma_1(t), \sigma_2(t)) = 0$ implies $\Xi = 0$. In Case 1: for $-2 < \tau \leq 0$ and $\tau \neq -1$, if the condition $\zeta_1 + \frac{\ln \mu}{T_a} + \mu^{-N_0}\zeta_2 + \mu^{-N_0}\zeta_3\sigma_2 < 0$ holds then it can be observed from (3.39) that the error bound $\bar{e} = 0$, which means that the trajectory of the error system (3.8) exponentially converges to zero with the convergence rate $\lambda > 0$, where λ is the unique solution of the continuous function $\psi(\lambda) = \lambda + \zeta_1 + \frac{\ln \mu}{T_a} + \mu^{-N_0}\zeta_2 e^{\lambda\sigma_1} + \mu^{-N_0}\zeta_3\frac{e^{\lambda\sigma_2}-1}{\lambda}$. Similarly, in Case 2: for $-\infty < \tau \leq -2$ or $\tau > 0$, if the condition $\zeta_1 + \frac{\ln \mu}{T_a} + \mu^{N_0}\zeta_2 + \mu^{N_0}\zeta_3\sigma_2 < 0$ holds then the trajectory of the error system (3.8) will be converging exponentially to zero with the rate of convergence $\lambda' > 0$, where λ' is the unique solution of the continuous function $\psi(\lambda') = \lambda' + \zeta_1 + \frac{\ln \mu}{T_a} + \mu^{N_0}\zeta_2 e^{\lambda'\sigma_1} + \mu^{N_0}\zeta_3\frac{e^{\lambda'\sigma_2}-1}{\lambda'}$.

Remark 3.3.3. The impulses are categorized into three categories: desynchronizing impulses, synchronizing impulses and inactive impulses, see [88]. In the previous published works [91, 92, 93, 94, 95], sufficient criteria for synchronization had been derived either for $\tau \in (0, 1)$ or for $\tau \in (-2, 0)$. Most of these works have considered either positive effects or negative effects of impulse that is very conservative study in the context of impulsive effects. This chapter focused on deriving sufficient criteria of weak projective synchronization in Theorem 3.1 for wider range of impulsive effects classified in two cases, one for $-2 < \tau \leq 0$ except $\tau \neq -1$ and another for $-\infty < \tau \leq -2$ or $\tau > 0$. The results obtained in Theorem 3.1 are the extended analysis of impulsive effects on the projective synchronization under the adverse influences (for synchronization) of parameter mismatches and projective factor. Additionally, we should discuss the effects for some special impulses $\tau = -2, -1, 0$. From equation (3.8), the followings are observed: (i) for $\tau = -2$ ($\mu = 1$), we have $e(t_k^+) = -e(t_k^-)$, the impulse could hinder the synchronization. But if we increase the value of feedback control gain $\gamma_i > 0$, then it will counteract the negative effect of impulse. (ii) For $\tau = -1$ ($\mu = 0$), the case $e(t_k^+) = 0$ is impossible for the impulsive control. (iii) For $\tau = 0$, we have $e(t_k^+) = e(t_k^-)$ which implies that the impulsive effect is inactive.

Remark 3.3.4. Introducing the concept of the average impulsive interval in Lemma 3.2.1, one can easily increase or decrease the distance between the impulsive times of the impulsive sequence ξ in the time interval (s, t), just by adjusting the positive constants N_0 and T_a. Normally, for T_a, we have $\inf_{k\in Z^+}\{t_k - t_{k-1}\} \leq T_a \leq \sup_{k\in Z^+}\{t_k - t_{k-1}\}$. This concept helps the controller to reduce control cost in controlling the response system (3.4).

3.4 Optimization of error bound

Since we have derived the error bound for the projective synchronization between the systems, so it is natural to have a question to optimize the error bound. In this section, we will proceed to obtain the optimal conditions so that the error bound will have minimum value under some constraints.

3.4.1 Optimized error bound for $-2 < \tau < 0$ except $\tau = -1$

From (3.39), we have

$$min \ \frac{\|Q^{-1}\|_q \|Q\|_q \Xi}{-\mu^{N_0}\left(\zeta_1 + \frac{\ln \mu}{T_a}\right) - \zeta_2 - \zeta_3\sigma_2}, \tag{3.46}$$

subject to the constraints

$$0 < \mu < 1, \qquad \zeta_1 + \frac{\ln \mu}{T_a} < 0, \qquad T_a > 0.$$

Since $\|Q^{-1}\|_q$, $\|Q\|_q$ and Ξ are known, then the desired synchronization error bound will depend on the variables μ, ζ_1, and T_a. Firstly, fixed the variables μ and T_a, then from (3.18), we observe that ζ_1 depends on the coupling strength matrix Γ =diag$(\gamma_1, \gamma_2, ..., \gamma_n) > 0$ which implies that we could obtain desired value of synchronization error bound for suitable large $\gamma_i > 0$ as it is discussed in neumerical verification section. Secondly, fixed the variable ζ_1, to minimize the objective function (3.46). We will have to maximize the denominator of the objective function (3.46). Now the objective function becomes

$$max \ -\mu^{N_0}\big(\zeta_1 + \frac{\ln \mu}{T_a}\big) - \zeta_2 - \zeta_3\sigma_2, \tag{3.47}$$

subject to the same constraints as given in (3.46). Suppose $F(\mu) = -\mu^{N_0}\big(\zeta_1 + \frac{\ln \mu}{T_a}\big) - \zeta_2 - \zeta_3\sigma_2$ and its derivative $F^{'}(\mu) = -\mu^{N_0 - 1}\big(\zeta_1 N_0 + \frac{1}{T_a} + \frac{N_0 \ln \mu}{T_a}\big)$. Using the concept of calculus, it is found that at a critical point $\mu^* = e^{\frac{-(\zeta_1 N_0 T_a + 1)}{N_0}}$, the function $F(\mu)$ will have its maximum for $T_a > 0$, where

$$F(\mu^*) = \frac{1}{N_0 T_a} e^{-(\zeta_1 N_0 T_a + 1)} - \zeta_2 - \zeta_3\sigma_2. \tag{3.48}$$

It is clear from (3.48) that $F(\mu^*)$ inversely depends on T_a. That is, the lowest value of T_a will give us the greatest value of $F(\mu^*)$. From (3.48), we have

$$\lim_{T_a \to 0} F(\mu^*) = \lim_{T_a \to 0} \frac{1}{N_0 T_a} e^{-(\zeta_1 N_0 T_a + 1)} - \zeta_2 - \zeta_3\sigma_2 = \infty, \tag{3.49}$$

that is, the objective function (3.47) does not have any finite optimal solution. Therefore, to maximize the optimization problem (3.47) ones should choose T_a as small as possible.

Suppose $T_a \geq \bar{T}_a$, then from (3.48), we have the optimized solution in the form of $\bar{T}_a$ as

$$F(\mu^*) = \frac{1}{N_0\bar{T}_a}e^{-(\zeta_1 N_0 \bar{T}_a+1)} - \zeta_2 - \zeta_3\sigma_2, \tag{3.50}$$

where $\bar{T}_a$ can be obtained from (3.9) as $T_a \geq \bar{T}_a = \frac{\inf_{k\in Z^+}\{t_k - s\}}{N_\xi(s,t)-N_0}$, provided $N_\xi(s,t) > N_0$.

3.4.2 Discussion about optimal error bound for $-\infty < \tau < -2$ or $\tau = 0$

From the inequality (3.45), we have the following optimization problem

$$min \ \frac{\|Q^{-1}\|_q\|Q\|_q\Xi}{-\mu^{-N_0}\big(\zeta_1 + \frac{\ln\mu}{T_a}\big) - \zeta_2 - \zeta_3\sigma_2}, \tag{3.51}$$

subject to the constraints

$$\mu > 1, \quad \zeta_1 + \frac{\ln\mu}{T_a} < 0, \quad T_a > 0. \tag{3.52}$$

For getting desired synchronization error bound, ζ_1 can be adjusted as given in sub-section A. For fixed ζ_1, consider the continuous function $\Phi(\mu) = -\mu^{-N_0}\big(\zeta_1 + \frac{\ln\mu}{T_a}\big) - \zeta_2 - \zeta_3\sigma_2$, whose derivative is $\Phi'(\mu) = \mu^{-N_0-1}\big[N_0\big(\zeta_1 + \frac{\ln\mu}{T_a}\big) - \frac{1}{T_a}\big] < 0$ for all $\mu > 1$. Thus, $\Phi(\mu)$ is strictly decreasing function for all $\mu > 1$, which implies that the function will have the supremum at $\mu = 1$. But from Remark 4, it is observed that for $\mu = 1(\tau = 0)$ the impulse is inactive. Therefore, there is no optimal error bound for $-\infty < \tau \leq -2$ or $\tau > 0$.

3.5 Numerical computation and discussion

In this section, we have verified our theoretical results obtained in Theorem 3.1 with the help of one example. A drive has been taken to indicate the merit of the results of this article compared to the existing results.

Now, consider the neural networks with mixed time-varying delays described by the following equation as

$$\begin{aligned} \dot{u}(t) =& -\bar{A}u(t) + \bar{B}f(u(t)) + \bar{C}g(u(t-\sigma_1(t))) + \bar{D}\int_{t-\sigma_2(t)}^{t} h(u(s))ds + I, \quad (3.53) \\ u(t) =& \phi(t) \in C([-\sigma, 0], \mathbb{R}^n), \end{aligned}$$

where

$$\bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \bar{B} = \begin{pmatrix} 2 & -0.1 \\ -5 & 4.5 \end{pmatrix},$$
$$\bar{C} = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{pmatrix}, \qquad \bar{D} = \begin{pmatrix} -0.3 & 0.1 \\ 0.1 & -0.2 \end{pmatrix},$$

and $\sigma_1(t) = 1,\ \sigma_2(t) = 0.2$, i.e., $\sigma = \max\{1,\ 0.2\} = 1$. The activation functions $f(u(t)) = g(u(t)) = h(u(t)) = [\tanh(u(t)),\ \tanh(u(t))\ ]^T$ have Lipschitz constants $L_f = 1,\ L_g = 1$ and $L_h = 1$ under Assumption 3.1 for $q = 2$. The initial value of the state vector $u(t) = [u_1(t),\ u_2(t)]^T$ is $\phi(s) = [0.01,\ 0.1], -\sigma \leq s \leq 0$. Suppose another state equation of neural networks with mixed time-varying delays described as

$$\dot{v}(t) = -\tilde{A}v(t) + \tilde{B}f(v(t)) + \tilde{C}g(v(t-\sigma_1(t))) + \tilde{D}\int_{t-\sigma_2(t)}^{t} h(v(s))ds + U(t) + J,$$
$$v(t) = \varphi(t) \in C([-\sigma, 0], \mathbb{R}^n), \tag{3.54}$$

where

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 1.8 & -0.15 \\ -5.2 & 3.5 \end{pmatrix},$$
$$\tilde{C} = \begin{pmatrix} -1.7 & -0.12 \\ -0.26 & -2.5 \end{pmatrix}, \qquad \tilde{D} = \begin{pmatrix} 0.6 & 0.15 \\ -2 & -0.12 \end{pmatrix}.$$

Clearly, it is seen that the parameters between the drive system (3.53) and the slave system (3.54) are mismatched. In the slave system (3.54), $U(t) = -\Gamma e(t) + \sum_{k=1}^{\infty} \tau e(t)\delta(t - t_k)$ is impulsive controller under which the projective synchronization is done. The initial value of the state vector $v(t) = [v_1(t), v_2(t)]^T$ is $\varphi(s) = [0.02, 0.01]$, $-\sigma \leq s \leq 0$. The phase portraits of the drive system (3.53) and the slave system (3.54) are drawn in Fig.3.1(a) and Fig. 3.1(b), respectively.

We consider the non-uniform distribution of impulses. We have used specific example of impulsive sequence [88] as $\bar{\zeta} = \{\epsilon,\ 2\epsilon, ...,\ (N_0 - 1)\epsilon,\ N_0T_a,\ N_0T_a + \epsilon,\ N_0T_a + 2\epsilon, ...,\ N_0T_a + (N_0 - 1)\epsilon,\ 2N_0T_a, ...\}$, which can also be written as $t_k - t_{k-1} = \epsilon$, if mod$(k, N_0) \neq 0$ and $N_0(T_a - \epsilon) + \epsilon$, if mod$(k, N_0) = 0$, where $0 < \epsilon < T_a$. To verify the results of the Theorem 3.1, we presume the following constants: the average impulsive interval $T_a = 0.02$, positive integer $N_0 = 2$, $\epsilon = 0.01$, projective factor $\alpha = 0.5$, $q = 2$ and non-singular matrix $Q = I(Identity\ matrix)$. The non-uniform distribution of impulses can be observed from the Fig.3.3(b). For the Case 1, when $\tau = -0.5\ or\ \mu = 0.5$, the impulse is synchronizing impulse. Suppose $\gamma_i = 0$. After some calculations, we found the value of the constants $\zeta_1 = 5.4671$, $\zeta_2 = 2.5440$, $\zeta_3 = 2.0940$,and $\frac{\ln\mu}{T_a} = -34.6574$. Using these known values, we could verify that $\zeta_1 + \frac{\ln\mu}{T_a} + \mu^{-N_0}\zeta_2 + \mu^{-N_0}\zeta_3\sigma_2 = -17.3391 < 0$. That is, the error system (3.7) exponentially converges into the small compact set $\bar{\Delta} = \{e(t) \in \mathbb{R}^n : \|e(t)\|_2 \leq \bar{e} = 0.4223\}$ at a convergence rate $\lambda = 0.953$. In Fig.3.2(a), one can find the experimental error bound 0.0695, which is less than the theoretical error bound $\bar{e} = 0.4223$. This further implies that the weak projective synchronization between the systems (3.53) and (3.54) could be achieved with a given error bound.

For case 2, when $\tau = 0.2$ *or* $\mu = 1.2$, then the impulse is desynchronizing impulse. It may destroy the synchronization. Therefore, we set the feedback gain by $\gamma_i = 20$ in the controller (3.7) to overcome the negative influence of the impulse. Substituting the values of the constants $\zeta_1 = -14.5399$, $\zeta_2 = 2.5440$, $\zeta_3 = 2.0940$, $\frac{\ln\mu}{T_a} = 9.1161$ in the equation (3.22), we get $\zeta_1 + \frac{\ln\mu}{T_a} + \mu^{N_0}\zeta_2 + \mu^{N_0}\zeta_3\sigma_2 = -1.1574 < 0$. That is, the error system (3.7) exponentially converges into the small compact set $\tilde{\Delta} = \{e(t) \in \mathbb{R}^n : \|e(t)\|_2 \leq \bar{e} = 0.2443\}$ at a convergence rate $\lambda = 0.224$. From Fig.3.2(b), one can observe that the experimental error bound 0.0225, which is less than the theoretical error bound $\bar{e} = 0.2443$. It implies that the weak projective synchronization between the systems (3.53) and (3.54) could be obtained with a given error bound even in the case of desynchronizing impulse.

Remark 3.5.1. From the above mentioned discussions, it is observed that the feedback term in controller (3.7) has meaningful significance in the case of desynchronizing impulse. In order to show the impact of the feedback term in the controller (3.7), we set $\gamma_i = 0$ when $\tau = 0.2$ *or* $\mu = 1.2$. After checking the solution curve of $\|e(t)\|_2$ in Fig.3.3(a), it is found that the projective synchronization is not possible within a small error bound. Therefore, the feedback term $-\Gamma e(t)$ in controller (3.7) plays an important role in the projective synchronization.

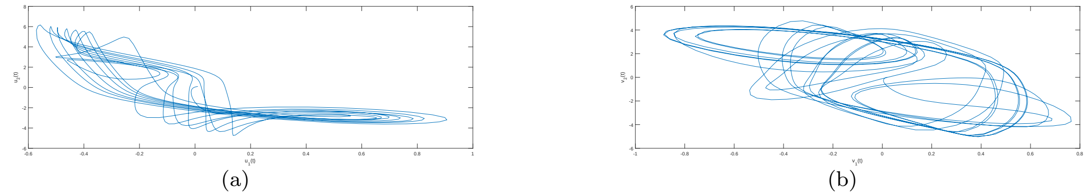

FIGURE 3.1: The chaotic attractors of master system (3.53) and response system (3.54).

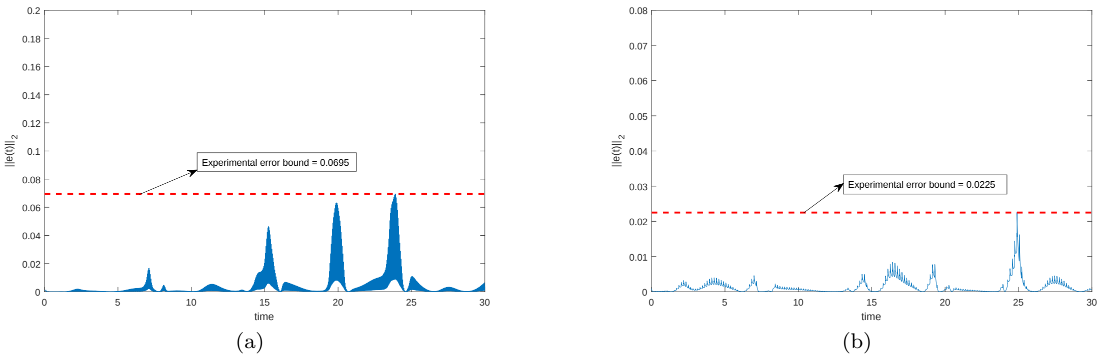

FIGURE 3.2: (a) and (b) presents error bound for $\tau = -0.5$ and $\tau = 0.2$, when $\gamma_i = 0$ and $\gamma_i = 20$ respectively.

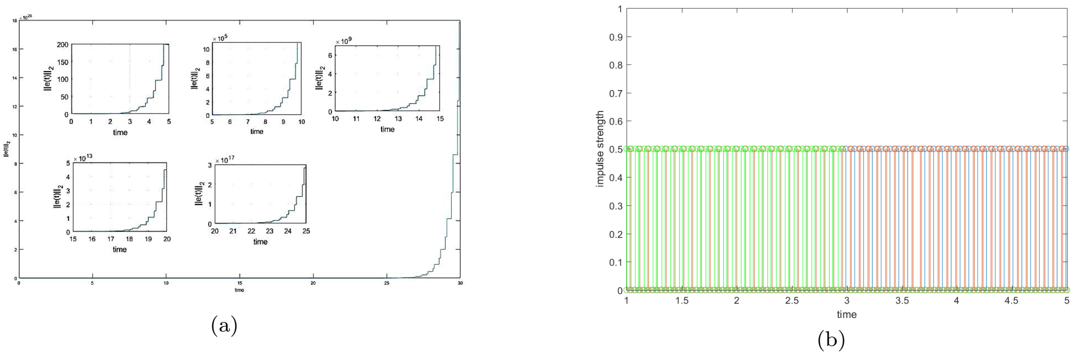

FIGURE 3.3: (c) demonstrate the error curve when $\tau = 0.2$ and $\gamma_i = 0$.(d) plot of nonuniform distribution of impulses.

Remark 3.5.2. As discussed in Remark 3.3.4 about the novelty of average impulsive interval T_a, in order to verify it numerically, we will use $\inf_{k \in Z^+}\{t_k - t_{k-1}\} = 0.01$

instead of $T_a = 0.02$ in inequality (3.22). Now,

$$\zeta_1 + \frac{\ln \mu}{\inf_{k \in Z^+}\{t_k - t_{k-1}\}} + \mu^{N_0}\zeta_2 + \mu^{N_0}\zeta_3\sigma_2 = 7.9587 > 0. \tag{3.55}$$

It is clear from (3.55) that the projective synchronization criteria obtained by infimum of impulsive interval does not hold. This implies that it fails to assure the weak projective synchronization between the systems (3.53) and (3.54).

Remark 3.5.3. It is clearly observed from (3.17) and (3.22) that the time delays in the system have negative effects on impulsive synchronization. If distributed delay is not considered in the systems then the inequalities (3.17) and (3.22) become $\zeta_1 + \frac{\ln \mu}{T_a} + \mu^{-N_0}\zeta_2 < 0$ and $\zeta_1 + \frac{\ln \mu}{T_a} + \mu^{N_0}\zeta_2 < 0$, respectively, whereas in the case of delays free system the sufficient criteria will be $\zeta_1 + \frac{\ln \mu}{T_a} < 0$ and $\zeta_1 + \frac{\ln \mu}{T_a} < 0$. It is worth to mention that in the case of delays free system, the rate of convergence increases rapidly from $\lambda = 0.0953$ and $\lambda' = 0.224$ to $\lambda = 19.02$ and $\lambda' = 1.761$, respectively.

3.6 Conclusion

In this chapter, we have done the analysis of positive and negative effects of impulses on projective synchronization between two nonidentical neural networks with mixed time-varying delays. Due to the existence of parameter mismatch and projective factor, we have investigated the weak projective synchronization under the impulsive controller. For deriving sufficient criteria of exponential synchronization within a small compact set containing the origin, the matrix measure technique is applied together with extended comparison principle and the formula of variation of parameters for distributed delayed impulsive system. Further, there is discussion on the convergence rates of the error system and derived synchronization error bounds for different ranges of impulse. Instead of using lower or upper bound of the impulsive interval, the concept of average impulsive interval is applied to enhance the novelty of the impulsive control method. Sufficient criteria are derived for optimal synchronization error bounds under different cases of impulsive effects. Finally, the numerical simulations have been done to validate the theoretical results obtained in the present chapter.

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