Chapter 2

Weak, modified, and function projective synchronization of Cohen-Grossberg neural networks

2.1 Introduction

In this chapter, we investigate MFPS of general Cohen-Grossberg neural networks (CGNNs), introduced in previous section 1.2.4, with parameter mismatches. The master and slave systems contain discrete and distributed delays that have been introduced in the subsection 1.3.1. The concept of projective synchronization was first studied by Mainieri and Rahacek [75] in 1999 in which they synchronized the slave and the response systems up to a constant scaling factor. Later on more general cases of projective synchronization, defined in the section 1.4, have been investigated in the literature [76, 77, 78, 79, 80, 81, 82].

2.2 Model description and mathematical preliminaries

We consider a CGNN with mixed time-varying delays as a drive system

$$\dot{x}_i(t) = -\alpha_i(x_i(t))\Bigg[\gamma_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_1(t))) - \sum_{j=1}^{n} d_{ij} \int_{t-\tau_2(t)}^{t} f_j(x_j(s))\, ds \ + I_i\Bigg], \quad (2.1)$$

where $i = 1, 2, ..., n$. This can be written in more concise form as

$$\dot{x}(t) = -\alpha(x(t))\Bigg[\Upsilon(x(t)) - Af(x(t)) - Bf(x(t-\tau_1(t))) - D\int_{t-\tau_2(t)}^{t} f(x(s))ds \ + I\Bigg], \quad (2.2)$$

where n represents the number of neurons in the neural network; $x(t) = [x_1(t), ..., x_n(t)]^T \in \mathbb{R}^n$ is the state vector associative with the neurons at time t;$\alpha(x(t)) = \text{diag}(\alpha_1(x(t)), \alpha_2(x(t)), ..., \alpha_n(x_n(t)))$ is a diagonal matrix of the state dependent amplification functions; $\Upsilon(t) = [\gamma_1(t), \gamma_2(t), ..., \gamma_n(t)]^T \in \mathbb{R}^n$ denotes a column vector of appropriately behaved functions ;$f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), ..., f_n(x_n(t))]^T \in \mathbb{R}^n$ denotes a column vector of activation functions at time t with $f(0) = 0$; $\tau_1(t)$ is discrete and $\tau_2(t)$ is distributed time-varying delays ; $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}$, and $D = (d_{ij})_{n\times n}$ denote connections weight matrices at time t, time-varying delay $\tau_1(t)$ and distributed delay $\tau_2(t)$ respectively. $I = [I_1, I_2, I_3, ..., I_n]$ is a column vector of external constant inputs to neurons. The initial condition of equation (2.2) is denoted by $x(t) = \phi(t) \in C([-\tau, 0], \mathbb{R}^n)$, where $C([-\tau, 0], \mathbb{R}^n)$ denotes the set of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ and $\phi(t) = [\phi_1(t), \phi_2(t), ..., \phi_n(t)]^T$ is a column vector of continuous functions. Throughout this chapter the following hypotheses will be considered.

Assumption 2.1. There exists $\bar{\eta}_i$,$\tilde{\eta}_i > 0$ such that

$$0 \leq \alpha_i(u_i(t)) \leq \bar{\eta}_i; \quad i = 1, 2, 3, ..., n.$$

$$0 \leq \beta_i(u_i(t)) \leq \tilde{\eta}_i; \quad i = 1, 2, 3, ..., n.$$

Assumption 2.2. For any $u, v \in \mathbb{R}^n$, there exist constants $\rho_i > 0, i = 1, 2, ..., n$,such that

$$0 \leq \frac{f_i(u) - f_i(v)}{u - v} \leq \rho_i, \quad i = 1, 2, ..., n.$$

Assumption 2.3. For any $u, v \in \mathbb{R}^n$, there exist constants $\sigma_i > 0,\ \rho_i > 0,\ \varrho_i > 0,\ i = 1, 2, ..., n$, such that

$$\begin{aligned} \|\gamma_i(u) - \gamma_i(v)\|_p \leq& \sigma_i \|u - v\|_p, \\ \|f_i(u) - f_i(v)\|_p \leq& \rho_i \|u - v\|_p, \\ \|\Gamma_i(u) - \Gamma_i(v)\|_p \leq& \varrho_i \|u - v\|_p. \end{aligned}$$

The p-norm is defined in the section 1.3.3. There is an another norm that is used in this chapter is $\omega-$norm defined as

$$\|M\|_\omega = \max_j \sum_{i=1}^{n} \frac{\omega_i}{\omega_j} |m_{ij}|, \tag{2.3}$$

and the matrix measure induced from the $\omega-$ norm is defined as

$$\mu_\omega(M) = \max_j \left[m_{jj} + \sum_{i=1, i \neq j}^{n} \frac{\omega_i}{\omega_j} |m_{ij}| \right]. \tag{2.4}$$

Assumption 2.4. Let us consider that $0 \leq \tau_1(t), \tau_2(t) \leq \tau, \forall t$.

The following lemma will be used to prove the theorems.

Lemma 2.2.1. [83] If $\Psi(t) \geq 0$, $\forall t \in (-\infty, \infty)$,

$$D^+\left(\Psi(t)\right) \leq \xi_1(t) + \xi_2(t)\Psi(t) + \xi_3(t) \sup_{t-\tau(t)\leq s\leq t} \Psi(s),$$

for $t > t_0$, where $\xi_1(t) \geq 0, \xi_2(t) \leq 0, \xi_3(t) \geq 0$ are continuous functions and $\tau(t) \geq 0$. $D^+\left(\Psi(t)\right) = \overline{lim_{h\to 0^+}} \dfrac{\Psi(t+h) - \Psi(t)}{h}$ is the upper right Dini's derivative of $\Psi(t)$. If $\exists \delta \geq 0$, such that

$$\xi_2(t) + \xi_3(t) \leq -\delta < 0, \quad t \geq t_0,$$

then $\Psi(t) \leq \frac{\xi^*}{\delta} + \sup_{-\infty\leq s\leq t_0} \Psi(s)e^{-\mu^*(t-t_0)}$, where $\xi^* = sup_{t_0\leq s<\infty \xi_1(s)}$ and $\mu^* = inf_{t\geq t_0}\{\mu(t)\colon \mu(t) + \xi_2(t) + \xi_3(t)e^{\mu(t)\tau(t)} = 0\}$.

2.3 Main results

In this section, the problem of the weak MFPS between coupled CGNNs systems with mixed time-varying delays and parameter mismatch is formulated.

Let us consider an another CGNN as a slave system whose parameters are different from the drive system (2.2) as

$$\begin{aligned}\dot{y}(t) = &-\beta(y(t))\left[\mathit{\Gamma}(y(t)) - Pf(y(t)) - Qf(y(t-\tau_1(t))) - R\int_{t-\tau_2(t)}^{t} f(y(s))ds \ + J\right] \\ &+ U(t), \text{and } y(t) = \varphi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{2.5}$$

where $y(t) = [y_1(t), y_2(t), ..., y_n(t)]^T \in \mathbb{R}^n$ is a state vector; $\beta(y(t)) \in \mathbb{R}^{n\times n}$ is a diagonal matrix for which Assumption 2.1 is already stated; $\mathit{\Gamma}(y(t))$ is appropriately behaved function satisfies Assumption 2.3 ; $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{n\times n}$ are constant matrices; $\varphi(t) \in C\left([-\tau, 0], \mathbb{R}^n\right)$ is the initial condition of equation (2.5). The controller is chosen as $U(t) = -\zeta(t)\left(x(t) - \Lambda(t)y(t)\right)$, where $\Lambda(t) = \operatorname{diag}\left(\nu_1(t), \nu_2(t), ..., \nu_n(t)\right)$ is a diagonal matrix of order n and $\nu_i(t)$ is continuously differentiable function with bound, $\nu_i(t) \neq 0, i = 1, 2, ..., n$, for all t. $\zeta(t)$ is coupling strength matrix.

In order to achieve a weak MFPS, we construct an error system from drive and response systems (2.2)-(2.5) as follows:

$$e(t) = y(t) - \Lambda(t)x(t). \tag{2.6}$$

From the equations (2.2) and (2.5), we get

$$\begin{aligned}\dot{e}(t) =& -\zeta(t)e(t) - \beta(y(t))\tilde{\Gamma}(e(t)) + \beta(y(t))P\tilde{f}(e(t)) + \beta(y(t))Q\tilde{f}(e(t - \tau_1(t))) \\ &+ \beta(y(t))R\int_{t-\tau_2(t)}^{t} \tilde{f}(e(s))ds + H(x(t), \Lambda(t), \tau_1(t), \tau_2(t)),\end{aligned} \tag{2.7}$$

where $\tilde{f}(e(t)) = f(y(t)) - f(\Lambda(t)x(t)), \tilde{\Gamma}(e(t)) = \Gamma(y(t)) - \Gamma(\Lambda(t)x(t))$, and

$$\begin{aligned}H\left(x(t), \Lambda(t), \tau_1, \tau_2\right) =& \beta(y(t))\Gamma(\Lambda(t)x(t)) + \Lambda(t)\alpha(x(t))\Upsilon(x(t)) \\ &+ \beta(y(t))Pf(\Lambda(t)x(t)) - \Lambda(t)\alpha(x(t))Af(x(t)) \\ &+ \beta(y(t))Qf(\Lambda(t)x(t - \tau_1(t))) - \Lambda(t)\alpha(x(t))Bf(x(t - \tau_1(t))) \\ &+ \beta(y(t))R\int_{t-\tau_2(t)}^{t} f(\Lambda(s)x(s))ds - \Lambda(t)\alpha(x(t)) \\ &\times D\int_{t-\tau_2(t)}^{t} f(x(s))ds + J(\Lambda(t)\alpha(x(t)) - \beta(y(t)).\end{aligned} \tag{2.8}$$

From literature survey [84, 85, 86] on dynamics of neural networks with parameter mismatches. It is seen that the parameter mismatches is unfavorable for the complete synchronization. Chen and Cao [85] have considered the mismatched parameters between the chaotic neural networks for projective synchronization problem. Our aim is to find weak MFPS between the CGNNs (2.2) and (2.5).

It is clear from equation (2.7) that the effect of parameter mismatches and the scaling function $\Lambda(t)$ cause the error system having non-zero equilibrium point. However, a small synchronization error bound can be obtained up to which a weak MFPS is achieved.

Definition 2.3.1. The weak MFPS is said to be achieved between the drive and slave systems (2.2)-(2.5) with an error bound $\epsilon > 0$ if there exists a scaling function matrix $\Lambda(t)$ and $T^{'} > 0$ such that

$$\|y(t) - \Lambda(t)x(t)\|_p \leq \epsilon,$$

for all $t \geq T^{'}$.

Remark 2.3.1. It can be observed from Definition (2.3.1) if the scaling function matrix is a zero matrix i.e.,$\Lambda(t) = 0$ then the problem becomes stability analysis of CGNN with mixed time-varying delays. In the case of $\Lambda(t) = I$ where I is an identity matrix, the problem is called quasi-synchronization.

Assumption 2.5. Let us assume that the states of driven system $x(t)$ is bounded i.e., $x(t) \in \{x(t) : \|x(t)\|_p \leq C_p\}, \forall t \geq -\tau$.

For $p = 1, \infty, \omega$, we have

$$\begin{aligned}\|H(x(t), \Lambda(t), \tau_1(t), \tau_2(t))\|_p \leq{}& \|\beta(y(t))\|_p \|\Gamma(\Lambda(t)x(t)\|_p + \|\Lambda(t)\|_p \|\alpha(x(t))\|_p \|\Upsilon(x(t))\|_p \\ &+ \|\beta(y(t))\|_p \|P\|_p \|f(\Lambda(t)x(t))\|_p \\ &+ \|\Lambda(t)\|_p \|\alpha(x(t))\|_p \|A\|_p \|f(x(t))\|_p \\ &+ \|\beta(y(t))\|_p \|Q\|_p \|f(\Lambda(t)x(t-\tau_1(t))\|_p \\ &+ \|\Lambda(t)\|_p \|\alpha(x(t))\|_p \|B\|_p \|f(x(t-\tau_1(t)))\|_p \\ &+ \|\beta(y(t))\|_p \|R\|_p \int_{t-\tau_2(t)}^{t} \|f(\Lambda(s)x(s)\|_p ds \\ &+ \|\Lambda(t)\|_p \|\alpha(x(t))\|_p \|D\|_p \int_{t-\tau_2(t)}^{t} \|f(x(s))\|_p ds \\ &+ \|I\|_p \|\Lambda(t)\|_p \|\alpha(x(t))\|_p + \|J\|_p \|\beta(y(t))\|_p.\end{aligned}$$

Using Assumptions (2.1)-(2.5), we get

$$\begin{aligned}\|H(x(t), \Lambda(t), \tau_1(t), \tau_2(t))\|_p \leq{}& (\|A\|_p + \|B\|_p)\, \bar{\eta}\rho C_p \|\Lambda(t)\|_p + (\|P\|_p + \|Q\|_p)\, \tilde{\eta}\rho C_p \|\Lambda(t)\|_p \\ &+ (\tilde{\eta}\|R\|_p + \bar{\eta}\|D\|_p)\rho C_p \tau \|\Lambda(t)\|_p + \left(\tilde{\eta}\varrho + \bar{\eta}\sigma\right) C_p \|\Lambda(t)\|_p \\ &+ \bar{\eta}\|\Lambda(t)\|_p \|I\|_P + \tilde{\eta}\|J\|_p, \end{aligned} \tag{2.9}$$

where $\bar{\eta} = \max_i \left\{\bar{\eta}_i\right\}, \rho = \max_i \{\rho_i\}, \sigma = \max_i \{\sigma_i\}, \tilde{\eta} = \max_i \{\tilde{\eta}_i\}, \varrho = \max_i \{\varrho_i\}$. From (2.9), $\|H(x(t), \Lambda(t), \tau_1(t), \tau_2(t)\|_p$ is bounded for $\forall t \geq -\tau$. Suppose that

$$\Xi = sup_{t\geq 0} \|H(x(t), \Lambda(t), \tau_1(t), \tau_2(t)\|_p, \tag{2.10}$$

where $\Xi < \infty$.

Theorem 2.1. *Suppose the Assumptions (2.1), (2.3) and (2.4) are true. Then the error system (2.7) is said to be exponential convergent within a small domain D containing the origin if there exists a matrix measure $\mu_p(.)(p = 1, \infty, \omega)$ and a* *non-singular matrix T such that*

$$\mu_p(-T\zeta(t)T^{-1}) + \varrho\tilde{\eta}\|T\|_p\|T^{-1}\| + \rho\tilde{\eta}\|P\|_p\|T\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|Q\|_p\|T\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|T\|_p\|T^{-1}\|_p\|R\|_p\tau_2(t) \leq -\delta_1 < 0, \forall t \geq 0, \tag{2.11}$$

where

$$D = \left\{ e \in \mathbb{R}^n | \|e(t)\|_p \leq \frac{\|T^{-1}\|_p\|T\|_p\Xi}{\delta_1} \right\},$$

and δ_1 is a positive real number.

Proof. Let T is nonsingular matrix then the chosen Lyapunov function is given by

$$V(e(t)) = \|Te(t)\|_p \tag{2.12}$$

The Dini derivative of the equation (2.12) with respect to t along the solution of error system (2.7) is

$$\begin{aligned} D^+(V(e(t))) =& \overline{\lim_{h\to 0^+}} \frac{\|Te(t+h)\|_p - \|Te(t)\|_p}{h} \\ =& \overline{\lim_{h\to 0^+}} \frac{\|Te(t) + hT\dot{e}(t) + o(h)\|_p - \|Te(t)\|_p}{h} \\ =& \overline{\lim_{h\to 0^+}} \frac{1}{h} \Bigg\{ \|Te(t) + hT\Big(- \zeta(t)e(t) - \beta(y(t))\tilde{\Gamma}(e(t)) \\ & + \beta(y(t))P\tilde{f}(e(t)) + \beta(y(t))Q\tilde{f}(e(t-\tau_1(t))) \\ & + \beta(y(t))R\int_{t-\tau_2(t)}^{t} \tilde{f}(e(s))ds + H(x(t), \Lambda(t), \tau_1(t), \tau_2(t))\Big) \\ & + O(h)\|_p - \|Te(t)\|_p \Bigg\}. \end{aligned}$$

$$\begin{aligned} D^+(V(e(t))) \leq & \overline{\lim_{h\to 0^+}} \frac{1}{h}\Big[\|Te(t)+h(-T\zeta(t)T^{-1})Te(t)\|_p - \|Te(t)\|_p\Big] \\ & + \|T\beta(y(t))\tilde{\Gamma}(e(t))\|_p + \|T\beta(y(t))P\tilde{f}(e(t))\|_p \\ & + \|T\beta(y(t))Q\tilde{f}(e(t-\tau_1(t)))\|_p \\ & + \|T\beta(y(t))R\int_{t-\tau_2(t)}^{t}\tilde{f}(e(s))ds\|_p \\ & + \|TH(x(t),\Lambda(t),\tau_1(t),\tau_2(t)\|_p. \end{aligned}$$

Using Assumptions (2.1), (2.3) and (2.4), we get

$$\begin{aligned} D^+(V(e(t))) \leq & \overline{\lim_{h\to 0^+}} \frac{\|I+h(-T\zeta(t)T^{-1})\|_p - 1}{h}\|Te(t)\|_p \\ & + \left(\tilde{\eta}\varrho\|T\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|T\|_p\|P\|_p\|T^{-1}\|_p\right)\|Te(t)\|_p \\ & + \Big(\tilde{\eta}\rho\|T\|_p\|Q\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|T\|_p\|R\|_p\|T^{-1}\|_p\tau_2(t)\Big) \\ & \times \sup_{s\in[t-\tau,t]}\|Te(s)\|_p + \|T\|_p\|H\|_p \end{aligned} \tag{2.13}$$

$$\begin{aligned} D^+(V(e(t))) \leq & \|T\|_p\|H\|_p + \Big(\mu_p(-T\zeta(t)T^{-1}) + \tilde{\eta}\varrho\|T\|_p\|T^{-1}\|_p \\ & + \tilde{\eta}\rho\|T\|_p\|P\|_p\|T^{-1}\|_p\Big) + \Big(\tilde{\eta}\rho\|T\|_p\|Q\|_p\|T^{-1}\|_p \\ & + \tilde{\eta}\rho\|T\|_p\|R\|_p\|T^{-1}\|_p\tau_2(t)\Big)\sup_{s\in[t-\tau,t]}\|Te(s)\|_p. \end{aligned} \tag{2.14}$$

Let

$$\begin{aligned} \xi_1(t) &= \|T\|_p\|H\|_p, \\ \xi_2(t) &= \mu_p(-T\zeta(t)T^{-1}) + \tilde{\eta}\varrho\|T\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|T\|_p\|P\|_p\|T^{-1}\|_p, \\ \xi_3(t) &= \tilde{\eta}\rho\|T\|_p\|Q\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|T\|_p\|R\|_p\|T^{-1}\|_p\tau_2(t). \end{aligned}$$

Then by using $\xi_1(t),\ \xi_2(t)$ and $\xi_3(t)$ in inequality (2.13), we get

$$D^+(V(e(t))) \le \xi_1(t) + \xi_2(t) + \xi_3(t) \sup_{t-\tau\le s\le t} \|Te(s)\|_p,\ t \ge 0.$$

Now, using (2.11) and Lemma (2.2.1) (Generalized Halanay inequality), we obtain

$$V(e(t)) \le \frac{\|T\|_p \Xi}{\delta_1} + \sup_{s\in[-\tau,0]} V(e(s)) e^{-\mu^* t}, \tag{2.15}$$

where $\mu^* = inf_{t\ge 0}\left\{\mu(t) : \mu(t) + \xi_2(t) + \xi_3(t)e^{\mu(t)\tau} = 0\right\}$. The equation (2.12) can be written as

$$\|T^{-1}\|_p V(e(t)) = \|T^{-1}\|_p \|Te(t)\|_p \ge \|e(t)\|_p; \tag{2.16}$$

$$V(e(t)) = \|Te(t)\|_p \le \|T\|_p \|e(t)\|_p. \tag{2.17}$$

By using the inequalities (2.16) and (2.17) in the inequality (2.3), we get

$$\|e(t)\|_p \le \frac{\|T^{-1}\|_p \|T\|_p \Xi}{\delta_1} + \|T^{-1}\|_p \|T\|_p \sup_{s\in[-\tau,0]} \|e(s)\|_p e^{-\mu^* t}. \tag{2.18}$$

Inequality (2.18) is showing the exponential convergence of the error system (2.7) within a small domain

$$D = \left\{ e(t) \in \mathbb{R}^n | \|e(t)\|_p \le \frac{\|T^{-1}\|_p \|T\|_p \Xi}{\delta_1} \right\}.$$

containing the origin. From (2.18), we observe that for an arbitrary $\epsilon > 0$, $\exists\ T' > 0$ such that

$$\|e(t)\|_p \le \epsilon + \frac{\|T^{-1}\|_p \|T\|_p \Xi}{\delta_1} \tag{2.19}$$

for any $t \ge T'$.

Thus from Definition (2.3.1), the weak MFPS is achieved between the systems (2.2) and (2.5) with a small synchronization error bound $\epsilon + \frac{\|T^{-1}\|_p \|T\|_p \Xi}{\delta_1}$. This completes the proof. □

Remark 2.3.2. The information about each of ρ_i for all $i = 1, 2, ..., n$ is not found in Theorem 2.1. So, we are going to derive the lemma based on Assumption 2.2 which will use ρ_i for each i. Based on this lemma we construct the Theorem 2.2 that will have more precise result than Theorem 2.1.

Lemma 2.3.1. Let the Assumption (2.1) is true, and $\|.\|_p$ be an induced matrix norm on $R^{n\times n}$ and suppose $\mu_p(.)$ be the respective matrix measure, then $\mu_p(\beta(y(t))PF(e(t))) \leq \mu_p(q\tilde{P}K)$, where

$$F(e(t)) = diag\left\{\frac{\tilde{f}_1(e_1(t))}{e_1(t)}, \frac{\tilde{f}_2(e_2(t))}{e_2(t)}, \frac{\tilde{f}_3(e_3(t))}{e_3(t)}, ..., \frac{\tilde{f}_n(e_n(t))}{e_n(t)}\right\},$$

$K = \text{diag}\{\rho_1, \rho_2, \rho_3, ..., \rho_n\}$, $\beta(y(t)) = \text{diag}\Big\{\beta_1(y_1(t)), \beta_2(y_2(t)), ..., \beta_n(y_n(t))\Big\}$, $q = \text{diag}\{\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_n\}$,

and

$$\tilde{P} = (\tilde{p_{ij}})_{n\times n} = \begin{cases} max(0, p_{ij}), & if i = j, \\ p_{ij}, & otherwise, \end{cases}$$

for $p = 1, \infty, \omega$.

Proof. We know from the definitions of P, $F(e(t))$, $\beta(y(t))$, Assumptions (2.1) and (2.2) that $0 \leq \frac{\tilde{f}_i(e_i(t))}{e_i(t)} \leq \rho_i, 1 \leq i \leq n$ and

$$\beta(y(t))PF(e(t)) = \begin{pmatrix} \beta_1(y_1(t))p_{11}\frac{\tilde{f}_1(e_1(t))}{e_1(t)} & ... \ \beta_1(y_1(t))p_{1n}\frac{\tilde{f}_n(e_n(t))}{e_n(t)} \\ \vdots & \vdots \\ \beta_n(y_n(t))p_{n1}\frac{\tilde{f}_1(e_1(t))}{e_1(t)} & ... \ \beta_n(y_n(t))p_{nn}\frac{\tilde{f}_n(e_n(t))}{e_n(t)} \end{pmatrix}.$$

For $p = 1$, Definition 1.3.4 will confirm the following

$$\mu_1(\beta(y(t))PF(e(t))) = \max_j \left[\beta_j(y_j(t))p_{jj}\frac{\tilde{f}(e_j(t))}{e_j(t)} + \sum_{i=1,i\neq j}^{n}\left|\beta_i(y_i(t))p_{ij}\frac{\tilde{f}(e_j(t))}{e_j(t)}\right|\right],$$

and

$$\begin{aligned}
&\beta_j(y_j(t))p_{jj}\frac{\tilde{f}(e_j(t))}{e_j(t)} \leq \tilde{\eta}_j max(0, p_{jj})\rho_j,\\
&\left|\beta_i(y_i(t))p_{ij}\frac{\tilde{f}(e_j(t))}{e_j(t)}\right| = |\beta_i(y_i(t))||p_{ij}|\left|\frac{\tilde{f}(e_j(t))}{e_j(t)}\right| \leq |\tilde{\eta}_i p_{ij}\rho_j|.
\end{aligned}$$

Thus, we obtain the following for $1 \leq j \leq n$

$$\beta_j(y_j(t))p_{jj}\frac{\tilde{f}(e_j(t))}{e_j(t)} + \sum_{i=1,i\neq j}^{n}\left|\beta_i(y_i(t))p_{ij}\frac{\tilde{f}(e_j(t))}{e_j(t)}\right| \leq \tilde{\eta}_j max(0, p_{jj})\rho_j + \sum_{i=1,i\neq j}^{n}|\tilde{\eta}_i p_{ij}\rho_j|. \tag{2.20}$$

From inequality (2.20), we can write $\mu_1(\beta(y(t))PF(e(t))) \leq \mu_1(q\tilde{P}K)$ where $\mu_1(q\tilde{P}K) = \max_j\left\{\tilde{\eta}_j max(0, p_{jj})\rho_j + \sum_{i=1,i\neq j}^{n}|\tilde{\eta}_i p_{ij}\rho_j|\right\}$. In similar way, we can show the proof of lemma for $p = \infty, \omega$. □

Theorem 2.2. *Based on Assumption (2.2) and Lemma (2.3.1), if there exists a matrix measure $\mu_p(.)$ $(p = 1, \infty, \omega)$ induced by a norm $\|.\|_p$ and a nonsingular matrix T such that*

$$\begin{aligned}
\mu_p(-T\zeta(t)T^{-1}) + \mu_p(Tq\tilde{P}KT^{-1}) + \tilde{\eta}\varrho\|T\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|T\|_p\|T^{-1}\|_p\|Q\|_p&\\
+\tilde{\eta}\|T\|_p\|T^{-1}\|_p\|R\|_p\tau_2(t) \leq -\delta_2 < 0, \forall t \geq 0.&
\end{aligned} \tag{2.21}$$

then the weak MFPS between the drive system (2.2) and the slave system (2.5) is achieved with an error bound $\epsilon + \frac{\|T\|_p\|T^{-1}\|_p\Xi}{\delta_2}$, where $\delta_2, \epsilon > 0$, $K = diag\{\rho_1, \rho_2, \rho_3, ..., \rho_n\}$,

$q = diag\left\{\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_n\right\}$, *and*

$$\tilde{P} = (\tilde{p_{ij}})_{n\times n} = \begin{cases} max(0, p_{ij}), & if\ i = j, \\ p_{ij}, & otherwise. \end{cases}$$

Proof. The Dini derivative of the chosen Lyapunov function in (2.12) with respect to t along the solution of error system (2.7) is

$$\begin{aligned} D^+(V(e(t))) =& \overline{\lim_{h\to 0^+}} \frac{\|Te(t+h)\|_p - \|Te(t)\|_p}{h} \\ =& \overline{\lim_{h\to 0^+}} \frac{\|Te(t) + hT\dot{e}(t) + o(h)\|_p - \|Te(t)\|_p}{h} \\ =& \overline{\lim_{h\to 0^+}} \frac{1}{h}\Bigg\{ \|Te(t) + hT\Big(- \zeta(t)e(t) - \beta(y(t))\tilde{\Gamma}(e(t)) \\ & + \beta(y(t))P\tilde{f}(e(t)) + \beta(y(t))Q\tilde{f}(e(t-\tau_1(t))) \\ & + \beta(y(t))R\int_{t-\tau_2(t)}^{t} \tilde{f}(e(s))ds + H(x(t), \Lambda(t), \tau_1(t), \tau_2(t))\Big) \\ & + O(h)\|_p - \|Te(t)\|_p \Bigg\} \\ \leq& \overline{\lim_{h\to 0^+}} \frac{1}{h}\left[\|Te(t) + hT\left(-\zeta(t)e(t) + \beta(y(t))P\tilde{f}(e(t))\right)\|_p - \|Te(t)\|_p\right] \\ & + \|T\beta(y(t))\tilde{\Gamma}(e(t))\|_p + \|T\beta(y(t))Q\tilde{f}(e(t-\tau_1(t)))\|_p + \|T\beta(y(t))R \\ & \times \int_{t-\tau_2(t)}^{t} \tilde{f}(e(s))ds\|_p + \|TH(x(t), \Lambda(t), \tau_1(t), \tau_2(t)\|_p. \end{aligned} \tag{2.22}$$

From Lemma 2.2.1, we have

$$F(e(t)) = diag\left\{\frac{\tilde{f}_1(e_1(t))}{e_1(t)}, \frac{\tilde{f}_2(e_2(t))}{e_2(t)}, ..., \frac{\tilde{f}_n(e_n(t))}{e_n(t)}\right\},$$

which can be written as $\tilde{f}(e(t)) = F(e(t))e(t)$. Now putting it in the inequality (2.22), we get

$$D^+(V(e(t))) \leq \overline{\lim_{h\to 0^+}} \frac{1}{h}\Big[\|I + h\big(-T\zeta(t)T^{-1} + T\beta(y(t))PF(e(t))T^{-1}\big)\|_p - 1\Big]\|Te(t)\|_p + \tilde{\eta}\varrho\|T\|_p\|T^{-1}\|_p\|Te(t)\|_p + \Big(\tilde{\eta}\rho\|T\|_p\|Q\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|T\|_p\|R\|_p \times\|T^{-1}\|_p\tau_2(t)\Big)\sup_{t-\tau\leq t}\|Te(s)\|_p + \|T\|_p\|H\|_p. \tag{2.23}$$

$$D^+(V(e(t))) \leq \big(\mu_p(-T\zeta(t)T^{-1}) + \mu_p(T\beta(y(t))PF(e(t))T^{-1})\big)\|Te(t)\|_p + \tilde{\eta}\varrho\|T\|_p \times \|T^{-1}\|_p\|Te(t)\|_p + \Big(\tilde{\eta}\rho\|T\|_p\|Q\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|T\|_p\|R\|_p\|T^{-1}\|_p\tau_2(t)\Big) \times \sup_{t-\tau\leq t}\|Te(s)\|_p + \|T\|_p\|H\|_p. \tag{2.24}$$

$$D^+(V(e(t))) \leq \|T\|_p\|H\|_p + \Big(\mu_p(-T\zeta(t)T^{-1}) + \mu_p(Tq\tilde{P}KT^{-1}) + \tilde{\eta}\varrho\|T\|_p\|T^{-1}\|_p\Big) \times \|Te(t)\|_p + \Big(\tilde{\eta}\rho\|T\|_p\|Q\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|T\|_p\|R\|_p\|T^{-1}\|_p\tau_2(t)\Big) \times \sup_{t-\tau\leq s\leq t}\|Te(s)\|_p. \tag{2.25}$$

Suppose

$$\xi_1(t) = \|T\|_p\|H\|_p,$$
$$\xi_2(t) = \mu_p(-T\zeta(t)T^{-1}) + \mu_p(Tq\tilde{P}KT^{-1}) + \tilde{\eta}\varrho\|T\|_p\|T^{-1}\|_p,$$
$$\xi_3(t) = \tilde{\eta}\rho\|T\|_p\|Q\|_p\|T^{-1}\|_p + \tilde{\eta}\rho\|T\|_p\|R\|_p\|T^{-1}\|_p\tau_2(t).$$

Putting these $\xi_1(t)$,$\xi_2(t)$ and $\xi_3(t)$ in inequality (2.25), we get

$$D^+(V(e(t))) \leq \xi_1(t) + \xi_2(t)V(e(t)) + \xi_3(t)\sup_{t-\tau\leq s\leq t} V(e(s)).$$

From inequality (2.21), we have $\xi_2(t) + \xi_3(t) \leq -\delta_2 < 0$. According to Lemma 2.2.1 (Generalized Halanay inequality), we will have

$$V(e(t)) \leq \frac{\|T\|_p \Xi}{\delta_2} + \sup_{s\in[-\tau,0]} V(e(s))e^{-\mu_1^* t},$$

where $\mu_1^* = inf_{t\geq 0}\left\{\mu(t) : \mu(t) + \xi_2(t) + \xi_3(t)e^{\mu(t)\tau} = 0\right\}$. Using the estimations (2.16) and (2.17), we will get the estimation of the error system (2.7) as

$$\|e(t)\|_p \leq \frac{\|T^{-1}\|_p\|T\|_p\Xi}{\delta_2} + \|T^{-1}\|_p\|T\|_p \sup_{s\in[-\tau,0]} \|e(s)\|_p e^{-\mu_1^* t}. \tag{2.26}$$

It can be observed from the inequality (2.26) that we get the small domain D of exponential convergence of the error system (2.7) as

$$D = \left\{ e \in \mathbb{R}^n | \|e(t)\|_p \leq \frac{\|T^{-1}\|_p\|T\|_p\Xi}{\delta_2} \right\}$$

and it is obvious from (2.26) that for $\epsilon > 0$, $\exists\, T^{'} > 0$ such that

$$\|e(t)\|_p \leq \epsilon + \frac{\|T^{-1}\|_p\|T\|_p\Xi}{\delta_2}, \forall t \geq T^{'}. \tag{2.27}$$

Thus, the weak MFPS between the systems (2.2) and (2.5) is achieved with a small error bound $\epsilon + \frac{\|T^{-1}\|_p\|T\|_p\Xi}{\delta_2}$. □

If we replace the non singular matrix T with identity matrix I in Theorem 2.1 and 2.2, then the following corollaries can be drawn.

Corollary 2.3.1. There exists a matrix measure $\mu_p(.)$ $(p = 1, \omega, \infty)$ induced by a norm $\|.\|_p$ such that

$$\mu_p(-\zeta(t)) + \varrho\tilde{\eta} + \rho\tilde{\eta}\|P\|_p + \tilde{\eta}\rho\|Q\|_p + \tilde{\eta}\rho\|R\|_p\tau_2(t) \leq -\delta_1 < 0, \forall t \geq 0. \tag{2.28}$$

holds then the trajectory of error system (2.7) converges exponentially in the domain

$$D = \left\{ e \in \mathbb{R}^n | \|e(t)\|_p \leq \frac{\Xi}{\delta_1} \right\}.$$

It means the slave system (2.5) achieve weak MFPS with the drive system (2.2) up to a small error bound $\epsilon + \frac{\Xi}{\delta_1}$.

Corollary 2.3.2. If there exists a matrix measure $\mu_p(.)$ $(p = 1, \infty, \omega)$ induced by a norm $\|.\|_p$ such that

$$\mu_p(-\zeta(t)) + \mu_p(q\tilde{P}K) + \tilde{\eta}\varrho + \tilde{\eta}\rho\|Q\|_p + \tilde{\eta}\|R\|_p\tau_2(t) \leq -\delta_2 < 0, \forall t \geq 0, \qquad (2.29)$$

then weak MFPS is achieved between the slave and drive systems (2.5)-(2.2) with an error bound $\epsilon + \frac{\Xi}{\delta_2}$. The trajectory of error system (2.7) will be exponentially converging to a small domain

$$D = \left\{ e \in \mathbb{R}^n | \|e(t)\|_p \leq \frac{\Xi}{\delta_2} \right\}.$$

Remark 2.3.3. The benefits of using ω-measure is if we assume $\omega_i << \omega_j$ then Ξ can be minimized to a desired synchronization error bound that can not be possible in others measures.

Remark 2.3.4. From Definition 1.3.4, we can see that matrix norm $\| - M\|_p = \|M\|_p, (i = 1, 2, \omega, \infty)$ is restricted to a non-negative value. On the other hand, matrix measure can be negative or positive or zero. Weights of interconnections among the neurons can be negative or positive depending on inhibitory or excitatory signals respectively. Thus, matrix measure approach gives more intuitive results than norm.

Remark 2.3.5. If the amplification functions and scaling matrix are constant then the obtained results will also be true for $p = 2$.

2.4 Results and discussion

In this section, the following example has been considered to validate the effectiveness of the results of our proposed theorems. Let us consider the numerical examples of the drive and slave systems as

$$\begin{aligned}\dot{x}(t) =& -\alpha\left[\Upsilon x(t) - Af(x(t)) - Bf(x(t-\tau_1(t))) - D\int_{t-\tau_2(t)}^{t} f(x(s))ds \ + I\right],\\ x(t) =& \phi(t), \quad t \in [-\tau, 0]\end{aligned} \tag{2.30}$$

and

$$\begin{aligned}\dot{y}(t) =& -\beta\left[\mathit{\Gamma} y(t) - Pf(y(t)) - Qf(y(t-\tau_1(t))) - R\int_{t-\tau_2(t)}^{t} f(y(s))ds \ + J\right] + U(t),\\ y(t) =& \varphi(t), \quad t \in [-\tau, 0].\end{aligned} \tag{2.31}$$

The parameters of both equations are taken as

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -0.11 \\ -5.0 & 3.2 \end{bmatrix},$$

$$B = \begin{bmatrix} -1.6 & -0.1 \\ -0.18 & -2.4 \end{bmatrix}, \quad D = \begin{bmatrix} -0.5 & 1 \\ 2 & -1.8 \end{bmatrix},$$

and

$$\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.97 & 0 \\ 0 & 1.1 \end{bmatrix}, \quad P = \begin{bmatrix} 2.1 & -0.1 \\ -5.1 & 3.19 \end{bmatrix}$$

$$Q = \begin{bmatrix} -1.5 & 0 \\ -0.15 & -2.3 \end{bmatrix}, R = \begin{bmatrix} -0.48 & 1.02 \\ 2.1 & -1.9 \end{bmatrix},$$

where $x(t) = [x_1(t), x_2(t)]^T$ and $y(t) = [y_1(t), y_2(t)]^T$ are the state vectors of the systems (2.30) and (2.31) respectively. $f(x(t)) = [\tanh x_1(t), \quad \tanh x_2(t)]^T$ is activation function which satisfies the Assumptions 2.2 and 2.3, $\tau_1(t) = \tau_2(t) = \frac{e^t}{1+e^t}$ with $\tau = 1$. Let $\nu_1(t) = 2 + 0.3sin(\frac{2\pi t}{40})$ and $\nu_2(t) = 2 + 0.28sin(\frac{2\pi t}{40})$ are the entries of the scaling function $\Lambda(t) = \text{diag}(\nu_1(t), \nu_2(t))$ and its norm $\|\Lambda(t)\|_1 = 2.3$. External inputs are $I = J = [0, 0]$.

If the coupling strength $\zeta(t) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for the linear controller $U(t) = \zeta(t)\big(y(t) - \Lambda(t)x(t)\big)$, then both systems will be independent of each other as we can see in Fig.2.1(b). Consider $\zeta(t) = \left[\begin{smallmatrix} 40 & 0 \\ 0 & 42 \end{smallmatrix}\right]$ and $T = I$, then the error system converges exponentially to a small domain $D = \{e \in \mathbb{R}^n | \|e(t)\|_p \leq 0.39\}$ which can be seen in Fig.2.2(a), i.e., the weak MFPS between the slave system (2.5) and the drive system (2.2) is achieved with an error bound 0.39. But, in Fig.(2.2(b)), for $\Lambda(t) = \left[\begin{smallmatrix} 0.5 & 0 \\ 0 & 0.4 \end{smallmatrix}\right]$ the small domain of convergence is reduced to an upper bound 0.14 i.e., $D = \{e \in \mathbb{R}^n | \|e(t)\|_p \leq 0.14\}$.This is happened because Ξ also depends on norm of $\Lambda(t)$. From Corollaries 2.3.1 and 2.3.2, we can see that error bound depends on Ξ and δ_1 or δ_2. Another way to reduce the error bound is stated in Remark 2.3.3. As we stated, for $\omega-$measure if we take $\omega_i << \omega_j, i \neq j$, the error bound will be tending to zero as shown in the Fig.2.3. We have $\tilde{\eta} = 1, \rho = 1, \varrho = 1.1, \|P\|_1 = 7.2, \|Q\|_1 = 2.3$

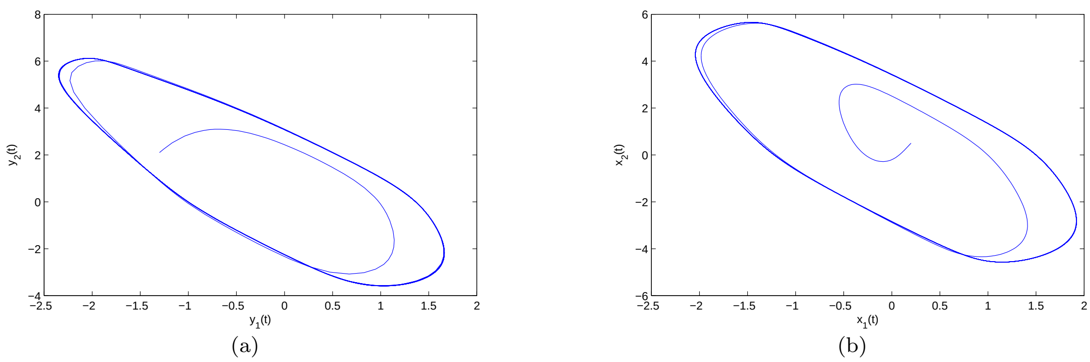
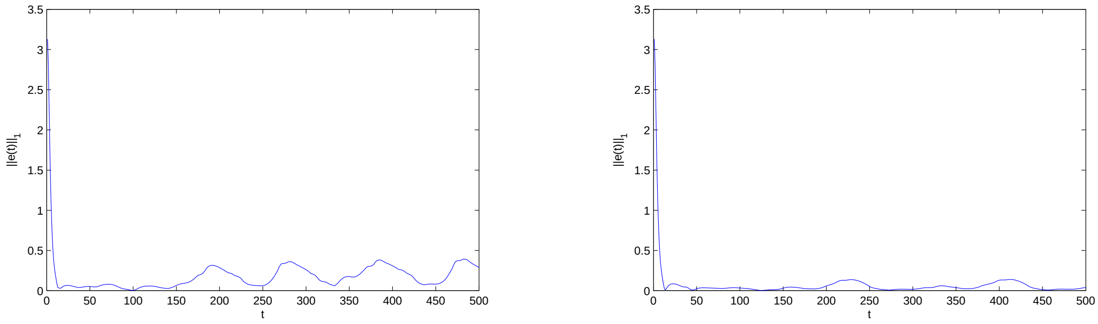


FIGURE 2.1: Periodic attractors of neural networks without coupling terms.



(a) (b)

FIGURE 2.2: Time evolution of the error system is shown in (a) for $\|\Lambda(t)\|_1 = 2.3$, and in (b) for $\|\Lambda(t)\|_1 = 0.5$.

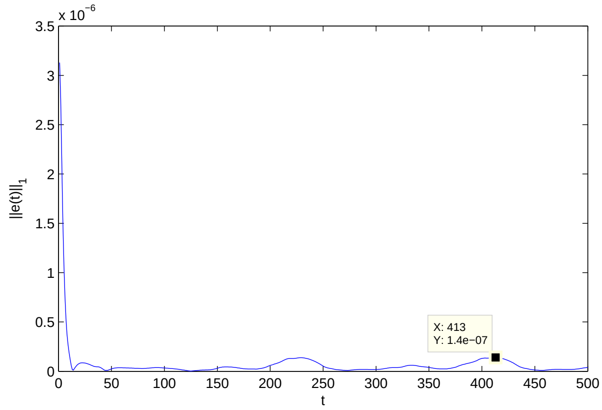


FIGURE 2.3: Time evaluation of the error system for $\omega_i/\omega_j \approx 0.000001, i \neq j$.

and $\|R\|_1 = 2.92$ for the systems (2.30) and (2.31). By putting these values in result of Theorem 2.1, we get $-\mu_1(-\zeta(t)) = 42 > 10.60$. For Theorem 2.2, It is obvious that $3.30 < -\mu_1(-\zeta(t)) - \mu_p(q\tilde{P}k) = 49.20$. Thus all the theorems are verified.

2.5 Conclusion

This chapter discussed a weak MFPS of two different CGNNs with time-varying delays. Due to the presence of parameter mismatches, the complete MFPS between the drive-response systems is not possible. Therefore, a new concept, viz.,weak MFPS is investigated with a small synchronization error bound. Based on the generalized Halanay inequality and Matrix measure, the state of the error system is estimated and several generic criteria are derived. The effectiveness of our proposed theory is validated through an example.

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