Chapter 2

Mathematical Formulation

2.1. Introduction

In this chapter, the formulation of the governing equations required for carrying out static and dynamic analysis of plate structures resting on an elastic foundation is presented in the framework of a plate theory. The problem of a multi-layered smart laminated composite plate with a piezoelectric actuator and a sensor under the action of electromechanical loads is considered for the analyses. A non-polynomial HSDT with ZZ kinematics is chosen as the plate theory for describing the in-plane and transverse displacements of any point inside the plate. The kinematic model employs a trigonometric function namely the secant function as the shear-strain function for introducing the non-linear transverse shear strains through the thickness of the smart composite plates. The model consists of only five primary variables like the FSDT and does not need any shear correction factor to be multiplied with the transverse shear stiffness coefficients. The inter-laminar continuity conditions of the transverse shear stresses are also satisfied at all the interfaces of the plate structure.

The governing equations of equilibrium are derived with Hamilton's principle which produces a system of five partial differential equations (PDEs) corresponding to the primary variables in terms of integrated quantities like the stress-resultants and inertia components. These quantities are defined over a unit length of the plate and are responsible for reducing the 3 D nature of the problem to a 2 D plate with the integrated values of the 3 D stresses and density 'ρ^k'of each discrete layer. The five governing PDEs are associated with fourteen stress-resultants which makes the problem indeterminate. The indeterminacy is removed with fourteen plate-constitutive equations which are expressed in terms of the integrated material properties of the discrete layers known as the rigidity matrices and spatial derivatives of the five unknown primary variables. Therefore nineteen unknowns are finally associated with nineteen unknowns, which make the problem determinate. The solutions of the final governing equations are obtained using an analytical and numerical approach. For the analytical solutions, Navier's solution technique is used in which the separation of the variables concept is applied to express the primary variables in terms of double trigonometric series in the spatial domain. The PDEs are then transformed to a system of ODEs in time with the assumed solutions in space, and the solutions of the ODEs in time are obtained with Newmark's time integration scheme. For the numerical solutions, the finite element method (FEM) in conjunction with Newmark's time integration scheme is employed to solve the governing equations. A C^0 continuous isoparametric formulation is developed by considering an eight-noded serendipity element for the spatial discretization of the physical domain. The overall framework of the solutions to the governing equations is still the same, *i.e*, first assuming some solutions of the primary variables in the physical domain. In the FEM, this is achieved with the help of shape functions defined for an element. The primary variables are discretized in terms of the shape functions and the unknown generalized coordinates. The discretized equations of the primary variables are further used to discretize additional relations which are involved in the formulation. With the help of Hamilton's principle, a discretized system of ODEs is obtained as the dynamic governing equations. Then a suitable time integration scheme, like in this case, the Newmark's time integration scheme is used to solve the ODEs.

The present chapter deals with the analytical and FE modeling of smart composite plates resting on an elastic foundation for static and dynamic analysis. The assumptions made in the present investigations, basic equations like the stress-strain relationships, strain displacement relationships, foundation model and the kinematic model are first presented followed by a detailed description of the formulation of the governing equations and the method of solutions.

2.2. Basic Assumptions

A smart laminated composite plate is considered with a piezoelectric actuator and sensor at the top and bottom of the plate, respectively, also shown in Figure 2.1. The Cartesian coordinate system (x_1, x_2, x_3) is considered throughout the formulation. The underlying assumptions made in the present mathematical formulations are as follows:

- The smart laminated composite plates considered in the formulation do not fall in the micro and nano-scale such that the small-scale effects are discarded.
- The discrete layers including the piezoelectric layers that are stacked in the thickness direction are homogeneous and orthotropic.
- The bonding in between the layers is sufficiently strong to prevent any slip and separation in between the layers.
- The midplane (z = 0) is considered as the reference plane.
- The materials in the present formulations obey Hooke's law.
- The lateral deflection is very small in comparison to the in-plane dimensions of the plate structures.
- The transverse normal stress is very small in comparison to the other stresses and therefore neglected.
- The transverse displacement is assumed to be constant across the thickness of the smart composite plates. Therefore, the thickness-stretching effects are not considered.

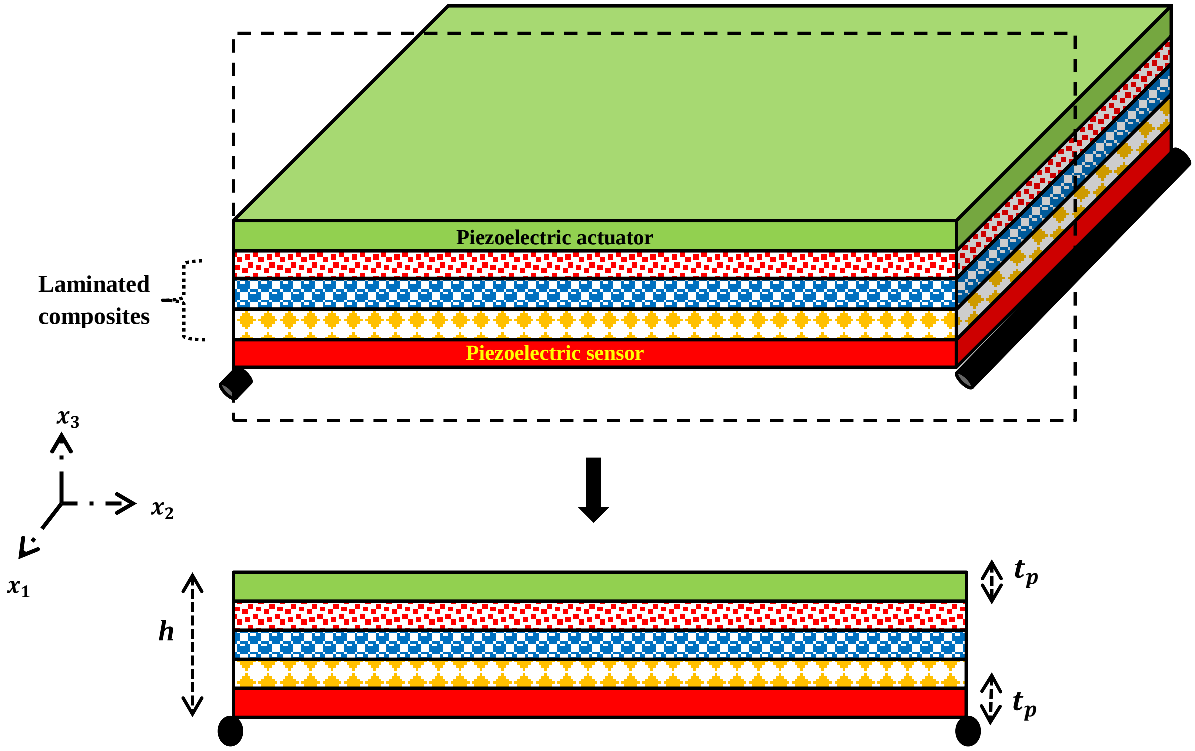

Figure 2.1. Laminated composite plate with a piezoelectric actuator and a piezoelectric sensor

2.3. Stress-Strain Constitutive Relations

The stress-strain relationships of the homogeneous orthotropic lamina with the material axes aligned with the global axes are written as

$$\{\bar{\sigma}\}^k = [Q]^k\{\bar{\varepsilon}\} \tag{2.1a}$$

where, $\{\bar{\sigma}\}^k$ and $\{\bar{\varepsilon}\}$ are the stress and strain vectors at any point in the k^{th} lamina defined in the material coordinate axes system.

The components of the stress and strain vectors are given below

$$\{\bar{\sigma}\}^k = \{\bar{\sigma}_{11} \quad \bar{\sigma}_{22} \quad \bar{\tau}_{12} \quad \bar{\tau}_{23} \quad \bar{\tau}_{13}\}^t \text{ and } \{\bar{\varepsilon}\} = \{\bar{\varepsilon}_{11} \quad \bar{\varepsilon}_{22} \quad \bar{\gamma}_{12} \quad \bar{\gamma}_{23} \quad \bar{\gamma}_{13}\}^t \tag{2.1b}$$

$[Q]^k$ is known as the reduced stiffness coefficient matrix and it is used to relate the stress and strain vectors of the k^{th} layer. The components of the reduced stiffness matrix are given below which are obtained from the plane stress condition.

$$[Q]^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix}^{(k)}$$

where, $Q_{11} = \frac{E_{11}}{1-\vartheta_{12}\vartheta_{21}}$; $Q_{22} = \frac{E_{22}}{1-\vartheta_{12}\vartheta_{21}}$; $Q_{12} = \frac{\vartheta_{21}E_{11}}{1-\vartheta_{12}\vartheta_{21}}$; $Q_{66} = G_{12}$

$Q_{44} = G_{23}$ and $Q_{55} = G_{13}$ 2.1c

The directions '1' and '2' are the directions along the fibers and perpendicular to the fibers, respectively, and also refers to the material axis system. When the material axis '1' of the fibers are aligned to the coordinate axis 'x_1' at an angle 'θ', then the modified stress-strain relationship of the orthotropic lamina is written as

$$\{\sigma\}^k = [\bar{Q}]^k\{\varepsilon\} \qquad 2.2$$

where, $\{\sigma\}^k = \{\sigma_{11} \quad \sigma_{22} \quad \tau_{12} \quad \tau_{23} \quad \tau_{13}\}^t$ and $\{\varepsilon\} = \{\varepsilon_{11} \quad \varepsilon_{22} \quad \gamma_{12} \quad \gamma_{23} \quad \gamma_{13}\}^t$ are the transformed stress and strain vector defined in the global coordinate system (x_1, x_2, x_3). $[\bar{Q}]^k$ is now denoted as the transformed reduced stiffness matrix and its components are given by

$$[\bar{Q}]^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}^{(k)} \qquad 2.3a$$

where,

$$\begin{Bmatrix} \bar{Q}_{11} \\ \bar{Q}_{12} \\ \bar{Q}_{22} \\ \bar{Q}_{16} \\ \bar{Q}_{26} \\ \bar{Q}_{66} \\ \bar{Q}_{44} \\ \bar{Q}_{45} \\ \bar{Q}_{55} \end{Bmatrix}^{(k)} = \begin{bmatrix} C^4 & 2C^2S^2 & S^4 & 4C^2S^2 & 0 & 0 \\ C^2S^2 & C^4+S^4 & C^2S^2 & -4C^2S^2 & 0 & 0 \\ S^4 & 2C^2S^2 & C^4 & 4C^2S^2 & 0 & 0 \\ C^3S & CS^3-C^3S & -CS^3 & -2CS(C^2-S^2) & 0 & 0 \\ CS^3 & C^3S-CS^3 & -C^3S & 2CS(C^2-S^2) & 0 & 0 \\ C^2S^2 & -2C^2S^2 & C^2S^2 & (C^2-S^2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & C^2 & S^2 \\ 0 & 0 & 0 & 0 & -CS & CS \\ 0 & 0 & 0 & 0 & S^2 & C^2 \end{bmatrix} \begin{Bmatrix} Q_{11} \\ Q_{12} \\ Q_{22} \\ Q_{66} \\ Q_{44} \\ Q_{55} \end{Bmatrix}^{(k)}$$

C = cosθ and S = sinθ 2.3b

The stress-strain constitutive relationships of the piezoelectric lamina in the material axis system is written as

$\{\bar{\sigma}\}^p = [Q]^p\{\bar{\varepsilon}\} - [e]^p\{\bar{E}\}^p$ 2.4a

where $[e]^p$ is the matrix containing the piezoelectric coefficients which couples the mechanical stress vector '$\{\bar{\sigma}\}^p$' with the electric field vector '$\{\bar{E}\}^p$'. The superscript '^{p}' denotes the piezoelectric layer. The piezoelectric layers are homogeneous and orthotropic, therefore, the coefficients of $\{\bar{\sigma}\}^p$, $\{\bar{\varepsilon}\}$ and $[Q]^p$ are same as defined in Eqs. 2.1(a-c). The components of the piezoelectric coefficient matrix and the electric field vector are given below.

$$[e]^p = \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & 0 \\ 0 & e_{24} & 0 \\ e_{15} & 0 & 0 \end{bmatrix}^P ; \{\bar{E}\}^p = \begin{Bmatrix} \bar{E}_{11} \\ \bar{E}_{22} \\ \bar{E}_{22} \end{Bmatrix}^p \quad 2.4b$$

Eq. 2.4a is also known as the converse law or the actuator law. Similarly, a direct law is available for the piezoelectric lamina which can also be called as the sensor law.

$\{\bar{D}\}^p = [e]^p\{\bar{\varepsilon}\} - [\epsilon]^p\{\bar{E}\}^p$ 2.5a where, $\{\bar{D}\}^p$ and $[\epsilon]^p$ are known as the electric displacement vector and electric permittivity matrix, respectively. The coefficients of the electric displacement vector and the electric permittivity matrix are given below

$$\{\bar{D}\}^p = \begin{Bmatrix} \bar{D}_{11} \\ \bar{D}_{22} \\ \bar{D}_{33} \end{Bmatrix}^p ; [\epsilon]^p = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \qquad 2.5b$$

The actuator law is used to determine the actuation in the responses when an electric voltage is applied on the piezoelectric layer whereas the sensor law is used to calculate the total charges accumulated on the electrodes of the piezoelectric layer when the layers experience mechanical strains. The transformed converse and direct relationships in the global coordinate system are expressed as follows:

$$\{\sigma\}^p = [\bar{Q}]^p\{\varepsilon\} - [\bar{e}]^p\{E\}^p \qquad 2.6a$$

$$\{D\}^p = [\bar{e}]^p\{\varepsilon\} - [\bar{\epsilon}]^p\{E\}^p \qquad 2.6b$$

The coefficients of $\{\sigma\}^p$, $\{\varepsilon\}$ and $[\bar{Q}]^p$ are same as shown in Eqs. (2.2, 2.3a and 2.3b). The coefficients of the piezoelectric coefficient matrix, electric permittivity matrix and the electric displacement vector in the global coordinate system are given below

$$[\bar{e}]^p = \begin{bmatrix} 0 & 0 & \bar{e}_{31} \\ 0 & 0 & \bar{e}_{32} \\ 0 & 0 & \bar{e}_{36} \\ \bar{e}_{14} & \bar{e}_{24} & 0 \\ \bar{e}_{15} & \bar{e}_{25} & 0 \end{bmatrix}^p ; [\bar{\epsilon}]^p = \begin{bmatrix} \bar{\epsilon}_{11} & \bar{\epsilon}_{12} & 0 \\ \bar{\epsilon}_{12} & \bar{\epsilon}_{22} & 0 \\ 0 & 0 & \bar{\epsilon}_{33} \end{bmatrix}^p \text{ and } \{D\}^p = \begin{Bmatrix} D_{11} \\ D_{22} \\ D_{33} \end{Bmatrix}^p$$

where,

$$\begin{Bmatrix} \bar{e}_{31} \\ \bar{e}_{32} \\ \bar{e}_{36} \\ \bar{e}_{14} \\ \bar{e}_{24} \\ \bar{e}_{15} \\ \bar{e}_{25} \end{Bmatrix}^p = \begin{bmatrix} C^2 & S^2 & 0 & 0 \\ S^2 & C^2 & 0 & 0 \\ CS & -CS & 0 & 0 \\ 0 & 0 & -CS & CS \\ 0 & 0 & C^2 & S^2 \\ 0 & 0 & S^2 & C^2 \\ 0 & 0 & -CS & CS \end{bmatrix}^p \begin{Bmatrix} e_{31} \\ e_{32} \\ e_{24} \\ e_{15} \end{Bmatrix}^p ; \begin{Bmatrix} \bar{\epsilon}_{11} \\ \bar{\epsilon}_{22} \\ \bar{\epsilon}_{12} \\ \bar{\epsilon}_{33} \end{Bmatrix}^p = \begin{bmatrix} C^2 & S^2 & 0 \\ S^2 & C^2 & 0 \\ CS & -CS & 0 \\ 0 & 0 & 1 \end{bmatrix}^p \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \end{Bmatrix}^p \qquad 2.6c$$

2.4. Strain displacement relationships

The strain displacement relations corresponding to the linear theory of elasticity as given by

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial U_1}{\partial x_1} \\ \dfrac{\partial U_2}{\partial x_2} \\ \left(\dfrac{\partial U_1}{\partial x_2} + \dfrac{\partial U_2}{\partial x_1}\right) \\ \left(\dfrac{\partial U_2}{\partial x_3} + \dfrac{\partial U_3}{\partial x_2}\right) \\ \left(\dfrac{\partial U_1}{\partial x_3} + \dfrac{\partial U_3}{\partial x_1}\right) \end{Bmatrix} \qquad 2.7$$

where, U_1, U_2 and U_3 are the displacements in x_1, x_2 and x_3-direction, respectively.

2.5. Plates on elastic foundation

In the past, the Winkler model has been effectively used to model the elastic soil under the beams, plates and shell structures. The Winkler model is a one parameter model based on the “Winkler’s hypothesis,” which states that the deflection at any point on a surface of the elastic soil is proportional to the load being applied onto the surface and is independent of the load being applied on any other points on the surface (Tanahashi, 2007). It is due to this hypothesis that leads to a mechanical model of the elastic soil by assuming mutually independent vertical springs. The shortcoming in this model is the discontinuity of the adjacent displacements in the mutually independent springs. In this research, an improved two parameter model which takes into account the proportional interaction between the pressure and deflection of any point on the surface of the elastic soil and also accommodates the continuity of the adjacent displacements by considering shear interactions among the points on the elastic soil. This model is also known as the Pasternak’s foundation model (Zenkour, 2010). The reaction-deflection relationship of the Pasternak’s model is written as

$$f_{EF} = k_w U_3 - k_{s1}\frac{\partial^2 U_3}{\partial {x_1}^2} - k_{s2}\frac{\partial^2 U_3}{\partial {x_2}^2} \qquad 2.8a$$

where, f_{EF} is the reaction, k_w is the modulus of the subgrade reaction or the stiffness coefficient of the springs and k_{s1}, k_{s2} are the shear moduli of the subgrade or the shear stiffness coefficients of the foundation (shear layer). If the soil is homogenous and isotropic, then both the shear moduli is considered to be equal. In that case, k_{s1}=k_{s2}=k_s. In the present formulation, the soil/foundation is assumed to be homogenous and isotropic. Thus the modified version of the reaction-deflection relationship in Eq. 2.8a is given by

$$f_{EF} = k_w U_3 - k_s\left(\frac{\partial^2 U_3}{\partial {x_1}^2} + \frac{\partial^2 U_3}{\partial {x_2}^2}\right) \qquad 2.8b$$

If the shear stiffness of the soil is neglected, then the Pasternak's model reduces to the Winkler model. A pictorial representation of a smart composite plate resting on an elastic foundation modeled using Pasternak's model is shown in Figure 2.2.

2.6. Kinematic field

In the present research, the Trigonometric ZZ theory is used to describe the deformation of any point inside the plate. This model, as the name suggests, incorporates a trigonometric function, namely the secant function for generating the nonlinear through-thickness profile of the transverse shear stresses. The kinematics for the in-plane displacement components are obtained by superposing a globally varying nonlinear field on a piecewise linearly varying zigzag field with slope discontinuities at the layer interfaces.

Present model consists of deformation modes (u_1,u_2, u_3, β_1, β_2) defined at the mid-plane and auxiliary variables (α^i_{1u}, α^j_{1l}, α^i_{2u}, α^j_{2l}) defined at the interfaces of the plate. The auxiliary variables represent the changes in the slope of the in-plane displacement components (U_1, U_2) across the thickness of the layered plate structure.

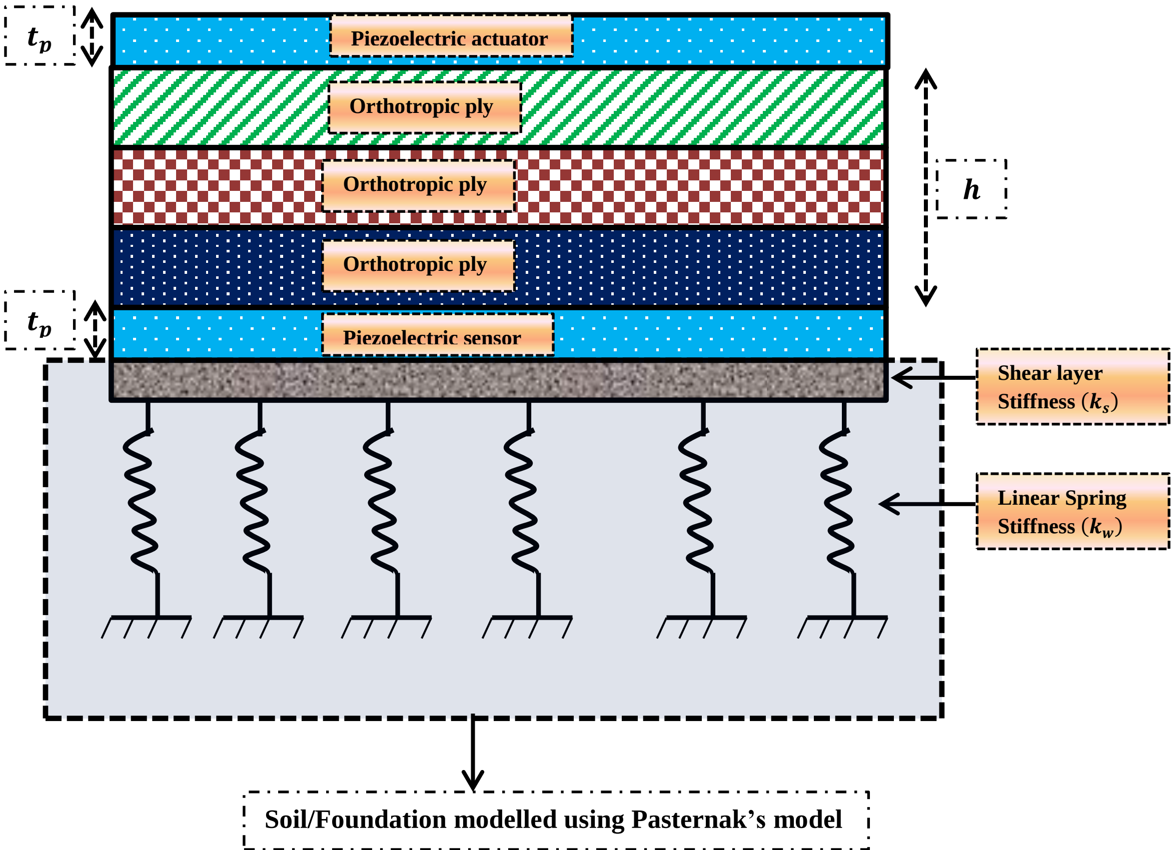

Figure 2.2. Smart composite plate resting on an elastic foundation

A detailed illustration of the kinematics is shown in Figure 2.3. The present model can be expressed as the sum of ESL field and ZZ field as shown below:

$$\{U_\aleph\} = \{U_\aleph{}^{\mathrm{ESL}}\} + \{U_\aleph{}^{\mathrm{ZZ}}\} \quad 2.9$$

where, $\aleph$ can take values 1 and 2, denoting the 3 D in-plane displacements, $U_1(x_1, x_2, x_3, t)$ and $U_2(x_1, x_2, x_3, t)$, respectively.

$\{U_\aleph{}^{\mathrm{ESL}}\}$ is used to denote the ESL field while $\{U_\aleph{}^{\mathrm{ZZ}}\}$ denotes the ZZ field.

The ESL field consists of a non-polynomial HSDT while the ZZ field consists of piecewise linear mathematical functions of the thickness coordinate. The ESL and the ZZ fields are given by

$$\begin{Bmatrix} U_1{}^{\mathrm{ESL}} \\ U_2{}^{\mathrm{ESL}} \end{Bmatrix} = \begin{Bmatrix} u_1(x_1,x_2,t) - x_3 u_{3,x_1}(x_1,x_2,t) + \left\{x_3 \sec\left(\frac{rx_3}{h}\right) + x_3\Omega_1\right\}\beta_1(x_1,x_2,t) \\ u_2(x_1,x_2,t) - x_3 u_{3,x_2}(x_1,x_2,t) + \left\{x_3 \sec\left(\frac{rx_3}{h}\right) + x_3\Omega_2\right\}\beta_2(x_1,x_2,t) \end{Bmatrix}$$

CPT

Refinements using non-polynomial functions

Non-polynomial HSDT

$$\begin{Bmatrix} U_1{}^{\mathrm{ZZ}} \\ U_2{}^{\mathrm{ZZ}} \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n_u-1}(x_3 - x_3{}^{iu})H(x_3 - x_3{}^{iu})\,\alpha_{1u}^{i}(x_1,x_2,t) + \sum_{j=1}^{n_l-1}(x_3 - x_3{}^{jl})H(-x_3 + x_3{}^{jl})\,\alpha_{1l}^{j}(x_1,x_2,t) \\ \sum_{i=1}^{n_u-1}(x_3 - x_3{}^{iu})H(x_3 - x_3{}^{iu})\,\alpha_{2u}^{i}(x_1,x_2,t) + \sum_{j=1}^{n_l-1}(x_3 - x_3{}^{jl})H(-x_3 + x_3{}^{jl})\,\alpha_{2l}^{j}(x_1,x_2,t) \end{Bmatrix} \qquad 2.10$$

The Heaviside step function in Eq. 2.10 is used to introduce the auxiliary variables at the respective interfaces and is also useful to create discontinuous transverse shear strains at the interfaces of the smart composite plate. The trigonometric function, '$x_3 \sec\left(\frac{rx_3}{h}\right)$' is the non-polynomial function used to refine the bending profile of the system. The displacement components, u_1 and u_2 represent the membrane deformation modes and u_3 is the transverse deformation mode. β_1 and β_2 are the slopes of the transverse normal to the mid-plane about the x_2 and x_1-direction, respectively. The value of 'r' is considered to be 0.1 (Sahoo and Singh, 2014). The in plane displacements at any point in the plate can now be written with the help of Eqs. 2.9 and 2.10 in the following manner:

$$U_1(x_1,x_2,x_3,t) = u_1(x_1,x_2,t) + x_3\sec(rx_3/h)\beta_1(x_1,x_2,t) + \sum_{i=1}^{n_u-1}(x_3 - x_3{}^{iu})H(x_3 - x_3{}^{iu})\,\alpha_{1u}^{i}(x_1,x_2,t)$$
$$+ \sum_{j=1}^{n_l-1}(x_3 - x_3{}^{jl})H(-x_3 + x_3{}^{jl})\,\alpha_{1l}^{j}(x_1,x_2,t) + x_3\left\{-u_{3,x_1}(x_1,x_2,t) + \Omega_1\beta_1(x_1,x_2,t)\right\}$$

$$U_2(x_1,x_2,x_3,t) = u_2(x_1,x_2,t) + x_3\sec(rx_3/h)\beta_2(x_1,x_2,t) + \sum_{i=1}^{n_u-1}(x_3 - x_3{}^{iu})H(x_3 - x_3{}^{iu})\,\alpha_{2u}^{i}(x_1,x_2,t)$$

$$+\sum_{j=1}^{n_l-1}(x_3 - x_3{}^{jl})H(-x_3 + x_3{}^{jl})\,\alpha_{2l}^{j}(x_1,x_2,t) + x_3\{-u_{3,x_2}(x_1,x_2,t) + \Omega_2\beta_2(x_1,x_2,t)\}$$

2.11a

The present model does not consider the thickness stretching of normal effects, therefore, the 3 D transverse displacement '$U_3(x_1,x_2,x_3,t)$' is assumed to be constant.

$$U_3(x_1,x_2,x_3,t) = u_3(x_1,x_2,t) \quad \text{2.11b}$$

Eqs. 2.11(a, b) represents the through-thickness variations of the 3 D displacements in the smart composite plate. Creating a displacement field is the initial step of any analysis when carried out in the framework of a plate theory. The present model is also a refinement over the CPT as observed in Eqs. 2.11(a, b).

The kinematic field in Eq. 2.11a is modified after enforcing the inter-laminar continuity conditions of tractions $(\tau_{i3}^{k-1} = \tau_{i3}^{k})_{x_3 = x_3^k}$ (i = 1 and 2; 'k' is the layer number)' at the interfaces of the smart composite plate. The modified kinematic field is expressed as follows:

$$U_1(x_1,x_2,x_3,t) = u_1(x_1,x_2,t) + x_3\{-u_{3,x_1}(x_1,x_2,t) + \Omega_1\beta_1(x_1,x_2,t)\} + p_1\beta_1(x_1,x_2,t)$$
$$U_2(x_1,x_2,x_3,t) = u_2(x_1,x_2,t) + x_3\{-u_{3,x_2}(x_1,x_2,t) + \Omega_2\beta_2(x_1,x_2,t)\} + p_2\beta_2(x_1,x_2,t)$$
$$U_3(x_1,x_2,x_3,t) = u_3(x_1,x_2,t) \quad \text{2.12a}$$

where,

$$p_1 = x_3\sec(rx_3/h) + \sum_{i=1}^{n_u-1}(x_3 - x_3{}^{iu})\,H(x_3 - x_3{}^{iu})\alpha_{1u}^{i} + \sum_{j=1}^{n_l-1}(x_3 - x_3{}^{jl})\,H(-x_3 + x_3{}^{iu})\alpha_{1l}^{j}$$
$$p_2 = x_3\sec(rx_3/h) + \sum_{i=1}^{n_u-1}(x_3 - x_3{}^{iu})\,H(x_3 - x_3{}^{iu})\alpha_{2u}^{i} + \sum_{j=1}^{n_l-1}(x_3 - x_3{}^{jl})\,H(-x_3 + x_3{}^{iu})\alpha_{2l}^{j}$$
$$q_1 = \frac{dp_2}{dx_3}\,;\; q_2 = \frac{dp_1}{dx_3} \quad \text{2.12b}$$

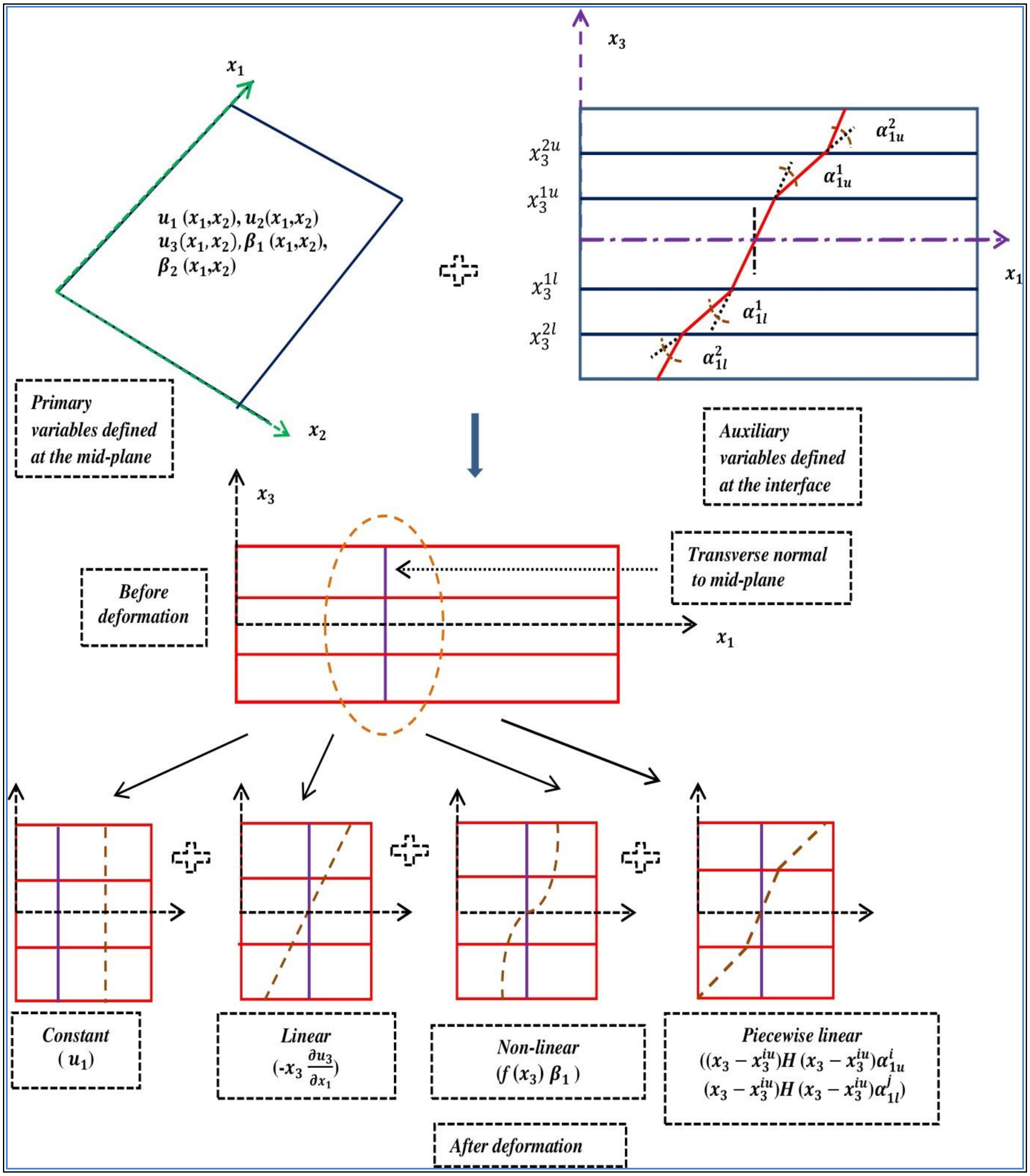

Figure 2.3. Kinematics of Trigonometric Zigzag Theory (TZZT)

Eq. 2.12a is further expressed as follows after some simplifications:

$$U_1(x_1,x_2,x_3,t) = u_1(x_1,x_2,t) - x_3\, u_{3,1}(x_1,x_2,t) + f(x_3)\beta_1(x_1,x_2,t)$$
$$U_2(x_1,x_2,x_3,t) = u_2(x_1,x_2,t) - x_3\, u_{3,2}(x_1,x_2,t) + g(x_3)\beta_2(x_1,x_2,t)$$
$$U_3(x_1,x_2,x_3,t) = u_3(x_1,x_2,t) \qquad 2.13$$

where $f(x_3)$ and $g(x_3)$ are expressed as '$p_1 + x_3\Omega_1$' and '$p_2 + x_3\Omega_2$' respectively. 'p_1 and p_2' are mathematical functions of the thickness coordinate. In the above equation, '$u_{3,1}$' and '$u_{3,2}$' denotes 'u_{3,x_1}' and 'u_{3,x_2}' which represents the differentiation with respect to independent variable 'x_1'and 'x_2' respectively. It can be shown that the function '$x_3 \sec(rx_3/h)$' implicitly accommodates the higher-order modes of polynomial HSDTs and is also responsible for refining the bending of the system. This can be mathematically shown in the following manner.

$$x_3 \sec(rx_3/h) = x_3\left(1 + \left(\frac{rx_3}{h}\right)^2\frac{1}{2} + \left(\frac{rx_3}{h}\right)^4\frac{5}{24} + \left(\frac{rx_3}{h}\right)^6\frac{61}{720} \ldots\ldots\ldots..\infty\right) \qquad 2.14$$

It is clearly observed in Eq. 2.14 that the above expansion consists of all the odd power terms of x_3 and shall therefore contribute in the refinement of the bending phenomenon.

2.7. Analytical Formulation

The strain displacement relations of the problem can be obtained with the help of Eqs. 2.7, 2.12a and b.

$$\{\varepsilon\} = \{\varepsilon\}^{(0)} + x_3\{\varepsilon\}^{(1)} + p_1\{\varepsilon\}^{(2)} + p_2\{\varepsilon\}^{(3)}$$
$$\{\gamma\} = \{\gamma\}^{(0)} + q_1\{\gamma\}^{(1)} + q_1\{\gamma\}^{(2)}$$

where, $\{\varepsilon\} = \begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{Bmatrix}$; $\{\varepsilon\}^{(0)} = \begin{Bmatrix}u_{1,1}\\ u_{2,2}\\ u_{1,2}+u_{2,1}\end{Bmatrix}$; $\{\varepsilon\}^{(1)} = \begin{Bmatrix}-u_{3,11}+\Omega_1\beta_{1,1}\\ -u_{3,22}+\Omega_2\beta_{2,2}\\ -2u_{3,12}+\Omega_1\beta_{1,2}+\Omega_2\beta_{2,1}\end{Bmatrix}$; $\{\varepsilon\}^{(2)} = \begin{Bmatrix}\beta_{1,1}\\ 0\\ \beta_{1,2}\end{Bmatrix}$;

$$\{\varepsilon\}^{(3)} = \begin{Bmatrix}0\\ \beta_{2,2}\\ \beta_{2,1}\end{Bmatrix};\ \{\gamma\} = \begin{Bmatrix}\gamma_{23}\\ \gamma_{13}\end{Bmatrix};\ \{\gamma\}^{(0)} = \begin{Bmatrix}\Omega_2\beta_2\\ \Omega_1\beta_1\end{Bmatrix};\ \{\gamma\}^{(1)} = \begin{Bmatrix}\beta_2\\ 0\end{Bmatrix};\ \{\gamma\}^{(2)} = \begin{Bmatrix}0\\ \beta_1\end{Bmatrix} \qquad 2.15$$

The basic equations like the kinematic field, stain displacement relations and the stress-strain constitutive relations required for the present investigation are presented above and also in the previous sections.

2.7.1. Equations of motion

The governing equations of motion that describes the dynamic behavior of a smart composite plate with piezoelectric actuator and sensor can be derived with Hamilton's principle, which states that

$$\int_{t_1}^{t_2}(\delta U + \delta U_F - \delta W - \delta K)\mathrm{d}t = 0 \tag{2.16}$$

where 'δU' and 'δK' represents small variation in the strain energy and the kinetic energy of the plate due to some variation in the primary variables. δU_F is the variation in the strain energy of the elastic foundation. δW is the variation in the work potential of the forces applied on the plate structure.

The variation in the strain energy of the smart composite plate is written as

$$\delta U = \int_{\Omega_o}\begin{pmatrix} N_{11}\delta u_{1,1} + M_{11}\{-\delta u_{3,11} + \Omega_1\delta\beta_{1,1}\} + N_{11}^*\delta\beta_{1,1} + N_{22}\delta u_{2,2} + M_{22}\{-\delta u_{3,22} + \Omega_2\delta\beta_{2,2}\} \\ M_{22}^*\delta\beta_{2,2} + N_{12}\{\delta u_{1,2} + \delta u_{2,1}\} + M_{12}\{-2\delta u_{3,12} + \Omega_1\delta\beta_{1,2} + \Omega_2\delta\beta_{2,1}\} + N_{12}^*\delta\beta_{1,2} + M_{12}^*\delta\beta_{2,1} \\ +Q_2\Omega_2\delta\beta_2 + T_2\delta\beta_2 + Q_1\Omega_1\delta\beta_1 + T_1^*\delta\beta_1 \end{pmatrix}\mathrm{d}\Omega_o \tag{2.17}$$

Eq. 2.17 is obtained by integrating the 3 D stresses across the thickness of the smart composite plates and replacing the integrated quantities with stress-resultants defined over unit length. The stress-resultants are defined as follows:

$$[N_{11} \quad N_{22} \quad N_{12}] = \int_{-\frac{h}{2}-t_p}^{\frac{h}{2}+t_p}[\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}]^k \mathrm{d}x_3;$$

$$[M_{11} \quad M_{22} \quad M_{12}] = \int_{-\frac{h}{2}-t_p}^{\frac{h}{2}+t_p} x_3[\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}]^k \mathrm{d}x_3;$$

$$[N_{11}^{*} \quad N_{12}^{*}] = \int_{-\frac{h}{2}-t_p}^{\frac{h}{2}+t_p} p_1[\sigma_{11} \quad \sigma_{12}]^k \mathrm{d}x_3;\ [M_{22}^{*} \quad M_{12}^{*}] = \int_{-\frac{h}{2}-t_p}^{\frac{h}{2}+t_p} p_2[\sigma_{22} \quad \sigma_{12}]^k \mathrm{d}x_3;$$

$$[Q_1 \quad Q_2] = \int_{-\frac{h}{2}-t_p}^{\frac{h}{2}+t_p} [\sigma_{13} \quad \sigma_{23}]^k \mathrm{d}x_3;\ [T_1^{*} \quad T_2] = \int_{-\frac{h}{2}-t_p}^{\frac{h}{2}+t_p} [q_2\sigma_{13} \quad q_1\sigma_{23}]^k \mathrm{d}x_3 \qquad 2.18$$

The stresses should be carefully integrated in the above equation depending on the type of material present at a particular layer as the constitutive relations are different for the orthotropic ply and the piezoelectric layer.

The variation in the work potential of the load is defined as

$$\delta W = \int_{\Omega_o} q\, \delta u_3\, \mathrm{d}\Omega_o \qquad 2.19$$

where, q is the time-dependent mechanical pressure acting on the top surface of the smart composite plate.

The variation in the kinetic energy of the smart composite plate is written as

$$\delta K = \int_{\Omega_o} \begin{pmatrix} \bar{I}_0\dot{u}_1\delta\dot{u}_1 - \bar{I}_1\dot{u}_1\delta\dot{u}_{3,1} + \bar{I}_3\dot{u}_1\delta\dot{\beta}_1 - \bar{I}_1\dot{u}_{3,1}\delta\dot{u}_1 + \bar{I}_2\dot{u}_{3,1}\delta\dot{u}_{3,1} - \bar{I}_4\dot{u}_{3,1}\delta\dot{\beta}_1 + \bar{I}_3\dot{\beta}_1\delta\dot{u}_1 - \bar{I}_4\dot{\beta}_1\delta\dot{u}_{3,1} \\ + \bar{I}_5\dot{\beta}_1\delta\dot{\beta}_1 + \bar{I}_0\dot{u}_2\delta\dot{u}_2 - \bar{I}_1\dot{u}_2\delta\dot{u}_{3,2} + \bar{I}_6\dot{u}_2\delta\dot{\beta}_2 - \bar{I}_1\dot{u}_{3,2}\delta\dot{u}_2 + \bar{I}_2\dot{u}_{3,2}\delta\dot{u}_{3,2} - \bar{I}_7\dot{u}_{3,2}\delta\dot{\beta}_2 + \\ \bar{I}_6\dot{\beta}_2\delta\dot{u}_2 - \bar{I}_7\dot{\beta}_2\delta\dot{u}_{3,2} + \bar{I}_8\dot{\beta}_2\delta\dot{\beta}_2 + \bar{I}_0\dot{u}_3\delta\dot{u}_3 \end{pmatrix} \mathrm{d}\Omega_o \qquad 2.20$$

where, $\bar{I}_0$, $\bar{I}_1$, $\bar{I}_2$,$\bar{I}_3$, $\bar{I}_4$, $\bar{I}_5$, $\bar{I}_6$, $\bar{I}_7$, $\bar{I}_8$ are the components of Inertia obtained by integrating the density 'ρ^k' of the material. The inertial components are further defined as follows:

$$\begin{bmatrix} \bar{I}_0 & \bar{I}_3 & \bar{I}_6 \\ \bar{I}_1 & \bar{I}_4 & \bar{I}_7 \\ \bar{I}_2 & \bar{I}_5 & \bar{I}_8 \end{bmatrix} = \left(\int_{-\frac{h}{2}-t_p}^{\frac{h}{2}+t_p} \left\{ \begin{bmatrix} \rho^k & f(x_3)\rho^k & g(x_3)\rho^k \\ x_3\rho^k & x_3 f(x_3)\rho^k & x_3 g(x_3)\rho^k \\ {x_3}^2\rho^k & f(x_3)^2\rho^k & g(x_3)^2\rho^k \end{bmatrix} \mathrm{d}x_3 \right\} \right) \qquad 2.21$$

The strain energy of the elastic foundation is written as

$$U_F = \frac{1}{2}\left\{ \int_{\Omega_o} \left\{ k_w {u_3}^2 + k_s \left\{ \left(\frac{\partial u_3}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_2}\right)^2 \right\} \right\} \mathrm{d}\Omega_o \right\} \qquad 2.22$$

The variation of the strain energy of the elastic foundation is written with the help of the Eq. 2.22 in the following manner.

$$\delta U_F = \int_{\Omega_o} \left\{ k_w u_3 \delta u_3 + k_s \left(\frac{\partial u_3}{\partial x_1}\frac{\partial \delta u_3}{\partial x_1} + \frac{\partial u_3}{\partial x_2}\frac{\partial \delta u_3}{\partial x_2} \right) \right\} \mathrm{d}\Omega_o \quad 2.23$$

Substituting the expressions of δU, δU_F, δV and δK from Eqs. 2.17, 2.19, 2.20 and 2.23 in Eq. 2.16, and the resulting equation is integrated by parts in space (x_1, x_2) and time (t_1, t_2). The variation of the primary variables obtained in space and at initial (t_1) and final time (t_2) are then set to zero to obtain the dynamic equilibrium equations corresponding to each variation of the primary variables.

$$\begin{aligned}
&\delta u_0 : N_{11,1} + N_{12,2} = \bar{I}_0 \ddot{u}_1 - \bar{I}_1 \ddot{u}_{3,1} + \bar{I}_3 \ddot{\beta}_1 \\
&\delta v_0 : N_{12,1} + N_{22,2} = \bar{I}_0 \ddot{u}_2 - \bar{I}_1 \ddot{u}_{3,2} + \bar{I}_6 \ddot{\beta}_2 \\
&\delta w_0 : M_{11,11} + 2M_{12,12} + M_{22,22} + q - f_{EF} = \bar{I}_1(\ddot{u}_{1,1} + \ddot{u}_{2,2}) - \bar{I}_2(\ddot{u}_{3,11} + \ddot{u}_{3,22}) + \bar{I}_0 \ddot{u}_3 + \bar{I}_4 \ddot{\beta}_{1,1} \\
&\qquad + \bar{I}_7 \ddot{\beta}_{2,2} \\
&\delta \beta_x : \Omega_1 M_{11,1} + N^*_{11,1} + \Omega_1 M_{12,2} + N^*_{12,2} - \Omega_1 Q_1 - T^*_1 = \bar{I}_3 \ddot{u}_1 - \bar{I}_4 \ddot{u}_{3,1} + \bar{I}_5 \ddot{\beta}_1 \\
&\delta \beta_y : \Omega_2 M_{22,2} + M^*_{22,2} + \Omega_2 M_{12,1} + M^*_{12,1} - \Omega_2 Q_2 - T_2 = \bar{I}_6 \ddot{u}_2 - \bar{I}_7 \ddot{u}_{3,2} + \bar{I}_8 \ddot{\beta}_2
\end{aligned} \quad 2.24a$$

The essential and natural boundary conditions of the problem are expressed as follows:

Boundaries parallel to x_2 axis, *i.e*, x_1 = 0 or *l*

1. Either N_{11} = 0 or u_1 is prescribed
2. Either N_{12} = 0 or u_2 is prescribed
3. Either M_{11} = 0 or $\frac{\partial u_3}{\partial x_1}$ is prescribed
4. Either $\left(\frac{\partial M_{11}}{\partial x_1} + 2\frac{\partial M_{12}}{\partial x_2} + k_s \frac{\partial u_3}{\partial x_1} - \bar{I}_1 \ddot{u}_1 + \bar{I}_2 \frac{\partial \ddot{u}_3}{\partial x_1} - \bar{I}_4 \ddot{\beta}_1\right)$ = 0 or u_3 is prescribed
5. Either $(\Omega_1 M_{11} + N^*_{11})$ = 0 or β_1 is prescribed
6. Either $(\Omega_2 M_{12} + M^*_{12})$ = 0 or β_2 is prescribed

Boundaries parallel to x_1 axis, *i.e*, x_2 = 0 or *b*

1. Either N_{12} = 0 or u_1 is prescribed
2. Either N_{22} = 0 or u_2 is prescribed
3. Either M_{22} = 0 or $\frac{\partial u_3}{\partial x_2}$ is prescribed
4. Either $\left(\frac{\partial M_{22}}{\partial x_2} + 2\frac{\partial M_{12}}{\partial x_1} + k_s \frac{\partial u_3}{\partial x_2} - \bar{I}_1 \ddot{u}_2 + \bar{I}_2 \frac{\partial \ddot{u}_3}{\partial x_2} - \bar{I}_7 \ddot{\beta}_2\right)$ = 0 or u_3 is prescribed
5. Either $(\Omega_1 M_{12} + N^*_{12})$ = 0 or β_1 is prescribed

6. Either $(\Omega_2 M_{22} + M^*_{22}) = 0$ or β_2 is prescribed

At the corners

Either $M_{12} = 0$ or u_3 is prescribed. 2.24b

Eq. 2.24a is indeterminate as there are fourteen stress-resultants in five equations. When solving a problem using elasticity formulations, the formulation starts with the equilibrium equations of elasticity in which 3 equations are associated with 6 unknown stresses. To make the problem determinate, the strain-displacement relations and stress-strain constitutive equations are utilized which results in 15 unknowns and 15 equations. Similarly, in the present problem, additional equations are defined with the help of Eq. 2.18 which are known as the plate constitutive relationships and then substituted for the stress-resultants in Eq. 2.24a. The plate constitutive relations for the laminated composite plate and the piezoelectric layer are defined as follows:

Laminated composite plate

$$\begin{Bmatrix} \{N\}_{(3x1)} \\ \{M\}_{(3x1)} \\ \{N^*\}_{(2x1)} \\ \{M^*\}_{(2x1)} \end{Bmatrix}^{El} = \begin{bmatrix} [A]_{(3x3)} & [B]_{(3x3)} & [C]_{(3x2)} & [D]_{(3x2)} \\ [B]_{(3x3)} & [G]_{(3x3)} & [H]_{(3x2)} & [I]_{(3x2)} \\ [C]_{(2x3)} & [H]_{(2x3)} & [L]_{(2x2)} & [M]_{(2x2)} \\ [D]_{(2x3)} & [I]_{(2x3)} & [M]_{(2x2)} & [P]_{(2x2)} \end{bmatrix}^{El} \begin{Bmatrix} \{\varepsilon\}^{(0)}{}_{(3x1)} \\ \{\varepsilon\}^{(1)}{}_{(3x1)} \\ \{\varepsilon\}^{(2)}{}_{(2x1)} \\ \{\varepsilon\}^{(3)}{}_{(2x1)} \end{Bmatrix};$$

$$\begin{Bmatrix} \{Q\}_{(2x1)} \\ \{T\}_{(1x1)} \\ \{T^*\}_{(1x1)} \end{Bmatrix}^{El} = \begin{bmatrix} [AA]_{(2x2)} & [EE]_{(2x1)} & [FF]_{(2x1)} \\ [EE]_{(1x2)} & [SS]_{(1x1)} & [TT]_{(1x1)} \\ [FF]_{(1x2)} & [TT]_{(1x1)} & [UU]_{(1x1)} \end{bmatrix}^{El} \begin{Bmatrix} \{\gamma\}^{(0)}{}_{(2x1)} \\ \{\gamma\}^{(1)}{}_{(1x1)} \\ \{\gamma\}^{(2)}{}_{(1x1)} \end{Bmatrix} \qquad 2.25a$$

Piezoelectric Layer

$$\begin{Bmatrix} \{N\}_{(3x1)} \\ \{M\}_{(3x1)} \\ \{N^*\}_{(2x1)} \\ \{M^*\}_{(2x1)} \end{Bmatrix}^{Pz} = \begin{bmatrix} [A]_{(3x3)} & [B]_{(3x3)} & [C]_{(3x2)} & [D]_{(3x2)} \\ [B]_{(3x3)} & [G]_{(3x3)} & [H]_{(3x2)} & [I]_{(3x2)} \\ [C]_{(2x3)} & [H]_{(2x3)} & [L]_{(2x2)} & [M]_{(2x2)} \\ [D]_{(2x3)} & [I]_{(2x3)} & [M]_{(2x2)} & [P]_{(2x2)} \end{bmatrix}^{Pz} \begin{Bmatrix} \{\varepsilon\}^{(0)}{}_{(3x1)} \\ \{\varepsilon\}^{(1)}{}_{(3x1)} \\ \{\varepsilon\}^{(2)}{}_{(2x1)} \\ \{\varepsilon\}^{(3)}{}_{(2x1)} \end{Bmatrix} + \begin{Bmatrix} \{A\}_{(3x1)}{}^{PzE} \\ \{C\}_{(3x1)}{}^{PzE} \\ \{E\}_{(2x1)}{}^{PzE} \\ \{G\}_{(2x1)}{}^{PzE} \end{Bmatrix} V(x_1, x_2, t)$$

$$\begin{Bmatrix} \{Q\}_{(2x1)} \\ \{T\}_{(1x1)} \\ \{T^*\}_{(1x1)} \end{Bmatrix}^{Pz} = \begin{bmatrix} [AA]_{(2x2)} & [EE]_{(2x1)} & [FF]_{(2x1)} \\ [EE]_{(1x2)} & [SS]_{(1x1)} & [TT]_{(1x1)} \\ [FF]_{(1x2)} & [TT]_{(1x1)} & [UU]_{(1x1)} \end{bmatrix}^{Pz} \begin{Bmatrix} \{\gamma\}^{(0)}{}_{(2x1)} \\ \{\gamma\}^{(1)}{}_{(1x1)} \\ \{\gamma\}^{(2)}{}_{(1x1)} \end{Bmatrix} + \begin{Bmatrix} \{L\}_{(2x1)}{}^{PzE} \\ \{N\}_{(1x1)}{}^{PzE} \\ \{P\}_{(1x1)}{}^{PzE} \end{Bmatrix} V(x_1, x_2, t) \qquad 2.25b$$

Superscripts '^{El}' and '^{Pz}' are used to denote the laminated composite plate and the piezoelectric layer, respectively.

where, $\{N\} = \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix}$; $\{M\} = \begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix}$; $\{N^*\} = \begin{Bmatrix} N_{11}^* \\ N_{12}^* \end{Bmatrix}$; $\{M^*\} = \begin{Bmatrix} M_{22}^* \\ M_{12}^* \end{Bmatrix}$; $\{Q\} = \begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix}$; $\{T\} = T_2$; $\{T^*\} = T_1^*$

The vectors containing the derivatives of the primary variables defined at the mid-plane in Eq. 2.25a are already defined earlier in Eq. 2.15. The terms associated with the electric voltage '*V*' are defined later in the section 2.7.2. Rigidity sub-matrices relating the stress-resultants with the mid-plane derivative variables are given below.

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \\ [G] & [H] \\ [I] & [L] \\ [M] & [P] \end{bmatrix}^{El,Pz} = \sum_{k=1}^{NL} \left\{ \int_{{x_3}^k}^{{x_3}^{k+1}} \left(\begin{bmatrix} 1 & x_3 \\ p_1 & p_2 \\ {x_3}^2 & x_3 p_1 \\ x_3 p_2 & {p_1}^2 \\ p_1 p_2 & {p_2}^2 \end{bmatrix} \left[\bar{Q}_{ij}\right]^k \mathrm{d}x_3 \right) \right\} \quad (i, j = 1, 2, 6); \text{ (NL denotes number of layers)}$$

$$\begin{bmatrix} [AA] & [EE] & [FF] \\ [SS] & [TT] & [UU] \end{bmatrix}^{El,Pz} = \sum_{k=1}^{NL} \left\{ \int_{{x_3}^k}^{{x_3}^{k+1}} \left(\begin{bmatrix} 1 & q_1 & q_2 \\ {q_1}^2 & q_1 q_2 & {q_2}^2 \end{bmatrix} \left[\bar{Q}_{ij}\right]^k \mathrm{d}x_3 \right) \right\} \quad (i, j = 4, 5) \qquad 2.26$$

Now there are fourteen plate-constitutive relations associated with five primary variables in Eqs. 2.25(a, b). The problem is therefore determinate as nineteen unknowns are now associated with nineteen equations. Substituting the plate constitutive relations in Eq. 2.24a gives a system of PDEs in terms of the primary variables.

$$A_{11} \frac{\partial^2 u_1}{\partial {x_1}^2} + A_{12} \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + B_{11} \left(-\frac{\partial^3 u_3}{\partial {x_1}^3} + \Omega_1 \frac{\partial^2 \beta_1}{\partial {x_1}^2} \right) + B_{12} \left(-\frac{\partial^3 u_3}{\partial x_1 \partial {x_2}^2} + \Omega_2 \frac{\partial^2 \beta_2}{\partial x_1 \partial x_2} \right) + C_{11} \frac{\partial^2 \beta_1}{\partial {x_1}^2}$$
$$D_{12} \frac{\partial^2 \beta_2}{\partial x_1 \partial x_2} + A_{66} \left(\frac{\partial^2 u_1}{\partial {x_2}^2} + \Omega_1 \frac{\partial^2 u_2}{\partial x_1 \partial x_2} \right) + B_{66} \left(-2 \frac{\partial^3 u_3}{\partial x_1 \partial {x_2}^2} + \Omega_1 \frac{\partial^2 \beta_1}{\partial {x_2}^2} + \Omega_2 \frac{\partial^2 \beta_2}{\partial x_1 \partial x_2} \right) + C_{66} \frac{\partial^2 \beta_1}{\partial {x_2}^2} +$$
$$D_{66} \frac{\partial^2 \beta_2}{\partial x_1 \partial x_2} + V_{mn} A_{31}^p \bar{F}(t) \frac{\partial \bar{F}(x_1, x_2)}{\partial x_1} - \bar{I}_0 \ddot{u}_1 + \bar{I}_1 \frac{\partial \ddot{u}_3}{\partial x_1} - \bar{I}_3 \ddot{\beta}_1 = 0$$

$$A_{12} \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + A_{22} \frac{\partial^2 u_2}{\partial {x_2}^2} + B_{12} \left(-\frac{\partial^3 u_3}{\partial {x_1}^2 \partial x_2} + \Omega_1 \frac{\partial^2 \beta_1}{\partial x_1 \partial x_2} \right) + B_{22} \left(-\frac{\partial^3 u_3}{\partial {x_2}^3} + \Omega_2 \frac{\partial^2 \beta_2}{\partial {x_2}^2} \right) + C_{12} \frac{\partial^2 \beta_1}{\partial x_1 \partial x_2}$$
$$D_{22} \frac{\partial^2 \beta_2}{\partial {x_2}^2} + A_{66} \left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial {x_1}^2} \right) + B_{66} \left(-2 \frac{\partial^3 u_3}{\partial {x_1}^2 \partial x_2} + \Omega_1 \frac{\partial^2 \beta_1}{\partial x_1 \partial x_2} + \Omega_2 \frac{\partial^2 \beta_2}{\partial {x_1}^2} \right) + C_{66} \frac{\partial^2 \beta_1}{\partial x_1 \partial x_2} +$$
$$D_{66} \frac{\partial^2 \beta_2}{\partial {x_1}^2} + V_{mn} A_{32}^p \bar{F}(t) \frac{\partial \bar{F}(x_1, x_2)}{\partial x_2} - \bar{I}_0 \ddot{u}_2 + \bar{I}_1 \frac{\partial \ddot{u}_3}{\partial x_2} - \bar{I}_6 \ddot{\beta}_2 = 0$$

$$B_{11}\frac{\partial^3 u_1}{\partial x_1{}^3} + B_{12}\frac{\partial^3 u_2}{\partial x_1{}^2\partial x_2} + G_{11}\left(-\frac{\partial^4 u_3}{\partial x_1{}^4} + \Omega_1\frac{\partial^3 \beta_1}{\partial x_1{}^3}\right) + G_{12}\left(-\frac{\partial^4 u_3}{\partial x_1{}^2\partial x_2{}^2} + \Omega_2\frac{\partial^3 \beta_2}{\partial x_1{}^2\partial x_2}\right) + H_{11}\frac{\partial^3 \beta_1}{\partial x_1{}^3}$$
$$+ I_{12}\frac{\partial^3 \beta_2}{\partial x_1{}^2\partial x_2} + 2\,B_{66}\left(\frac{\partial^3 u_1}{\partial x_1\partial x_2{}^2} + \frac{\partial^3 u_2}{\partial x_1{}^2\partial x_2}\right) + 2G_{66}\left(-2\frac{\partial^4 u_3}{\partial x_1{}^2\partial x_2{}^2} + \Omega_1\frac{\partial^3 \beta_1}{\partial x_1\partial x_2{}^2} + \Omega_2\frac{\partial^3 \beta_2}{\partial x_1{}^2\partial x_2}\right)$$
$$+ 2H_{66}\frac{\partial^3 \beta_1}{\partial x_1\partial x_2{}^2} + 2I_{66}\frac{\partial^3 \beta_2}{\partial x_1{}^2\partial x_2} + B_{12}\frac{\partial^3 u_1}{\partial x_1\partial x_2{}^2} + B_{22}\frac{\partial^3 u_2}{\partial x_2{}^3} + G_{12}\left(-\frac{\partial^4 u_3}{\partial x_1{}^2\partial x_2{}^2} + \Omega_1\frac{\partial^3 \beta_1}{\partial x_1\partial x_2{}^2}\right)$$
$$+ G_{22}\left(-\frac{\partial^4 u_3}{\partial x_2{}^4} + \Omega_2\frac{\partial^3 \beta_2}{\partial x_2{}^3}\right) + H_{12}\frac{\partial^3 \beta_1}{\partial x_1\partial x_2{}^2} + I_{22}\frac{\partial^3 \beta_2}{\partial x_2{}^3} + V_{mn}C_{31}^{p}\bar{F}(t)\frac{\partial^2 \bar{F}(x_1,x_2)}{\partial x_1{}^2} + V_{mn}C_{32}^{p}\bar{F}(t)\frac{\partial^2 \bar{F}(x_1,x_2)}{\partial x_2{}^2}$$
$$-\bar{I}_1\left(\frac{\partial \ddot{u}_1}{\partial x_1} + \frac{\partial \ddot{u}_2}{\partial x_2}\right) + \bar{I}_2\left(\frac{\partial^2 \ddot{u}_3}{\partial x_1{}^2} + \frac{\partial^2 \ddot{u}_3}{\partial x_2{}^2}\right) - \bar{I}_4\frac{\partial \ddot{\beta}_1}{\partial x_1} - \bar{I}_7\frac{\partial \ddot{\beta}_2}{\partial x_2} - \bar{I}_0\ddot{u}_3 + q(t) - f_{EF} = 0$$

$$\Omega_1 B_{11}\frac{\partial^2 u_1}{\partial x_1{}^2} + \Omega_1 B_{12}\frac{\partial^2 u_2}{\partial x_1\partial x_2} + \Omega_1 G_{11}\left(-\frac{\partial^3 u_3}{\partial x_1{}^3} + \Omega_1\frac{\partial^2 \beta_1}{\partial x_1{}^2}\right) + \Omega_1 G_{12}\left(-\frac{\partial^3 u_3}{\partial x_1\partial x_2{}^2} + \Omega_2\frac{\partial^2 \beta_2}{\partial x_1\partial x_2}\right) +$$
$$\Omega_1 H_{11}\frac{\partial^2 \beta_1}{\partial x_1{}^2} + \Omega_1 I_{12}\frac{\partial^2 \beta_2}{\partial x_1\partial x_2} + C_{11}\frac{\partial^2 u_1}{\partial x_1{}^2} + C_{12}\frac{\partial^2 u_2}{\partial x_1\partial x_2} + H_{11}\left(-\frac{\partial^3 u_3}{\partial x_1{}^3} + \Omega_1\frac{\partial^2 \beta_1}{\partial x_1{}^2}\right)$$
$$+ H_{12}\left(-\frac{\partial^3 u_3}{\partial x_1\partial x_2{}^2} + \Omega_2\frac{\partial^2 \beta_2}{\partial x_1\partial x_2}\right) + L_{11}\frac{\partial^2 \beta_1}{\partial x_1{}^2} + M_{12}\frac{\partial^2 \beta_2}{\partial x_1\partial x_2} + \Omega_1 B_{66}\left(\frac{\partial^2 u_1}{\partial x_2{}^2} + \frac{\partial^2 u_2}{\partial x_1\partial x_2}\right) + \Omega_1 H_{66}\frac{\partial^2 \beta_1}{\partial x_2{}^2}$$
$$+ \Omega_1 G_{66}\left(-2\frac{\partial^3 u_3}{\partial x_1\partial x_2{}^2} + \Omega_1\frac{\partial^2 \beta_1}{\partial x_2{}^2} + \Omega_2\frac{\partial^2 \beta_2}{\partial x_1\partial x_2}\right) + \Omega_1 I_{66}\frac{\partial^2 \beta_2}{\partial x_1\partial x_2} + C_{66}\left(\frac{\partial^2 u_1}{\partial x_2{}^2} + \frac{\partial^2 u_2}{\partial x_1\partial x_2}\right) + L_{66}\frac{\partial^2 \beta_1}{\partial x_2{}^2}$$
$$+ H_{66}\left(-2\frac{\partial^3 u_3}{\partial x_1\partial x_2{}^2} + \Omega_1\frac{\partial^2 \beta_1}{\partial x_2{}^2} + \Omega_2\frac{\partial^2 \beta_2}{\partial x_1\partial x_2}\right) + M_{66}\frac{\partial^2 \beta_2}{\partial x_1\partial x_2} - \Omega_1{}^2 AA_{22}\beta_1 - 2\Omega_1 FF_{22}\beta_1 -$$
$$UU_{22}\beta_1 - \bar{I}_3\ddot{u}_1 + \bar{I}_4\frac{\partial \ddot{u}_3}{\partial x_1} - \bar{I}_5\,\ddot{\beta}_1 + V_{mn}\Omega_1 C_{31}^{p}\bar{F}(t)\frac{\partial \bar{F}(x_1,x_2)}{\partial x_1} - V_{mn}\Omega_1 L_{15}^{p}\bar{F}(t)\frac{\partial \bar{F}(x_1,x_2)}{\partial x_1}$$
$$- V_{mn}P_{15}^{p}\bar{F}(t)\frac{\partial \bar{F}(x_1,x_2)}{\partial x_1} + V_{mn}E_{31}^{p}\bar{F}(t)\frac{\partial \bar{F}(x_1,x_2)}{\partial x_1} = 0$$

$$\Omega_2 B_{12}\frac{\partial^2 u_1}{\partial x_1\partial x_2} + \Omega_2 B_{22}\frac{\partial^2 u_2}{\partial x_2{}^2} + + \Omega_2 G_{12}\left(-\frac{\partial^3 u_3}{\partial x_1{}^2\partial x_2} + \Omega_1\frac{\partial^2 \beta_1}{\partial x_1\partial x_2}\right) + \Omega_2 G_{22}\left(-\frac{\partial^3 u_3}{\partial x_2{}^3} + \Omega_2\frac{\partial^2 \beta_2}{\partial x_2{}^2}\right) +$$
$$\Omega_2 H_{12}\frac{\partial^2 \beta_1}{\partial x_1\partial x_2} + \Omega_2 I_{22}\frac{\partial^2 \beta_2}{\partial x_2{}^2} + D_{12}\frac{\partial^2 u_1}{\partial x_1\partial x_2} + D_{22}\frac{\partial^2 u_2}{\partial x_2{}^2} + I_{12}\left(-\frac{\partial^3 u_3}{\partial x_1{}^2\partial x_2} + \Omega_1\frac{\partial^2 \beta_1}{\partial x_1\partial x_2}\right) + M_{12}\frac{\partial^2 \beta_1}{\partial x_1\partial x_2}$$
$$+ I_{22}\left(-\frac{\partial^3 u_3}{\partial x_2{}^3} + \Omega_2\frac{\partial^2 \beta_2}{\partial x_2{}^2}\right) + P_{22}\frac{\partial^2 \beta_2}{\partial x_2{}^2} + \Omega_2 B_{66}\left(\frac{\partial^2 u_1}{\partial x_1\partial x_2} + \frac{\partial^2 u_2}{\partial x_1{}^2}\right) + \Omega_2 H_{66}\frac{\partial^2 \beta_1}{\partial x_1\partial x_2} + \Omega_2 I_{66}\frac{\partial^2 \beta_2}{\partial x_1{}^2}$$
$$+ \Omega_2 G_{66}\left(-2\frac{\partial^3 u_3}{\partial x_1{}^2\partial x_2} + \Omega_1\frac{\partial^2 \beta_1}{\partial x_1\partial x_2} + \Omega_2\frac{\partial^2 \beta_2}{\partial x_1{}^2}\right) + M_{66}\frac{\partial^2 \beta_1}{\partial x_1\partial x_2} + P_{66}\frac{\partial^2 \beta_2}{\partial x_1{}^2} - \Omega_2{}^2 AA_{11}\beta_2$$
$$+ D_{66}\left(\frac{\partial^2 u_1}{\partial x_1\partial x_2} + \frac{\partial^2 u_2}{\partial x_1{}^2}\right) + I_{66}\left(-2\frac{\partial^3 u_3}{\partial x_1{}^2\partial x_2} + \Omega_1\frac{\partial^2 \beta_1}{\partial x_1\partial x_2} + \Omega_2\frac{\partial^2 \beta_y}{\partial x_1{}^2}\right) - 2\Omega_2\,EE_{11}\beta_2 - SS_{11}\beta_2$$
$$- \bar{I}_6\ddot{u}_2 + \bar{I}_7\frac{\partial \ddot{u}_3}{\partial x_2} - \bar{I}_8\,\ddot{\beta}_2 + V_{mn}\Omega_2 C_{32}^{p}\bar{F}(t)\frac{\partial \bar{F}(x_1,x_2)}{\partial x_2} - V_{mn}\Omega_2 L_{24}^{p}\bar{F}(t)\frac{\partial \bar{F}(x_1,x_2)}{\partial x_2}$$
$$- V_{mn}N_{24}^{p}\bar{F}(t)\frac{\partial \bar{F}(x_1,x_2)}{\partial x_2} + V_{mn}G_{32}^{p}\bar{F}(t)\frac{\partial \bar{F}(x_1,x_2)}{\partial x_2} = 0 \qquad 2.27$$

2.7.2. Electric Potential

The electric potential ‘Φ’ is related to the electric field by the following relationships.

$$\begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} = -\begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \Phi(x_1, x_2, x_3, t) \qquad 2.28$$

The thickness of the PFRC layer is very small in comparison to that of the laminated composite plate. Therefore, the electric potential (Φ) is approximated with linear interpolation functions of the thickness co-ordinate. The edges of the laminated plate are also grounded. The variation of the electric potential function is given by

$\Phi(x_1, x_2, x_3, t) = \bar{F}(x_3)\, V(x_1, x_2, t)$; where $\bar{F}(x_3) = \left(x_3 - \frac{h}{2}\right)\frac{1}{t_p}$ 2.29

The electric voltage (V) is assumed to vary sinusoidally in the spatial domain (x_1, x_2) and in the time domain, the variations are shown in Figure 2.4.

Therefore, $V(x_1,\ x_2,\ t) = V_{mn}\,\bar{F}(x_1, x_2)\,\bar{F}(t)$, where $\bar{F}(x_1, x_2) = \sin\left(\frac{\pi x_1}{l}\right)\sin\left(\frac{\pi x_2}{b}\right)$ and $\bar{F}(t)$ is known functions of time. The integrated quantities associated with the piezoelectric force vector are expressed as follows:

$$\{A\}^{PzE} = \begin{Bmatrix} A_{31}^p \\ A_{32}^p \\ 0 \end{Bmatrix} = \int_{x_3{}^k}^{x_3{}^{k+1}} \left(\begin{Bmatrix} e_{31}\frac{\partial \bar{F}(x_3)}{\partial x_3} \\ e_{32}\frac{\partial \bar{F}(x_3)}{\partial x_3} \\ 0 \end{Bmatrix} \mathrm{d}x_3 \right);\ \{C\}^{PzE} = \begin{Bmatrix} C_{31}^p \\ C_{32}^p \\ 0 \end{Bmatrix} = \int_{x_3{}^k}^{x_3{}^{k+1}} \left(\begin{Bmatrix} e_{31}x_3\frac{\partial \bar{F}(x_3)}{\partial x_3} \\ e_{32}x_3\frac{\partial \bar{F}(x_3)}{\partial x_3} \\ 0 \end{Bmatrix} \mathrm{d}x_3 \right)$$

$$\{E\}^{PzE} = \begin{Bmatrix} E_{31}^p \\ 0 \\ 0 \end{Bmatrix} = \int_{x_3{}^k}^{x_3{}^{k+1}} \left(\begin{Bmatrix} e_{31}p_1\frac{\partial \bar{F}(x_3)}{\partial x_3} \\ 0 \\ 0 \end{Bmatrix} \mathrm{d}x_3 \right);\ \{G\}^{PzE} = \begin{Bmatrix} 0 \\ G_{32}^p \\ 0 \end{Bmatrix} = \int_{x_3{}^k}^{x_3{}^{k+1}} \left(\begin{Bmatrix} 0 \\ e_{32}p_2\frac{\partial \bar{F}(x_3)}{\partial x_3} \\ 0 \end{Bmatrix} \mathrm{d}x_3 \right);$$

$$\{L\}^{PzE} = \begin{Bmatrix} L_{24}^p \\ L_{15}^p \end{Bmatrix} = \int_{x_3{}^k}^{x_3{}^{k+1}} \left(\begin{Bmatrix} e_{24}\bar{F}(x_3) \\ e_{15}\bar{F}(x_3) \end{Bmatrix} \mathrm{d}x_3 \right);$$

$$\{N\}^{PzE} = \begin{Bmatrix} N_{24}^p \\ 0 \end{Bmatrix} = \int_{x_3{}^k}^{x_3{}^{k+1}} \left(\begin{Bmatrix} e_{24}q_1\bar{F}(x_3) \\ 0 \end{Bmatrix} \mathrm{d}x_3 \right);$$

$$\{P\}^{PzE} = \begin{Bmatrix} 0 \\ P_{15}^p \end{Bmatrix} = \int_{x_3{}^k}^{x_3{}^{k+1}} \left(\begin{Bmatrix} 0 \\ e_{15}q_2\bar{F}(x_3) \end{Bmatrix} \mathrm{d}x_3 \right) \qquad 2.30$$

2.7.3. Solution Scheme

The system of PDEs in Eq. 2.27 consists of the spatial derivatives and time derivatives of the primary variables. To solve the equations, the boundary conditions and the initial conditions of the problem are required. The general boundary conditions of TZZT in terms of the primary variables and the stress-resultants are derived and presented in Eq. 2.24b. Based on the boundary conditions of the problem, the displacements and stress-resultants are required to be specified at all the edges. At a particular edge, both the forces and displacements cannot be specified. For analytical solutions of Eq. 2.27, the Navier-based solution technique in terms of trigonometric closed-form solutions is employed. The plate is assumed to be diaphragm-supported at all the edges. The boundary conditions at all the edges are defined as follows:

$u_2\,(0,x_2) = u_2\,(l,x_2) = u_3\,(0,x_2) = u_3\,(l,x_2) = \beta_2(0,x_2) = \beta_2\,(l,x_2) = 0$

$N_{11}(0,x_2) = N_{11}(l,x_2) = M_{11}(0,x_2) = M_{11}(l,x_2) = (\Omega_1 M_{11}(0,x_2) + N_{11}^*(0,x_2)) = (\Omega_1 M_{11}(l,x_2) + N_{11}^*(l,x_2)) = 0$

$u_1\,(x_1, 0) = u_1\,(x_1, b) = u_3(x_1, 0) = u_3\,(x_1, b) = \beta_1(x_1, 0) = \beta_1\,(x_1, b) = 0$

$N_{22}(x_1, 0) = N_{22}(x_1, b) = M_{22}(x_1, 0) = M_{22}(x_1, b) = (\Omega_2 M_{22}(x_1, 0) + M_{22}^*(x_1, 0))$

$= (\Omega_2 M_{22}(x_1, b) + M_{22}^*(x_1, b)) = 0$ 2.31

Eq. 2.27 also has the 2nd order derivatives of the primary variables with time, and therefore two initial conditions are required, *i.e*, conditions of displacements and velocities at all the points in the space (x_1, x_2) at time, $t = 0$.

2.7.3.1. Static Analysis

In the static analysis, the inertia components are neglected from Eq. 2.27 and the smart composite plate is subjected to electromechanical loads which are not time-dependent. The primary variables are expressed in terms of double trigonometric series by satisfying the boundary conditions in Eq. 2.31. The mathematical functions for the primary variables are assumed as follows:

$$u_1 = \sum_{m=1,3..}^{\infty} \sum_{n=1,3..}^{\infty} U_{1_{mn}} \cos\left(\frac{m\pi x_1}{l}\right) \sin\left(\frac{n\pi x_2}{b}\right)$$

$$u_2 = \sum_{m=1,3..}^{\infty} \sum_{n=1,3..}^{\infty} U_{2_{mn}} \sin\left(\frac{m\pi x_1}{l}\right) \cos\left(\frac{n\pi x_2}{b}\right)$$

$$u_3 = \sum_{m=1,3..}^{\infty} \sum_{n=1,3..}^{\infty} U_{3_{mn}} \sin\left(\frac{m\pi x_1}{l}\right) \sin\left(\frac{n\pi x_2}{b}\right)$$

$$\beta_1 = \sum_{m=1,3..}^{\infty}\sum_{n=1,3..}^{\infty}\beta_{1_{mn}}\cos\left(\frac{m\pi x_1}{l}\right)\sin\left(\frac{n\pi x_2}{b}\right)$$

$$\beta_2 = \sum_{m=1,3..}^{\infty}\sum_{n=1,3..}^{\infty}\beta_{2_{mn}}\sin\left(\frac{m\pi x_1}{l}\right)\cos\left(\frac{n\pi x_2}{b}\right) \qquad 2.32$$

The external mechanical load and electric voltage are assumed to be sinusoidal and uniform in the spatial domain. The expression of the loads in the spatial domain are given below

$$q = \sum_{m=1,3..}^{\infty}\sum_{n=1,3..}^{\infty}q_{mn}\sin\left(\frac{m\pi x_1}{l}\right)\sin\left(\frac{n\pi x_2}{b}\right)$$

$$V = \sum_{m=1,3..}^{\infty}\sum_{n=1,3..}^{\infty}V_{mn}\sin\left(\frac{m\pi x_1}{l}\right)\sin\left(\frac{n\pi x_2}{b}\right) \qquad 2.33$$

For sinusoidal variation, q_{mn} = q_0 and V_{mn} = V_0 and for uniform variation, $q_{mn} = \frac{16}{\pi^2 mn}q_0$ and $V_{mn} = \frac{16}{\pi^2 mn}V_0$, where q_0 and V_0 are the amplitude of the mechanical and electrical load in the sinusoidal and uniform variation.

The assumed solutions in Eq. 2.32 and 2.33 are substituted in the partial differential equations presented in Eq. 2.27 and after some simplifications, a system of algebraic equations is obtained in terms of the field variables. The system of algebraic equations is given by

$$[\bar{K}]_{(5x5)}\{\Delta\}_{(5x1)} = \{\bar{F}_M\}_{(5x1)} + \{\bar{F}_E\}_{(5x1)} \qquad 2.34$$

where $[\bar{K}]$, $\{\Delta\}$, $\{\bar{F}_M\}$ and $\{\bar{F}_E\}$ are the stiffness matrix, vector containing the field variables, the external mechanical and electrical force vector, respectively. The details of the stiffness matrix are presented in Appendix A.

2.7.3.2. Free Vibration Analysis

In the free vibration analysis, the external loads, 'q and V' are neglected from the PDEs in Eq. 2.27. The field variables are expressed in terms of known mathematical functions of time and space based on the concept of separation of variables. The mathematical functions in the spatial domain are assumed based on Navier's solution scheme and a periodic solution is assumed in time.

The mathematical functions for the primary variables are given by:

$$u_1 = \sum_{m=1,2..}^{\infty} \sum_{n=1,2..}^{\infty} U_{1_{mn}}(t) \cos\left(\frac{m\pi x_1}{l}\right) \sin\left(\frac{n\pi x_2}{b}\right)$$

$$u_2 = \sum_{m=1,2..}^{\infty} \sum_{n=1,2..}^{\infty} U_{2_{mn}}(t) \sin\left(\frac{m\pi x_1}{l}\right) \cos\left(\frac{n\pi x_2}{b}\right)$$

$$u_3 = \sum_{m=1,2..}^{\infty} \sum_{n=1,2..}^{\infty} U_{3_{mn}}(t) \sin\left(\frac{m\pi x_1}{l}\right) \sin\left(\frac{n\pi x_2}{b}\right)$$

$$\beta_1 = \sum_{m=1,2..}^{\infty} \sum_{n=1,2..}^{\infty} \beta_{1_{mn}}(t) \cos\left(\frac{m\pi x_1}{l}\right) \sin\left(\frac{n\pi x_2}{b}\right)$$

$$\beta_2 = \sum_{m=1,2..}^{\infty} \sum_{n=1,2..}^{\infty} \beta_{2_{mn}}(t) \sin\left(\frac{m\pi x_1}{l}\right) \cos\left(\frac{n\pi x_2}{b}\right) \quad 2.35$$

The periodic functions in time are assumed in the following manner:

$$\left[U_{mn}(t) \quad V_{mn}(t) \quad W_{mn}(t) \quad \beta_{x_{mn}}(t) \quad \beta_{y_{mn}}(t)\right] = \left[U_{mn} \quad V_{mn} \quad W_{mn} \quad \beta_{x_{mn}} \quad \beta_{y_{mn}}\right] e^{i\omega t} \quad 2.36$$

where, $i = \sqrt{-1}$ and ω is the frequency of the natural vibration.

The assumed solutions in Eq. 2.35 and Eq. 2.36 are substituted in the partial differential equations in Eq. 2.27 and a system of homogeneous algebraic equations is obtained in the following form:

$$\left\{[\bar{K}]_{(5x5)} - \omega^2 [\bar{M}]_{(5x5)}\right\}\{\Delta\}_{(5x1)} = \{0\}_{(5x1)} \quad 2.37$$

where $[\bar{M}]$ and $\{0\}$ are the mass matrix and a vector containing zeros respectively. The details of the mass matrix are presented in Appendix B.

2.7.3.3. Transient Analysis

In the transient analysis, the entire system of governing equations presented in Eq. 2.27 is used. The field variables are first expressed in terms of known mathematical functions of space following the Navier's solution scheme. The solutions are then substituted in the PDEs in Eq. 2.27 and reduced to a system of ordinary differential equations (ODEs) in time. The mathematical functions for the primary variables and the external forces are given by

$$u_1 = \sum_{m=1,2..}^{\infty} \sum_{n=1,2..}^{\infty} U_{1_{mn}}(t) \cos\left(\frac{m\pi x_1}{l}\right) \sin\left(\frac{n\pi x_2}{b}\right)$$
$$u_2 = \sum_{m=1,2..}^{\infty} \sum_{n=1,2..}^{\infty} U_{2_{mn}}(t) \sin\left(\frac{m\pi x_1}{l}\right) \cos\left(\frac{n\pi x_2}{b}\right)$$
$$u_3 = \sum_{m=1,2..}^{\infty} \sum_{n=1,2..}^{\infty} U_{3_{mn}}(t) \sin\left(\frac{m\pi x_1}{l}\right) \sin\left(\frac{n\pi x_2}{b}\right)$$
$$\beta_1 = \sum_{m=1,2..}^{\infty} \sum_{n=1,2..}^{\infty} \beta_{1_{mn}}(t) \cos\left(\frac{m\pi x_1}{l}\right) \sin\left(\frac{n\pi x_2}{b}\right)$$
$$\beta_2 = \sum_{m=1,2..}^{\infty} \sum_{n=1,2..}^{\infty} \beta_{2_{mn}}(t) \sin\left(\frac{m\pi x_1}{l}\right) \cos\left(\frac{n\pi x_2}{b}\right)$$
$$q = \sum_{m=1..}^{\infty} \sum_{n=1..}^{\infty} q\,(t) \sin\left(\frac{m\pi x_1}{l}\right) \sin\left(\frac{n\pi x_2}{b}\right)$$
$$V = \sum_{m=1..}^{\infty} \sum_{n=1..}^{\infty} V\,(t) \sin\left(\frac{m\pi x_1}{l}\right) \sin\left(\frac{n\pi x_2}{b}\right) \qquad 2.38$$

The system of coupled ODEs in time is defined as follows:

$$[\bar{M}]_{(5x5)}\{\ddot{\Delta}\}_{(5x1)} + [\bar{K}]_{(5x5)}\{\Delta\}_{(5x1)} = \{\bar{F}(t)\}_{(5x1)} + \{\bar{F}_E(t)\}_{(5x1)} \qquad 2.39$$

where, $\{\ddot{\Delta}\}$ is the vector containing the double derivatives of the field variables in time. The functions of time assumed for the external forces are presented in Figure 2.4. Eq. 2.39 is also subjected to the following initial conditions of displacements and velocities:

$$u_1(x_1, x_2, 0) = \bar{u}_1(x_1, x_2);\ u_2(x_1, x_2, 0) = \bar{u}_2(x_1, x_2);\ u_3(x_1, x_2, 0) = \bar{u}_3(x_1, x_2)$$
$$\beta_1(x_1, x_2, 0) = \bar{\beta}_1(x_1, x_2);\ \beta_2(x_1, x_2, 0) = \bar{\beta}_2(x_1, x_2)$$
$$\dot{u}_1(x_1, x_2, 0) = \bar{\bar{u}}_1(x_1, x_2);\ \dot{u}_2(x_1, x_2, 0) = \bar{\bar{u}}_2(x_1, x_2);\ \dot{u}_3(x_1, x_2, 0) = \bar{\bar{u}}_3(x_1, x_2)$$
$$\dot{\beta}_1(x_1, x_2, 0) = \bar{\bar{\beta}}_1(x_1, x_2);\ \dot{\beta}_2(x_1, x_2, 0) = \bar{\bar{\beta}}_2(x_1, x_2) \qquad 2.40$$

The functions $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, $\bar{\beta}_1$, $\bar{\beta}_2$, $\bar{\bar{u}}_1$, $\bar{\bar{u}}_2$, $\bar{\bar{u}}_3$, $\bar{\bar{\beta}}_1$ and $\bar{\bar{\beta}}_2$ are further expressed in terms of the double fourier series like the field variables in the static analysis. The initial displacements and the initial velocities are obtained with the help of Eq. 2.40

Initial displacements

$$\begin{Bmatrix} U_{1mn} \\ U_{2mn} \\ U_{3mn} \\ \beta_{1mn} \\ \beta_{2mn} \end{Bmatrix} = \begin{Bmatrix} U_{1mn}(0) \\ U_{2mn}(0) \\ U_{3mn}(0) \\ \beta_{1mn}(0) \\ \beta_{2mn}(0) \end{Bmatrix} \qquad 2.41a$$

Initial velocities

$$\begin{Bmatrix} \dot{U}_{1mn} \\ \dot{U}_{2mn} \\ \dot{U}_{3mn} \\ \dot{\beta}_{1mn} \\ \dot{\beta}_{2mn} \end{Bmatrix} = \begin{Bmatrix} \dot{U}_{1mn}(0) \\ \dot{U}_{2mn}(0) \\ \dot{U}_{3mn}(0) \\ \dot{\beta}_{1mn}(0) \\ \dot{\beta}_{2mn}(0) \end{Bmatrix} \qquad 2.41b$$

Newmark's constant average acceleration method is adopted in this article to solve Eq. 2.39.

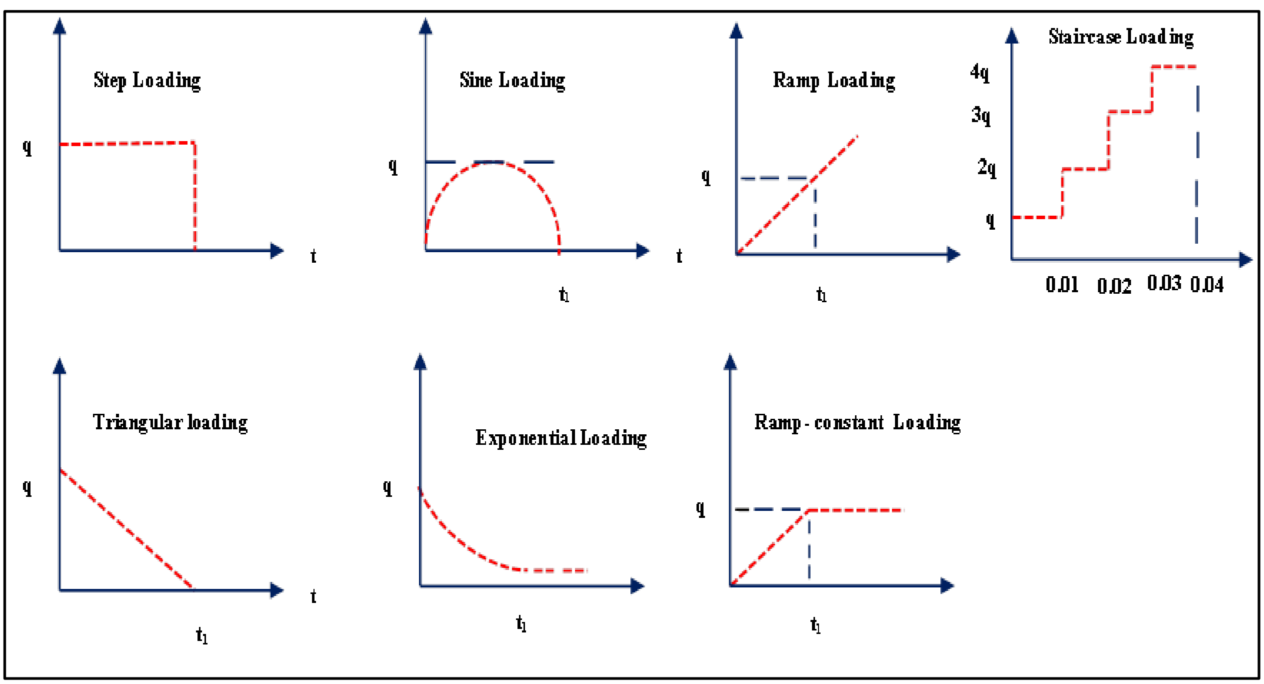

Figure 2.4. Different types of time-dependent electrical and mechanical loads

2.8. Finite Element (FE) Formulation

In the present FE formulation, an eight-noded isoparametric serendipity element shown in Figure 2.5 is used to discretize the physical domain (x_1,x_2) of the plate. The primary variables are written as a linear combination of the shape functions and the generalized nodal coordinates for an element. The geometry of the element is also expressed in terms of the same shape functions and generalized geometrical coordinates as the element is isoparametric. The interpolation shape functions for an eight-noded element are given below:

$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i + \eta\eta_i + 1)$ for i = 1, 3, 5, 7

$N_i = \frac{1}{2}(1-\xi^2)(\eta\eta_i + 1)$ for i = 2, 6

$N_i = \frac{1}{2}(1-\eta^2)(\xi\xi_i + 1)$ for i = 4, 8 2.42

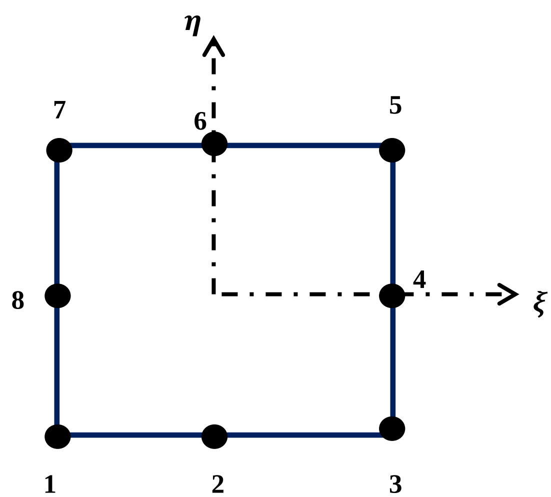

Figure 2.5. An eight-noded serendipity element

The discretized expressions of the primary variables in general and the element geometry are given by

$q = \sum_{i=1}^{NN} N_i q_i$ 2.43

where, q denotes the generalized primary variable and q_i is the value of the primary variable at the i^{th} node. NN is used to denote the number of nodes of the element.

Similarly, $x_1 = \sum_{i=1}^{NN} N_i x_{1i}$ and $x_2 = \sum_{i=1}^{NN} N_i x_{2i}$ 2.44

where, x_1 and x_2 are the coordinates used to define the geometry of the element. x_{1i} and x_{2i} are the coordinates of the i^{th} node in the x_1 and x_2-direction, respectively.

The above equations are used to discretize the basic equations like the kinematic field, strain-displacement relations and the stress-strain constitutive relations in terms of the generalized nodal coordinates.

2.8.1. Discretized Kinematic field

It can be observed in the equation of the kinematic field (Eq. 2.12a) and in the strain-displacement equations (Eq. 2.15) that the kinematic model requires C^1 continuity of the transverse displacement at the element boundaries due to the presence of first and second-order derivatives of transverse displacement in the kinematic field and the strain-displacement relations, respectively. Therefore a C^1-continuous FE formulation is required for the FE modeling of the smart composite plate using the present kinematic field. However, C^1-continuous FE formulations require more computational efforts than the C^0-continuous formulations. To reduce the continuity conditions of the transverse displacement in the present FE formulation, addition constraint equations are imposed which are written as

$u_{3,x_1} = \theta_{x_1} \Rightarrow \left(u_{3,x_1} - \theta_{x_1}\right) = 0$ and $u_{3,x_2} = \theta_{x_2} \Rightarrow \left(u_{3,x_2} - \theta_{x_2}\right) = 0$ 2.45

The constraint equations in Eq. 2.45 reduces the continuity requirements of transverse displacement, however, increase the number of field variables from five (ref: Eq. 2.12a) to seven. The modified displacement field after enforcing the constraint equation in Eq. 2.12a is now written as follows:

$$U_1(x_1,x_2,x_3,t) = u_1(x_1,x_2,t) + x_3\{-\theta_{x_1}(x_1,x_2,t) + \Omega_1\beta_1(x_1,x_2,t)\} + p_1\beta_1(x_1,x_2,t)$$
$$U_2(x_1,x_2,x_3,t) = u_2(x_1,x_2,t) + x_3\{-\theta_{x_2}(x_1,x_2,t) + \Omega_2\beta_2(x_1,x_2,t)\} + p_2\beta_2(x_1,x_2,t)$$
$$U_3(x_1,x_2,x_3,t) = u_3(x_1,x_2,t) \qquad 2.46$$

In the present formulation, the constraint conditions are enforced with the help of penalty functions. The detailed calculations for enforcing the constraint equations are presented in the section. 2.8.4. In this section, the discretized form of Eq. 2.46 is constructed and presented below.

First, the 3 D displacements 'U_1, U_2 and U_3' are written in a matrix-vector form containing the mathematical functions of the thickness coordinate and the surface-dependent primary variables.

$$\{U\}_{(3x1)} = [Z]_{(3x7)}\{\bar{U}\}_{(7x1)} \qquad 2.47a$$

where, $\{U\} = \{U_1 \quad U_2 \quad U_3\}^t$; $[Z] = \begin{bmatrix} 1 & 0 & 0 & (p_1 + x_3\Omega_2) & 0 & -x_3 & 0 \\ 0 & 1 & 0 & 0 & (p_2 + x_3\Omega_2) & 0 & -x_3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

and $\{\bar{U}\} = \{u_1 \quad u_2 \quad u_3 \quad \beta_1 \quad \beta_2 \quad \theta_1 \quad \theta_2\}^t$ 2.47b

Superscript '^{t}' is used to denote the transpose of the vectors. Further, the vector '$\{\bar{U}\}$' containing the primary variables are written in terms of the shape-functions defined in Eq. 2.42 and the generalized nodal coordinates with the help of Eq. 2.43.

$$\{\bar{U}\}_{(7x1)} = [\bar{N}]_{(7x56)}\{d^e\}_{(56x1)} \qquad 2.48a$$

where, the matrix '$[\bar{N}]$' contains the shape-functions of the eight-noded element and is given by

$$[\bar{N}] = \left[[\overline{N_1}]_{(7x7)} \quad [\overline{N_2}]_{(7x7)} \quad [\overline{N_3}]_{(7x7)} \quad [\overline{N_4}]_{(7x7)} \quad [\overline{N_5}]_{(7x7)} \quad [\overline{N_6}]_{(7x7)} \quad [\overline{N_7}]_{(7x7)} \quad [\overline{N_8}]_{(7x7)}\right] \qquad 2.48b$$

where, the individual submatrices '$[\bar{N}_i]_{(7x7)}$' is defined as follows:

$$[\bar{N}_i] = \begin{bmatrix} N_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_i \end{bmatrix} \qquad 2.48c$$

$\{d^e\}$ is the vector containing the generalized nodal coordinates defined at all the nodes of the element. The components of $\{d^e\}$ are given below

$$\{d^e\} = \left\{\{\bar{d}^e\}_1 \quad \{\bar{d}^e\}_2 \quad \{\bar{d}^e\}_3 \quad \{\bar{d}^e\}_4 \quad \{\bar{d}^e\}_5 \quad \{\bar{d}^e\}_6 \quad \{\bar{d}^e\}_7 \quad \{\bar{d}^e\}_8\right\}^t \qquad 2.48d$$

where, $\{\bar{d}^e\}_i = \{u_{1_i} \quad u_{2_i} \quad u_{3_i} \quad \beta_{1_i} \quad \beta_{2_i} \quad \theta_{1_i} \quad \theta_{2_i}\}^t$ 2.48e

Subscript '$_i$' is used to denote the i^{th} node and superscript '^{e}' denotes the e^{th} element of the physical domain.

Substituting for $\{\bar{U}\}$ from Eq. 2.48a in 2.47a, the final discretized equations for the 3 D displacement components are written as follows:

$$\{U\}_{(3x1)} = [Z]_{(3x7)}[\bar{N}]_{(7x56)}\{d^e\}_{(56x1)} \qquad 2.49$$

2.8.2. Discretized Strain displacement relationships

The discretized equations of the strain-displacement relationships are constructed with the help of Eqs. 2.7 and 2.49. The strain-displacement relations are first expressed in a matrix-vector form in which the components of the matrix are the mathematical functions of the thickness-coordinate and the components of the vector are the mid-plane derivatives of the primary variables.

$$\{\varepsilon\}_{(5x1)} = [H]_{(5x14)}\{\bar{\varepsilon}\}_{(14x1)} \qquad 2.50a$$

where, $\{\varepsilon\} = \{\varepsilon_{11} \quad \varepsilon_{22} \quad \gamma_{12} \quad \gamma_{23} \quad \gamma_{13}\}^t$;

$$[H]=\begin{bmatrix}1&0&0&x_3&0&0&p_1&0&0&0&0&0&0&0\\0&1&0&0&x_3&0&0&p_2&0&0&0&0&0&0\\0&0&1&0&0&x_3&0&0&p_1&p_2&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1&0&q_1&0\\0&0&0&0&0&0&0&0&0&0&0&1&0&q_2\end{bmatrix}$$ and

$$\{\bar{\varepsilon}\}=\{\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad \varepsilon_4 \quad \varepsilon_5 \quad \varepsilon_6 \quad \varepsilon_7 \quad \varepsilon_8 \quad \varepsilon_9 \quad \varepsilon_{10} \quad \varepsilon_{11} \quad \varepsilon_{12} \quad \varepsilon_{13} \quad \varepsilon_{14}\}^t$$

$$\varepsilon_1=\frac{\partial u_1}{\partial x_1};\ \varepsilon_2=\frac{\partial u_2}{\partial x_2};\ \varepsilon_3=\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right);\ \varepsilon_4=-\left(\frac{\partial \theta_1}{\partial x_1}+\Omega_1\frac{\partial \beta_1}{\partial x_1}\right);\ \varepsilon_5=-\left(\frac{\partial \theta_2}{\partial x_2}+\Omega_2\frac{\partial \beta_2}{\partial x_2}\right);$$

$$\varepsilon_6=-\left(\frac{\partial \theta_1}{\partial x_2}+\frac{\partial \theta_2}{\partial x_1}\right)+\Omega_1\frac{\partial \beta_1}{\partial x_2}+\Omega_2\frac{\partial \beta_2}{\partial x_1};\ \varepsilon_7=\frac{\partial \beta_1}{\partial x_1};\ \varepsilon_8=\frac{\partial \beta_2}{\partial x_2};\ \varepsilon_9=\frac{\partial \beta_1}{\partial x_2};\ \varepsilon_{10}=\frac{\partial \beta_2}{\partial x_1};$$

$$\varepsilon_{11}=-\theta_2+\frac{\partial u_3}{\partial x_2}+\Omega_2\beta_2;\ \varepsilon_{12}=-\theta_1+\frac{\partial u_3}{\partial x_1}+\Omega_1\beta_1;\ \varepsilon_{13}=\beta_2;\ \varepsilon_{14}=\beta_1 \qquad 2.50b$$

The mid-plane derivative variables in $\{\bar{\varepsilon}\}$ can be further expressed in terms of the derivatives of the shape-functions and the generalized nodal coordinates by the following discretized relation.

$$\{\bar{\varepsilon}\}_{(14x1)}=[B]_{(14x56)}\{d^e\}_{(56x1)} \qquad 2.51a$$

where,

$$[B]=\left[[\overline{B_1}]_{(14x7)} \quad [\overline{B_2}]_{(14x7)} \quad [\overline{B_3}]_{(14x7)} \quad [\overline{B_4}]_{(14x7)} \quad [\overline{B_5}]_{(14x7)} \quad [\overline{B_6}]_{(14x7)} \quad [\overline{B_7}]_{(14x7)} \quad [\overline{B_8}]_{(14x7)}\right] \qquad 2.51b$$

The components of the various submatrices in Eq. 2.51 are given below

$$\bar{B}_{1,1_i}=\bar{B}_{3,2_i}=\bar{B}_{7,4_i}=\bar{B}_{10,5_i}=\bar{B}_{12,3_i}=\frac{\partial N_i}{\partial x_1};\ \bar{B}_{2,2_i}=\bar{B}_{3,1_i}=\bar{B}_{8,5_i}=\bar{B}_{9,4_i}=\bar{B}_{11,3_i}=\frac{\partial N_i}{\partial x_2};$$

$$\bar{B}_{4,6_i}=\bar{B}_{6,7_i}=-\frac{\partial N_i}{\partial x_1};\ \bar{B}_{5,7_i}=\bar{B}_{6,6_i}=-\frac{\partial N_i}{\partial x_2};\ \bar{B}_{4,4_i}=\Omega_1\frac{\partial N_i}{\partial x_1};\ \bar{B}_{5,5_i}=\Omega_2\frac{\partial N_i}{\partial x_2};\ \bar{B}_{6,4_i}=\Omega_1\frac{\partial N_i}{\partial x_2};$$

$$\bar{B}_{6,5_i}=\Omega_2\frac{\partial N_i}{\partial x_1};\ \bar{B}_{12,4_i}=\Omega_1 N_i;\ \bar{B}_{11,5_i}=\Omega_2 N_i;\ \bar{B}_{11,7_i}=\bar{B}_{12,6_i}=-N_i;\ \bar{B}_{13,5_i}=\bar{B}_{14,4_i}=N_i$$

All the other entries in $[B]$ are zero. The derivatives of the shape functions in the above equation are with the x_1 and x_2-coordinates, however, the shape functions are functions of ξ and η. Therefore, the derivatives of the shape functions are required to evaluated using the chain rule of differentiation. The complete form of the matrix '$[B]$' for the i^{th} node can be written as follows:

$$[\bar{B}_i] = \begin{bmatrix} \bar{B}_{1,1_i} & \bar{B}_{1,2_i} & \bar{B}_{1,3_i} & \bar{B}_{1,4_i} & \bar{B}_{1,5_i} & \bar{B}_{1,6_i} & \bar{B}_{1,7_i} \\ \bar{B}_{2,1_i} & \bar{B}_{2,2_i} & \bar{B}_{2,3_i} & \bar{B}_{2,4_i} & \bar{B}_{2,5_i} & \bar{B}_{2,6_i} & \bar{B}_{2,7_i} \\ \bar{B}_{3,1_i} & \bar{B}_{3,2_i} & \bar{B}_{3,3_i} & \bar{B}_{3,4_i} & \bar{B}_{3,5_i} & \bar{B}_{3,6_i} & \bar{B}_{3,7_i} \\ \bar{B}_{4,1_i} & \bar{B}_{4,2_i} & \bar{B}_{4,3_i} & \bar{B}_{4,4_i} & \bar{B}_{4,5_i} & \bar{B}_{4,6_i} & \bar{B}_{4,7_i} \\ \bar{B}_{5,1_i} & \bar{B}_{5,2_i} & \bar{B}_{5,3_i} & \bar{B}_{5,4_i} & \bar{B}_{5,5_i} & \bar{B}_{5,6_i} & \bar{B}_{5,7_i} \\ \bar{B}_{6,1_i} & \bar{B}_{6,2_i} & \bar{B}_{6,3_i} & \bar{B}_{6,4_i} & \bar{B}_{6,5_i} & \bar{B}_{6,6_i} & \bar{B}_{6,7_i} \\ \bar{B}_{7,1_i} & \bar{B}_{7,2_i} & \bar{B}_{7,3_i} & \bar{B}_{7,4_i} & \bar{B}_{7,5_i} & \bar{B}_{7,6_i} & \bar{B}_{7,7_i} \\ \bar{B}_{8,1_i} & \bar{B}_{8,2_i} & \bar{B}_{8,3_i} & \bar{B}_{8,4_i} & \bar{B}_{8,5_i} & \bar{B}_{8,6_i} & \bar{B}_{8,7_i} \\ \bar{B}_{9,1_i} & \bar{B}_{9,2_i} & \bar{B}_{9,3_i} & \bar{B}_{9,4_i} & \bar{B}_{9,5_i} & \bar{B}_{9,6_i} & \bar{B}_{9,7_i} \\ \bar{B}_{10,1_i} & \bar{B}_{10,2_i} & \bar{B}_{10,3_i} & \bar{B}_{10,4_i} & \bar{B}_{10,5_i} & \bar{B}_{10,6_i} & \bar{B}_{10,7_i} \\ \bar{B}_{11,1_i} & \bar{B}_{11,2_i} & \bar{B}_{11,3_i} & \bar{B}_{11,4_i} & \bar{B}_{11,5_i} & \bar{B}_{11,6_i} & \bar{B}_{11,7_i} \\ \bar{B}_{12,1_i} & \bar{B}_{12,2_i} & \bar{B}_{12,3_i} & \bar{B}_{12,4_i} & \bar{B}_{12,5_i} & \bar{B}_{12,6_i} & \bar{B}_{12,7_i} \\ \bar{B}_{13,1_i} & \bar{B}_{13,2_i} & \bar{B}_{13,3_i} & \bar{B}_{13,4_i} & \bar{B}_{13,5_i} & \bar{B}_{13,6_i} & \bar{B}_{13,7_i} \\ \bar{B}_{14,1_i} & \bar{B}_{14,2_i} & \bar{B}_{14,3_i} & \bar{B}_{14,4_i} & \bar{B}_{14,5_i} & \bar{B}_{14,6_i} & \bar{B}_{14,7_i} \end{bmatrix} \quad (i = 1,2\ldots8) \qquad 2.51c$$

The final discretized strain-displacement relationship is given below

$$\{\varepsilon\}_{(5x1)} = [H]_{(5x14)}[B]_{(14x56)}\{d^e\}_{(56x1)} \qquad 2.52$$

2.8.3. Discretized Stress-Strain Constitutive relationships

The discretized stress-strain relationships for the traditional laminated composite plates is given by

$$\{\sigma\}_{(5x1)} = [Q]^{(k)}_{(5x5)}[H]_{(5x14)}[B]_{(14x56)}\{d^e\}_{(56x1)} \qquad 2.53$$

In the constitutive relationships of the piezoelectric material, there is coupling in between the mechanical stresses and the electric fields. The relationships between the electric field and the electric potential are presented earlier in Eq. 2.28. To obtain the discretized constitutive relationships of the piezoelectric materials, the relations presented in Eq. 2.28 are required to be discretized first. The electric voltage 'V' is expressed with the shape-functions and the generalized voltage coordinates defined at the nodes of an element.

$$V = \sum_{i=1}^{NN} N_i \, V_i \, (t)^{\,e} \qquad 2.54$$

where, $V_i \, (t)^{\,e}$ is the time-dependent generalized voltage coordinates at the nodes of the e^{th} element. The electric potential 'Φ' is discretized as follows with the help of Eq. 2.54.

$$\Phi = \left(x_3 - \frac{h}{2}\right)\frac{1}{t_p}\sum_{i=1}^{NN} N_i\, V_i\;(t)^{\;e} \tag{2.55}$$

The discretized relations of the electric field are given by

$$\{E\}_{(3x1)} = [\bar{Z}]_{(3x3)}[\bar{N}]_{(3x8)}\{V(t)^e\}_{(8x1)} \tag{2.56a}$$

where, $\{E\} = \{E_1 \quad E_2 \quad E_3\}^t$; $[\bar{Z}] = \begin{bmatrix} -\left(x_3 - \frac{h}{2}\right)\frac{1}{t_p} & 0 & 0 \\ 0 & -\left(x_3 - \frac{h}{2}\right)\frac{1}{t_p} & 0 \\ 0 & 0 & -\frac{1}{t_p} \end{bmatrix}$;

$[\bar{N}] = \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_2}{\partial x_1} & \frac{\partial N_3}{\partial x_1} & \frac{\partial N_4}{\partial x_1} & \frac{\partial N_5}{\partial x_1} & \frac{\partial N_6}{\partial x_1} & \frac{\partial N_7}{\partial x_1} & \frac{\partial N_8}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} & \frac{\partial N_2}{\partial x_2} & \frac{\partial N_3}{\partial x_2} & \frac{\partial N_4}{\partial x_2} & \frac{\partial N_5}{\partial x_2} & \frac{\partial N_6}{\partial x_2} & \frac{\partial N_7}{\partial x_2} & \frac{\partial N_8}{\partial x_2} \\ N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 \end{bmatrix}$ and

$$\{V(t)^e\} = \{V_1(t) \quad V_2(t) \quad V_3(t) \quad V_4(t) \quad V_5(t) \quad V_6(t) \quad V_7(t) \quad V_8(t)\}^t \tag{2.56b}$$

The discretized stress-strain relationship for the piezoelectric layer is presented below

$$\{\sigma\}_{(5x1)} = [Q]^{(k)}_{(5x5)}[H]_{(5x14)}[B]_{(14x56)}\{d^e\}_{(56x1)} - [e]_{(5x3)}[\bar{Z}]_{(3x3)}[\bar{N}]_{(3x8)}\{V(t)^e\}_{(8x1)} \tag{2.57}$$

2.8.4. Discretized governing equations of motion

The discretized governing equations of motion are derived in this section with Hamilton’s principle which is presented earlier in Eq. 2.16. To derive the equations of motion with Hamilton's principle, the variation in the strain energy, work potential and kinetic energy are required to be discretized.

The discretized equation of the variation in the strain energy ‘δU’ of the smart composite plate is derived as follows:

$$\delta U = \int_{\Omega_0}\int_{-\frac{h}{2}-t_p}^{\frac{h}{2}+t_p}\{\delta\varepsilon\}^t\{\sigma\}\,\mathrm{d}x_3\mathrm{d}\Omega_0 \tag{2.58}$$

Substituting the discretized strain-displacement relations and the stress-strain constitutive relations from Eq. 2.51 and Eq. 2.57 in Eq. 2.58, we get

$$
\begin{aligned}
\delta U = & \int_{\Omega_0}\int_{-\frac{h}{2}-t_p}^{-\frac{h}{2}}\left(\{\delta d^e\}^t[B]^t[H]^t[Q]^{(s)}[H][B]\{d^e\}\right)\mathrm{d}x_3\mathrm{d}\Omega_0 \\
& +\int_{\Omega_0}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\{\delta d^e\}^t[B]^t[H]^t[Q]^{(k)}[H][B]\{d^e\}\right)\mathrm{d}x_3\mathrm{d}\Omega_0 \\
& +\int_{\Omega_0}\int_{\frac{h}{2}}^{\frac{h}{2}+t_p}\left(\{\delta d^e\}^t[B]^t[H]^t[Q]^{(a)}[H][B]\{d^e\}\right)\mathrm{d}x_3\mathrm{d}\Omega_0 \\
& -\int_{\Omega_0}\int_{-\frac{h}{2}-t_p}^{-\frac{h}{2}}\left(\{\delta d^e\}^t[B]^t[H]^t[e]^{(s)}[\bar{Z}][\bar{N}]\{V(t)^e\}\right)\mathrm{d}x_3\mathrm{d}\Omega_0 \\
& -\int_{\Omega_0}\int_{\frac{h}{2}}^{\frac{h}{2}+t_p}\left(\{\delta d^e\}^t[B]^t[H]^t[e]^{(a)}[\bar{Z}][\bar{N}]\{V(t)^e\}\right)\mathrm{d}x_3\mathrm{d}\Omega_0
\end{aligned}
\qquad 2.59
$$

Superscripts '^{s}, ^{k}, ^{a}' are used to denote the sensor, laminated composites and the actuator layer, respectively. The thickness integration of the material properties in Eq. 2.59 are carried out as follows:

$$
[D] = \int_{-\frac{h}{2}}^{\frac{h}{2}}\left([H]^t[Q]^{(k)}[H]\right)\mathrm{d}x_3;\ \left[D^{(s)}\right] = \int_{-\frac{h}{2}-t_p}^{-\frac{h}{2}}\left([H]^t[Q]^{(s)}[H]\right)\mathrm{d}x_3;\ \left[D^{(a)}\right] = \int_{\frac{h}{2}}^{\frac{h}{2}+t_p}\left([H]^t[Q]^{(a)}[H]\right)\mathrm{d}x_3
$$

$$
\left[Z^{(s)}\right] = \int_{-\frac{h}{2}-t_p}^{-\frac{h}{2}}\left([H]^t[e]^{(s)}[\bar{Z}]\right)\mathrm{d}x_3;\ \left[Z^{(a)}\right] = \int_{\frac{h}{2}}^{\frac{h}{2}+t_p}\left([H]^t[e]^{(a)}[\bar{Z}]\right)\mathrm{d}x_3 \qquad 2.60
$$

The modified equation of the variation in the strain energy is now written as

$$
\begin{aligned}
\delta U = & \int_{\Omega_0}\left(\{\delta d^e\}^t[B]^t\left[D^{(s)}\right][B]\{d^e\}\right)\mathrm{d}\Omega_0+\int_{\Omega_0}(\{\delta d^e\}^t[B]^t[D][B]\{d^e\})\,\mathrm{d}\Omega_0 \\
& +\int_{\Omega_0}\left(\{\delta d^e\}^t[B]^t\left[D^{(a)}\right][B]\{d^e\}\right)\mathrm{d}\Omega_0-\int_{\Omega_0}\left(\{\delta d^e\}^t[B]^t\left[Z^{(s)}\right][\bar{N}]\{V(t)^e\}\right)\mathrm{d}\Omega_0 \\
& -\int_{\Omega_0}\left(\{\delta d^e\}^t[B]^t\left[Z^{(a)}\right][\bar{N}]\{V(t)^e\}\right)\mathrm{d}\Omega_0
\end{aligned}
\qquad 2.61
$$

The discretized equation of the variation in the kinetic energy 'δK' of the plate is derived as follows:

$$
\begin{aligned}
\delta K = & \int_{\Omega_0}\int_{-\frac{h}{2}-t_p}^{-\frac{h}{2}}\{\delta\dot{u}\}^t\rho^{(s)}\{\dot{u}\}\,\mathrm{d}x_3\mathrm{d}\Omega_0+\int_{\Omega_0}\int_{-\frac{h}{2}}^{\frac{h}{2}}\{\delta\dot{u}\}^t\rho^{(k)}\{\dot{u}\}\,\mathrm{d}x_3\mathrm{d}\Omega_0 \\
& +\int_{\Omega_0}\int_{\frac{h}{2}}^{\frac{h}{2}+t_p}\{\delta\dot{u}\}^t\rho^{(a)}\{\dot{u}\}\,\mathrm{d}x_3\mathrm{d}\Omega_0
\end{aligned}
\qquad 2.62
$$

Substituting the discretized equations of the kinematic field from Eq. 2.49 in the above equation, the following equation of the variation in the kinetic energy is obtained.

$$
\delta K=\int_{\Omega_0}\int_{-\frac{h}{2}-t_p}^{-\frac{h}{2}}\left(\{\delta\dot{d}^e\}^t[\bar{N}]^t[Z]^t\rho^{(s)}[Z][\bar{N}]\{\dot{d}^e\}\right)\mathrm{d}x_3\mathrm{d}\Omega_0+\int_{\Omega_0}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\{\delta\dot{d}^e\}^t[\bar{N}]^t[Z]^t\rho^{(k)}[Z][\bar{N}]\{\dot{d}^e\}\right)\mathrm{d}x_3\mathrm{d}\Omega_0
$$

$$+ \int_{\Omega_0} \int_{\frac{h}{2}}^{\frac{h}{2}+t_p} \left(\{\delta\dot{d}^e\}^t [\bar{N}]^t [Z]^t \rho^{(a)} [Z][\bar{N}]\{\dot{d}^e\}\right) \mathrm{d}x_3 \mathrm{d}\Omega_0 \tag{2.63}$$

The thickness integration of the densities of each layer in Eq. 2.63 is carried out as follows:

$$[I] = \int_{-\frac{h}{2}-t_p}^{-\frac{h}{2}} \left([Z]^t \rho^{(s)} [Z]\right) \mathrm{d}x_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \left([Z]^t \rho^{(k)} [Z]\right) \mathrm{d}x_3 + \int_{\frac{h}{2}}^{\frac{h}{2}+t_p} \left([Z]^t \rho^{(a)} [Z]\right) \mathrm{d}x_3 \tag{2.64}$$

The modified equation of the variation in the kinetic energy is written as follows with the help of Eq. 2.64.

$$\delta K = \int_{\Omega_0} \left(\{\delta\dot{d}^e\}^t [\bar{N}]^t [I][\bar{N}]\{\dot{d}^e\}\right) \mathrm{d}\Omega_0 \tag{2.65}$$

The discretized equation of the variation in the work potential 'δW' is derived as follows:

$$\delta W = \int_{\Omega_0} (\{\delta U\}^t \{f\}) \, \mathrm{d}\Omega_0 \tag{2.66}$$

where, $\{f\}$ is the external surface-force vector having components f_1, f_2 and f_3 in the x_1, x_2 and x_3-direction, respectively. In the present formulation, only f_3 is acting on the top surface of the plate. Therefore, the surface-force vector is written as

$$\{f\} = \begin{Bmatrix} 0 \\ 0 \\ f_3(x_1, x_2, t) \end{Bmatrix} \tag{2.67}$$

Eq. 2.66 is further modified by substituting the discretized equation of the kinematic field.

$$\delta W = \int_{\Omega_0} \left(\{\delta d^e\}^t [\bar{N}]^t [Z]^t{}_{(x_3 = \frac{h}{2} + t_p)} \{f\}\right) \mathrm{d}\Omega_0 \tag{2.68}$$

The product of the matrices '$[Z]^t\{f\}$' is carried out separately and denoted by a new vector '$\{f_s\}$'.

The final discretized equation of the work potential is written as follows:

$$\delta W = \int_{\Omega_0} (\{\delta d^e\}^t [\bar{N}]^t \{f_s\}) \, \mathrm{d}\Omega_0 \tag{2.69}$$

The variation in the strain energy of the elastic foundation is given earlier in Eq. 2.23. Eq. 2.23 can be written in a matrix-vector form in the following manner

$$\delta U_F = \int_{\Omega_o}\left\{\left\{\delta u_3 \quad \frac{\partial \delta u_3}{\partial x_1} \quad \frac{\partial \delta u_3}{\partial x_1}\right\}\begin{bmatrix} k_w & 0 & 0 \\ 0 & k_s & 0 \\ 0 & 0 & k_s \end{bmatrix}\begin{Bmatrix} u_3 \\ \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_3}{\partial x_2} \end{Bmatrix}\right\} \mathrm{d}\Omega_o \qquad 2.70a$$

$$= \int_{\Omega_o}\left\{\{\delta dd\}^t\left[\overline{\overline{K}}_{EF}\right]\{dd\}\right\} \mathrm{d}\Omega_o \qquad 2.70b$$

The primary variable 'u_3' and its derivatives in the vector '$\{dd\}$' in Eq. 2.70 can be expressed in the following manner

$$\{dd\} = [B_{EF}]\{d^e\} \qquad 2.71a$$

where, $[B_{EF}] = [[B_{EF1}] \quad [B_{EF2}] \quad [B_{EF3}] \quad [B_{EF4}] \quad [B_{EF5}] \quad [B_{EF6}] \quad [B_{EF7}] \quad [B_{EF8}]]$

$$[B_{EFi}] = \begin{bmatrix} 0 & 0 & N_i & & & & \\ 0 & 0 & \frac{\partial N_i}{\partial x_1} & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial x_2} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (i=1, 2\ldots.8) \qquad 2.71b$$

Eq. 2.70a is now modified with the help of Eq. 2.71, and written in the following manner

$$\delta U_F = \int_{\Omega_o}\left(\{\delta d^e\}^t[B_{EF}]^t\left[\overline{\overline{K}}_{EF}\right][B_{EF}]\{d^e\}\right)\mathrm{d}\Omega_o \qquad 2.72$$

Finally, all the discretized equations required for deriving the governing equations of motion using Hamilton's principle are derived. We are now only left with the satisfaction of the constraint equations in Eq. 2.45 using the penalty approach. In the penalty approach, a penalty function is created with the constraint equations and added to the total potential energy of an element. The penalty function is given as

$$P_e = \frac{\gamma}{2}\int_{\Omega_0}\left\{\left(\frac{\partial u_3}{\partial x_1} - \theta_1\right)^t\left(\frac{\partial u_3}{\partial x_1} - \theta_1\right) + \left(\frac{\partial u_3}{\partial x_2} - \theta_2\right)^t\left(\frac{\partial u_3}{\partial x_2} - \theta_2\right)\right\} \mathrm{d}\Omega_0 \qquad 2.73$$

where, γ is denoted as the penalty number.

Eq. 2.73 is written in terms of the nodal variables in the following manner

$$\left(\frac{\partial u_3}{\partial x_1} - \theta_1\right) = \{P_1\}_{(1x56)}\,\{d^e\}_{(56x1)}; \;\left(\frac{\partial u_3}{\partial x_2} - \theta_2\right) = \{P_2\}_{(1x56)}\,\{d^e\}_{(56x1)} \qquad 2.74a$$

where, $\{P_1\} = \{\{P_{1_1}\} \quad \{P_{1_2}\} \quad \{P_{1_3}\} \quad \{P_{1_4}\} \quad \{P_{1_5}\} \quad \{P_{1_6}\} \quad \{P_{1_7}\} \quad \{P_{1_8}\}\}$

$\{P_2\} = \{\{P_{2_1}\} \quad \{P_{2_2}\} \quad \{P_{2_3}\} \quad \{P_{2_4}\} \quad \{P_{2_5}\} \quad \{P_{2_6}\} \quad \{P_{2_7}\} \quad \{P_{2_8}\}\}$

$\{P_{1_i}\} = \left\{0 \quad 0 \quad \frac{\partial N_i}{\partial x_1} \quad 0 \quad 0 \quad -N_i \quad 0\right\}$ (i= 1, 2….8);

$\{P_{2_j}\} = \left\{0 \quad 0 \quad \frac{\partial N_j}{\partial x_2} \quad 0 \quad 0 \quad 0 \quad -N_i\right\}$ (j= 1, 2….8) 2.74b

Substituting the above discretized equations in Eq. 2.73, we get the discretized equation of the penalty function.

$$P_e = \frac{\gamma}{2}\int_{\Omega_0}\{\{d^e\}^t\{P_1\}^t\{P_1\}\{d^e\} + \{d^e\}^t\{P_2\}^t\{P_2\}\{d^e\}\}\,\mathrm{d}\Omega_0 \qquad 2.75$$

The corresponding variations of the penalty function can be written as follows:

$$\delta P_e = \gamma\int_{\Omega_0}\{\{\delta d^e\}^t\{P_1\}^t\{P_1\}\{d^e\} + \{\delta d^e\}^t\{P_2\}^t\{P_2\}\{d^e\}\}\,\mathrm{d}\Omega_0 \qquad 2.76$$

The variation of the Lagrangian function is written as

$\delta L = (\delta U + \delta U_F - \delta W) + \delta P_e - \delta K$

$\Rightarrow \delta L =$

$$\begin{pmatrix} \int_{\Omega_0}(\{\delta d^e\}^t[B]^t[D^{(s)}][B]\{d^e\})\,\mathrm{d}\Omega_0 + \int_{\Omega_0}(\{\delta d^e\}^t[B]^t[D][B]\{d^e\})\,\mathrm{d}\Omega_0 + \int_{\Omega_0}(\{\delta d^e\}^t[B]^t[D^{(a)}][B]\{d^e\})\,\mathrm{d}\Omega_0 \\ -\int_{\Omega_0}(\{\delta d^e\}^t[B]^t[Z^{(s)}]\,[\bar{N}]\{V(t)^e\})\,\mathrm{d}\Omega_0 - \int_{\Omega_0}(\{\delta d^e\}^t[B]^t[Z^{(a)}][\bar{N}]\{V(t)^e\})\,\mathrm{d}\Omega_0 - \int_{\Omega_0}(\{\delta d^e\}^t[\bar{N}]^t\{f_s\})\,\mathrm{d}\Omega_0 \\ +\gamma\int_{\Omega_0}\{\{\delta d^e\}^t\{P_1\}^t\{P_1\}\{d^e\} + \{\delta d^e\}^t\{P_2\}^t\{P_2\}\{d^e\}\}\,\mathrm{d}\Omega_0 + \int_{\Omega_o}(\{\delta d^e\}^t[B_{EF}]^t[\bar{\bar{K}}_{EF}][B_{EF}]\{d^e\})\mathrm{d}\Omega_o \\ -\int_{\Omega_0}\left(\{\delta\dot{d}^e\}^t[\bar{N}]^t[I][\bar{N}]\{\dot{d}^e\}\right)\,\mathrm{d}\Omega_0 \end{pmatrix}$$

2.77

In the above equation, the variation of the penalty function is added to the variation in the potential energy (δU – δW) of the system. If the value of γ in the penalty function is considered to be zero, then the constraints are not satisfied. As the value increases, then the value of the primary variables '$\{d^e\}$' changes in such a way that the constraints are more nearly satisfied, *i.e*, $\left(\frac{\partial u_3}{\partial x_1} - \theta_1\right) = 0$ and $\left(\frac{\partial u_3}{\partial x_2} - \theta_2\right) = 0$. The value of γ is considered to be 10^6.

The integrations in the spatial domain for evaluating the mass, stiffness and load vector of an element are carried out using the Gauss quadrature method. A selective integration scheme is employed for evaluating the stiffness matrix for a thin element in which the bending terms of the stiffness matrix are evaluated by taking (3x3) gauss points and the shear terms are evaluated by considering (2x2) gauss points. Such an approach helps in discarding the shear locking phenomenon for a thin plate. For a thick plate, a full integration scheme is employed in which the bending and the shear terms are evaluated by considering (3x3) gauss points.

The various integrals of the spatial domain in Eq. 2.77 are denoted as follows after carrying out the numerical integration

$$\left[K^{(s)}\right] = \int_{\Omega_0}\left([B]^t\left[D^{(s)}\right][B]\right) \mathrm{d}\Omega_0;\ [K] = \int_{\Omega_0}([B]^t[D][B])\, \mathrm{d}\Omega_0;\ \left[K^{(a)}\right] = \int_{\Omega_0}\left([B]^t\left[D^{(a)}\right][B]\right) \mathrm{d}\Omega_0;$$
$$[K_{ds}] = \int_{\Omega_0}\left([B]^t\left[Z^{(s)}\right][\bar{N}]\right) \mathrm{d}\Omega_0;\ [K_{da}] = \int_{\Omega_0}\left([B]^t\left[Z^{(a)}\right][\bar{N}]\right) \mathrm{d}\Omega_0;\ \{F_M\} = \int_{\Omega_0}([\bar{N}]^t\{f_s\})\, \mathrm{d}\Omega_0;$$
$$\left[K_{pe}\right] = \gamma \int_{\Omega_0}\left\{\{P_1\}^t\{P_1\} + \{P_2\}^t\{P_2\}\right\} \mathrm{d}\Omega_0;\ [M] = \int_{\Omega_0}([\bar{N}]^t[I][\bar{N}])\, \mathrm{d}\Omega_0;\ \left[K^{(F)}\right] = \int_{\Omega_o}\left([B_{EF}]^t\left[\bar{\bar{K}}_{EF}\right][B_{EF}]\right)\mathrm{d}\Omega_o \tag{2.78}$$

The discretized Lagrangian function is modified with the help of Eq. 2.78 and written as

$$\delta L = \begin{pmatrix} \{\delta d^e\}^t\left[K^{(s)}\right]\{d^e\} + \{\delta d^e\}^t[K]\{d^e\} + \{\delta d^e\}^t\left[K^{(F)}\right]\{d^e\} + \{\delta d^e\}^t\left[K^{(a)}\right]\{d^e\} \\ -\{\delta d^e\}^t[K_{ds}]\{V(t)^e\} - \{\delta d^e\}^t[K_{da}]\{V(t)^e\} - \{\delta d^e\}^t\{F_M\} \\ + \{\delta d^e\}^t\left[K_{pe}\right]\{d^e\} - \left\{\delta \dot{d}^e\right\}^t[M]\left\{\dot{d}^e\right\} \end{pmatrix} \tag{2.79}$$

Substituting for the variation in the Lagrangian function in Eq. 2.16 with the above equation, the following integral equation in time is obtained.

$$\int_{t_1}^{t_2}\begin{pmatrix} \{\delta d^e\}^t\left[K^{(s)}\right]\{d^e\} + \{\delta d^e\}^t[K]\{d^e\} + \{\delta d^e\}^t\left[K^{(F)}\right]\{d^e\} + \{\delta d^e\}^t\left[K^{(a)}\right]\{d^e\} \\ -\{\delta d^e\}^t[K_{ds}]\{V(t)^e\} - \{\delta d^e\}^t[K_{da}]\{V(t)^e\} - \{\delta d^e\}^t\{F_M\} \\ + \{\delta d^e\}^t\left[K_{pe}\right]\{d^e\} - \left\{\delta \dot{d}^e\right\}^t[M]\left\{\dot{d}^e\right\} \end{pmatrix} \mathrm{d}t = 0 \tag{2.80}$$

The last term of the equation is integrated by parts in time to get the following integral equation.

$$\left\{\begin{matrix} \int_{t_1}^{t_2}\begin{pmatrix} \{\delta d^e\}^t\left[K^{(s)}\right]\{d^e\} + \{\delta d^e\}^t[K]\{d^e\} + \{\delta d^e\}^t\left[K^{(F)}\right]\{d^e\} + \{\delta d^e\}^t\left[K^{(a)}\right]\{d^e\} \\ -\{\delta d^e\}^t[K_{ds}]\{V(t)^e\} - \{\delta d^e\}^t[K_{da}]\{V(t)^e\} - \{\delta d^e\}^t\{F_M\} \\ + \{\delta d^e\}^t\left[K_{pe}\right]\{d^e\} + \{\delta d^e\}^t[M]\left\{\ddot{d}^e\right\} \end{pmatrix}\mathrm{d}t \\ -\left|\{\delta d^e\}^t[M]\left\{\dot{d}^e\right\}\right|_{t=t_1}^{t=t_2} \end{matrix}\right\} = 0 \tag{2.81}$$

The variation of the primary variables '$\{\delta d^e\}$' at the initial and the final time is assumed to be zero by which the last term of the above equation gets vanished. The resulting equation is written as

$$\int_{t_1}^{t_2} \{\delta d^e\}^t \begin{pmatrix} [K^{(s)}]\{d^e\} + [K]\{d^e\} + [K^{(F)}]\{d^e\} + [K^{(a)}]\{d^e\} \\ -[K_{ds}]\{V(t)^e\} - [K_{da}]\{V(t)^e\} - \{F_M\} \\ + [K_{pe}]\{d^e\} + [M]\{\ddot{d}^e\} \end{pmatrix} \mathrm{d}t = 0 \qquad 2.82$$

$\{\delta d^e\}$ is a variation which is applied to the primary variables '$\{d^e\}$', and thus $\{\delta d^e\} \neq 0$. Therefore the terms inside the integral is equated to zero to get the discretized governing equations of an element of the smart composite plate.

$$[M]\{\ddot{d}^e\} + \left([K^{(s)}] + [K] + [K^{(a)}] + [K^{(F)}] + [K_{pe}]\right)\{d^e\} = \{F_M\} + [K_{da}]\{V(t)^e\} + [K_{ds}]\{V(t)^e\} \qquad 2.83$$

where, $[M]$ is the mass matrix, $[K^{(s)}]$, $[K]$, $[K^{(F)}]$, $[K_{pe}]$ and $[K^{(a)}]$ are the stiffness matrix of the sensor, laminated composite plate, foundation, penalty terms and the actuator layer, respectively. $\{F_M\}$ is the time-dependent mechanical force vector. $[K_{da}]\{V(t)^e\}$ and $[K_{ds}]\{V(t)^e\}$ are the electrical force vectors of the actuator and the sensor layer, respectively. In general, the electric voltage '$\{V(t)^e\}$' is applied on the outer electrode of the actuator and the voltage from the sensor is obtained from the outer electrodes of the sensor. In that case, the external voltage applied on the sensor is zero and Eq. 2.83 reduces to

$$[M]\{\ddot{d}^e\} + \left([K^{(s)}] + [K] + [K^{(a)}] + [K^{(F)}] + [K_{pe}]\right)\{d^e\} = \{F_M\} + [K_{da}]\{V(t)^e\} \qquad 2.84$$

The details of the voltage calculation are not presented here and can be found in the next section where the FE formulation of the Active Vibration Control is presented.

It is important to note that the matrices and the vectors in Eq. 2.84 are all obtained for an element 'e'. To obtain the governing equations for the entire system, the FE assembling of the matrices and the vectors are necessary. The final governing equations of the smart composite plate are written as follows:

$$[\bar{\mathbf{M}}]\{\ddot{\Delta}\} + \left([\bar{\mathbf{K}}^{(s)}] + [\bar{\mathbf{K}}] + [\bar{\mathbf{K}}^{(F)}] + [\bar{\mathbf{K}}^{(a)}] + [\bar{\mathbf{K}}^{(pe)}]\right)\{\Delta\} = \{\bar{\mathbf{F}}_{\boldsymbol{M}}\} + [\bar{\mathbf{K}}_{\boldsymbol{da}}]\{\mathbf{V}(\boldsymbol{t})\} + [\bar{\mathbf{K}}_{\boldsymbol{ds}}]\{\mathbf{V}(\boldsymbol{t})\} \quad 2.85$$

2.8.4.1. Static Analysis

In the static analysis, the inertia effects are neglected from Eq. 2.89 and the force vectors are not time-dependent. The static governing equations describing the bending responses of a smart composite plate with piezoelectric actuator and sensor is given below

$$\left([\bar{\mathbf{K}}^{(s)}] + [\bar{\mathbf{K}}] + [\bar{\mathbf{K}}^{(F)}] + [\bar{\mathbf{K}}^{(a)}] + [\bar{\mathbf{K}}^{(pe)}]\right)\{\Delta\} = \{\bar{\mathbf{F}}_{\boldsymbol{M}}\} + [\bar{\mathbf{K}}_{\boldsymbol{da}}]\{\mathbf{V}\} + [\bar{\mathbf{K}}_{\boldsymbol{ds}}]\{\mathbf{V}\} \quad 2.86$$

The sensor layer is not subjected to any external electric voltage, therefore, the last term of Eq. 2.86 gets automatically removed. Eq. 2.86 cannot be solved now as the stiffness matrix is invertible due to the non-availability of the constraint conditions. The constraint conditions or the boundary conditions are required to be imposed on the system to remove the rigid-body motion which makes the matrices invertible. The boundary conditions of the problem are presented as follows:

Simply-Supported boundary condition

For boundaries parallel to x_2 axis, x_1 = 0, l

$$u_2 = u_3 = \beta_2 = \theta_2 = 0 \quad 2.87a$$

For boundaries parallel to x_1 axis, x_2 = 0, b

$$u_1 = u_3 = \beta_1 = \theta_1 = 0 \quad 2.88b$$

Clamped-Clamped boundary condition

For boundaries parallel to x_1 and x_2-axis

$$u_1 = u_2 = u_3 = \beta_1 = \beta_2 = \theta_1 = \theta_2 = 0 \quad 2.89$$

After imposing the boundary conditions in Eq. 2.86, it is solved for the unknown field variables. The stresses and strains are then calculated with the results of the field variables at any desired location in the plate. The stresses are first evaluated at the gauss points and then extrapolated to the nodes with extrapolation functions (Cook *et al*., 2007). The nodes are shared by the adjacent elements in a FE mesh and the stresses at the common node from the adjacent element are not the same. Therefore, a nodal averaging technique is applied to get an average value of the stresses from the adjacent elements at the common nodes.

2.8.4.2. Free Vibration Analysis

In the free vibration analysis, the external force vector is not considered in Eq. 2.85 and the governing equation reduces to

$$[\bar{\mathbf{M}}]\{\ddot{\Delta}\} + \left([\bar{\mathbf{K}}^{(s)}] + [\bar{\mathbf{K}}] + [\bar{\mathbf{K}}^{(F)}] + [\bar{\mathbf{K}}^{(a)}] + [\bar{\mathbf{K}}^{(pe)}]\right)\{\Delta\} = \{0\} \tag{2.90}$$

The primary variables are assumed to be periodic in the case of free vibration.

$$\{\Delta\} = \{\Delta^0\}e^{i\omega t} \tag{2.91}$$

where, $\{\Delta^0\}$ is the amplitude vector independent of time and ω is the natural frequency of the plate. Substituting Eq. 2.91 in Eq. 2.90 yields a system of homogeneous algebraic equations in the following manner.

$$\left(\left([\bar{\mathbf{K}}^{(s)}] + [\bar{\mathbf{K}}] + [\bar{\mathbf{K}}^{(F)}] + [\bar{\mathbf{K}}^{(a)}] + [\bar{\mathbf{K}}^{(pe)}]\right) - \omega^2[\bar{\mathbf{M}}]\right) = \{0\} \tag{2.92}$$

After imposing the boundary conditions, Eq. 2.92 is solved as an eigen-value problem in which the eigen-values denote the natural frequencies and the eigen vectors denote the mode shape of the vibration.

2.8.4.3. Transient Analysis

In the transient analysis, the responses of the smart composite plates are obtained for time-dependent electromechanical loads. The governing equations of motion for the transient analysis are presented below:

$$[\bar{\mathbf{M}}]\{\ddot{\Delta}\} + \left([\bar{\mathbf{K}}^{(s)}] + [\bar{\mathbf{K}}] + [\bar{\mathbf{K}}^{(F)}] + [\bar{\mathbf{K}}^{(a)}] + [\bar{\mathbf{K}}^{(pe)}]\right)\{\Delta\} = \{\bar{\mathbf{F}}_{\boldsymbol{M}}\} + [\bar{\mathbf{K}}_{\boldsymbol{da}}]\{\mathbf{V}(\boldsymbol{t})\} \tag{2.93}$$

Eq. 2.93 is a system of second-order ODEs in time. The boundary conditions are imposed on Eq. 2.93 and then solved for the time-dependent responses. It should be noted that the primary variables in Eq. 2.93 are only discretized in space. To completely reduce Eq. 2.93 to a system of algebraic equations after time discretization, it is essential to approximate the time derivatives of the primary variables. In this research, the Newmark's time integration scheme mainly, the Newmark's constant average acceleration method is used. The vector of the primary variables '$\{\Delta\}$' is subjected to the following initial conditions of displacement and velocity.

$$\{\Delta\}_{t=0} = \{\Delta\}_0 \text{ and } \{\dot{\Delta}\}_{t=0} = \{\dot{\Delta}\}_0 \qquad 2.94$$

2.8.4.4. Active Vibration Control (AVC) of smart composite plates

In the AVC of smart composite plate structures, the mechanical vibration of the plate structures is suppressed by coupling the piezoelectric actuators and the sensors with a feedback controller. It can be observed in Eq. 2.93 also that the mechanical vibration of a smart composite plate under time-dependent mechanical excitation '$\{\bar{\mathbf{F}}_{\boldsymbol{M}}\}$' can be controlled by the time-dependent electrical load '$\{\mathbf{V}(\boldsymbol{t})\}$'. The electrical voltage is externally applied and by virtue of the piezoelectric coefficients, counteracting electrical forces are generated which helps in reducing the amplitude of vibration. Thus the magnitude of the voltage required should be known a priori. In the AVC, the electrical voltage is calculated from the charges accumulated at the electrodes of the sensors due to the mechanical deformation. The electric flux can be calculated due to the mechanical strains by the direct piezoelectric law and the total charges accumulated at the electrodes of the sensor can be determined by spatial integration of the electric flux over the total area of the electrodes. In the present research, a negative velocity feedback controller is used. Therefore, the voltage generated at the sensor can be determined with the strain rate of the sensors, *i.e*, rate of change of charge with time. The sensor voltage is then fed back to the actuator and a control algorithm is activated. In this way, the mechanical vibration of the structures is suppressed with an active control strategy. A pictorial presentation of a smart composite plate coupled with a feedback controller is shown in Figure 2.6. The entire process is also shown mathematically in this section and it can be observed in the final governing equations that a damping matrix is generated due to the negative feedback controller which is responsible for the damping of the mechanical vibration.

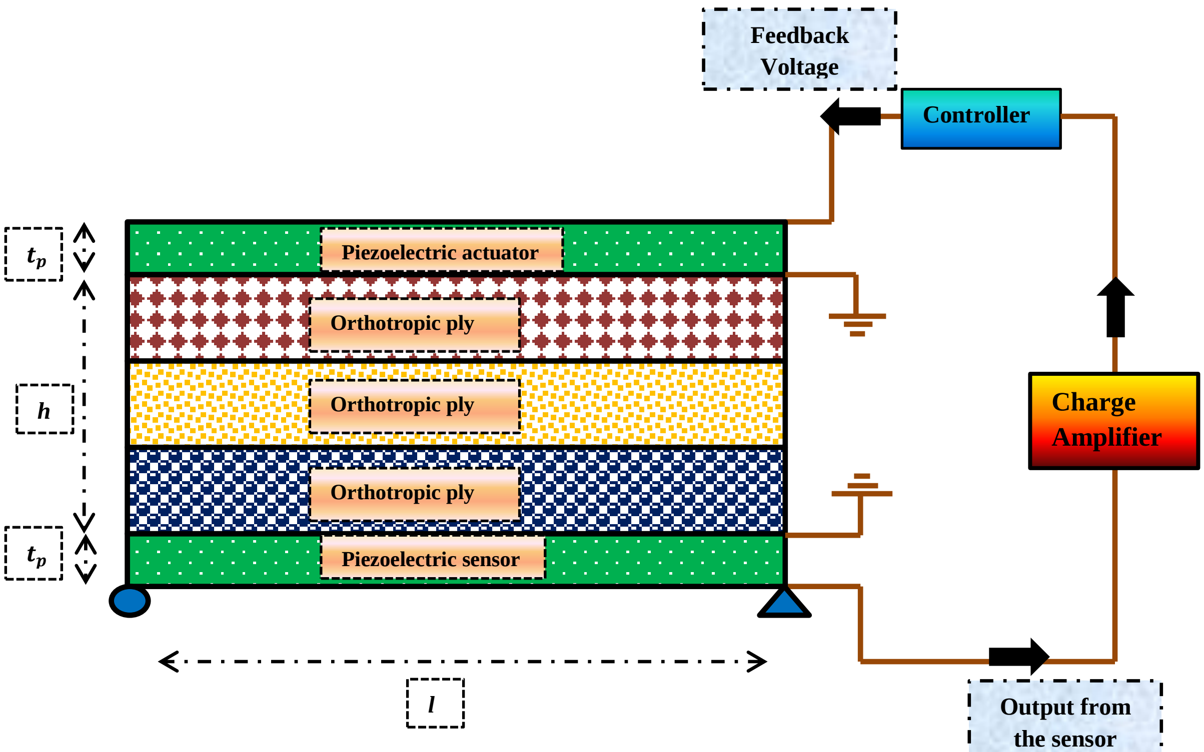

Figure 2.6. A smart composite plate coupled with a feedback controller

As mentioned earlier that the electric flux generated due to mechanical strains can be derived with the direct piezoelectric law. Therefore, the formulation starts by stating the direct piezoelectric law.

$$\begin{Bmatrix} D_{11} \\ D_{22} \\ D_{33} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} \\ 0 & 0 & 0 & e_{24} & 0 \\ e_{31} & e_{32} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix} + \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{Bmatrix} \quad 2.95$$

In the present formulation, the electric field is only applied in the thickness-direction, therefore, $E_{11} = E_{22} = 0$. The electrodes in the sensor are also placed at the top and bottom of the sensors, therefore, the charges can only get accumulated in the thickness-direction. Thus the third equation in Eq. 2.99 is only used in the present formulation for calculating the electric flux. The total charge accumulated at the electrodes of the sensors can be calculated as follows:

$$Q = \sum_{e=1}^{NE} \bar{Q}^e \quad 2.96$$

where, Q is the total charge accumulated at the electrodes of the sensor and $\bar{Q}^e$ is the total charge accumulated at the electrodes of an element. In the present formulation, the electrode covers the entire surface area of the sensors. $\bar{Q}^e$ can be calculated by integrating the electric flux 'D_{33}' over the surface area of the electrodes in the top and bottom surface of the sensors.

$$\bar{Q}^e = 0.5\int_0^{l_e}\int_0^{b_e} D_{33}\left(x_1, x_2, -\frac{h}{2}\right) dA^e + 0.5\int_0^{l_e}\int_0^{b_e} D_{33}\left(x_1, x_2, -\frac{h}{2} - t_p\right) dA^e \quad 2.97$$

The discretized relation of D_{33} is written as follows:

$$D_{33} = \{\bar{e}\}_{(1x5)} \left[H\left(-\frac{h}{2}\right)\right]_{(5x14)} [B]_{(14x56)} \{d^e\}_{(56x1)} + \{\bar{e}\}_{(1x5)} \left[H\left(-\frac{h}{2} - t_p\right)\right]_{(5x14)} [B]_{(14x56)} \{d^e\}_{(56x1)} \quad 2.98$$

The discretized expression of Eq. 2.97 is given by

$$\bar{Q}^e = \{\bar{k}_s\}\{d^e\} \quad 2.99a$$

where,

$\{\bar{k}_s\}$ =

$$0.5\int_0^{l_e}\int_0^{b_e}\{\bar{e}\}_{(1x5)}\left[H\left(-\frac{h}{2}\right)\right]_{(5x14)}[B]_{(14x56)}dA^e \;+\; 0.5\int_0^{l_e}\int_0^{b_e}\{\bar{e}\}_{(1x5)}\left[H\left(-\frac{h}{2}-t_p\right)\right]_{(5x14)}[B]_{(14x56)}dA^e \qquad 2.99b$$

Eq. 2.99b is numerically integrated in the spatial domain using the gauss quadrature method. The total charge accumulated over the top and bottom surface of the sensor is calculated using Eq. 2.96. The discretized expression of the total charge is given below

$$Q = \{\bar{\boldsymbol{K}}_s\}\,\{\Delta\} \qquad 2.100$$

The output voltage in the sensor 'V_s' is proportional to the rate of change of charge with time.

$$V_s = G_c\,\frac{dQ}{dt} \qquad 2.101$$

where, G_c is the constant gain of the amplifier. The sensor voltage is fed back to the amplifier with a change in polarity.

$$V_a = -GG_c\,\frac{dQ}{dt} \qquad 2.102$$

where, G is the gain of the amplifier. Substituting for Q from Eq. 2.100 in Eq. 2.102, we get

$$V_a = -GG_c\{\bar{\boldsymbol{K}}_s\}\,\{\dot{\Delta}\} \qquad 2.103$$

As the top surface of the actuator is electroplated, therefore all the nodes on the actuator surface will be equipotential. Therefore, all the elements in the voltage vector '$\{\mathbf{V}(\boldsymbol{t})\}$' will be the same and equal to V_a. Substituting Eq. 2.103 in Eq. 2.93, the following system of governing ODE for the AVC of smart composite plate structures is obtained.

$$[\bar{\mathbf{M}}]\{\ddot{\Delta}\} + [\boldsymbol{C}_{cnt}]\{\dot{\Delta}\} + \left(\left[\bar{\mathbf{K}}^{(s)}\right] + [\bar{\mathbf{K}}] + \left[\bar{\mathbf{K}}^{(a)}\right]\right)\{\Delta\} = \{\bar{\mathbf{F}}_{\boldsymbol{M}}\} \qquad 2.104$$

where, $[\boldsymbol{C}_{cnt}] = [\bar{\mathbf{K}}_{da}]GG_c\{\bar{\boldsymbol{K}}_s\}$ is the damping matrix generated due to the control algorithm. Every structure is inherently characterized by some damping known as the structural damping. Therefore structural damping matrix is also included in the formulation in addition to the active control damping. For creating the structural damping matrix, the Rayleigh damping is used which is given by

$$[C_R] = \alpha[\bar{\mathbf{M}}] + \beta\left([\bar{\mathbf{K}}^{(s)}] + [\bar{\mathbf{K}}] + [\bar{\mathbf{K}}^{(a)}]\right) \qquad 2.105$$

where, α and β are the Rayleigh damping coefficients.

Eq. 2.104 is further modified after including the Rayleigh damping matrix from the above equation. The modified equation is presented below

$$[\bar{\mathbf{M}}]\{\ddot{\Delta}\} + ([\boldsymbol{C_{cnt}}] + [C_R])\{\dot{\Delta}\} + \left([\bar{\mathbf{K}}^{(s)}] + [\bar{\mathbf{K}}] + [\bar{\mathbf{K}}^{(a)}]\right)\{\Delta\} = \{\bar{\mathbf{F}}_{\boldsymbol{M}}\} \qquad 2.106$$

The above equation is the final governing equation for the AVC of smart composite plates. It is now clear from the above equation that the control algorithm has generated active damping in addition to the structural damping which is responsible for the vibration suppression. When the active control is not considered then the gain 'GG_c' is equal to zero. Thus all the components in the active damping matrix '$[\boldsymbol{C_{cnt}}]$' becomes zero and the vibration corresponds to an uncontrolled mechanical vibration. However, the amplitude of the vibration will still decrease with time due to the presence of the structural damping '$[C_R]$'. When the active control algorithm is activated by providing some suitable gain to the system then the amplitude of the vibration will lower down faster due to the extra damping generated by the controller.

2.9. Material Properties

The material properties used in the numerical examples are listed below:

Table 2.1. Material properties of the composite and core layers

Material Model	Elastic properties							
	E_{11}	E_{22}	E_{33}	G_{12}	G_{13}	G_{23}	ν_{12}	ρ
MM1	25	1	-	0.5	0.5	0.2	0.25	1
MM2	0.04	0.04	-	0.016	0.06	0.06	0.25	1
MM3	172.369	6.895	-	3.448	3.448	1.379	0.25	-
MM4	172.9	6.916	-	3.458	3.458	1.383	0.25	1600
MM5	181	10.3	10.3	7.17	7.17	2.87	0.28	1578
MM6	Open	E_{22}	E_{22}	$0.6E_{22}$	$0.6E_{22}$	$0.5E_{22}$	0.25	1
MM7	276	6.9	6.9	6.9	6.9	6.9	0.25	681.8
MM8	0.5776	0.5776	0.5776	0.1079	0.1079	0.22215	0.0025	1000
MM9	131	10.34	6.9	6.9	6.2	6.9	0.22	1626
MM10	$6.9\text{x}10^{-3}$	$6.9\text{x}10^{-3}$	$6.9\text{x}10^{-3}$	$3.45\text{x}10^{-3}$	$3.45\text{x}10^{-3}$	$3.45\text{x}10^{-3}$	0	97
MM11	172.369	6.895	-	3.448	3.448	1.379	0.25	1603.03
MM12	$25E_{22}$	E_{22}	-	$0.5E_{22}$	$0.5E_{22}$	$0.5E_{22}$	0.25	$8\text{x}10^{-6}$
MM13	$25E_{22}$	E_{22}	-	$0.5E_{22}$	$0.5E_{22}$	$0.2E_{22}$	0.25	$8\text{x}10^{-6}$
MM14	$25E_{22}$	210	-	$0.5E_{22}$	$0.5E_{22}$	$0.2E_{22}$	0.25	800
MM15	172.5	6.9	-	3.45	3.45	3.45	0.25	1600
MM16	40	1	-	0.6	0.6	0.5	0.25	1

Table 2.2. Material properties of the piezoelectric layer

Material Model	Mechanical Properties							
MP1	E_{11}	E_{22}	E_{33}	G_{12}	G_{13}	G_{23}	ν_{12}	ρ
	2	2	2	0.775	0.775	0.775	0.29	1800
MP2	C_{11}	C_{12}	C_{22}	C_{66}	C_{44}	C_{55}	ν_{12}	ρ
	32.6	4.3	7.2	1.29	1.05	1.29	-	3640
	Electrical Properties							
MP1	e_{31}	e_{32}	e_{24}	e_{15}	ϵ_{11}	ϵ_{22}	ϵ_{33}	-
	0.046	0.046	-	-	$0.1062\text{x}10^{-9}$	$0.1062\text{x}10^{-9}$	$0.1062\text{x}10^{-9}$	-
MP2	-6.76	-	-	-	$0.037\text{x}10^{-9}$	$0.037\text{x}10^{-9}$	$10.64\text{x}10^{-9}$	-

2.10. Non-dimensional parameters

The various non-dimensional parameters used for presenting the results are given below:

Static Analysis and Transient Analysis

ND1: $[\bar{U}_1 \quad \bar{U}_3] = \left[\frac{E_{22}}{q_{mn}S^3h}U_1 \quad \frac{100E_{22}}{q_{mn}S^4h}U_3\right]$; $[\bar{\sigma}_{11} \quad \bar{\sigma}_{22} \quad \bar{\tau}_{12}] = \left[\frac{\sigma_{11}}{q_{mn}S^2} \quad \frac{\sigma_{22}}{q_{mn}S^2} \quad \frac{\tau_{12}}{q_{mn}S^2}\right]$

$[\bar{\tau}_{13} \quad \bar{\tau}_{23}] = \left[\frac{\tau_{13}}{q_{mn}S} \quad \frac{\tau_{23}}{q_{mn}S}\right]$;

ND7: $\bar{U}_3 = \frac{E_{22}}{V_{mn}e_{31}} U_3$

ND8: $\bar{K}_w = \bar{K}_1 = \frac{k_w b^4}{E_2 h^3}$; $\bar{K}_s = \bar{K}_2 = \frac{k_s b^2}{E_2 h^3}$

ND9: $\bar{U}_3 = \frac{0.999781}{hq} U_3\left(\frac{l}{2}, \frac{b}{2}, 0\right)$; $\bar{\sigma}_{11}^1 = \frac{1}{q}\sigma_{11}^1\left(\frac{l}{2}, \frac{b}{2}, -\frac{h}{2}\right)$; $\bar{\sigma}_{11}^2 = \frac{1}{q}\sigma_{11}^1\left(\frac{l}{2}, \frac{b}{2}, -\frac{2h}{5}\right)$

$\bar{\sigma}_{11}^3 = \frac{1}{q}\sigma_{11}^2\left(\frac{l}{2}, \frac{b}{2}, -\frac{h}{2}\right)$; $\bar{\sigma}_{22}^1 = \frac{1}{q}\sigma_{22}^1\left(\frac{l}{2}, \frac{b}{2}, -\frac{h}{2}\right)$; $\bar{\sigma}_{22}^2 = \frac{1}{q}\sigma_{22}^1\left(\frac{l}{2}, \frac{b}{2}, -\frac{2h}{5}\right)$;

$\bar{\sigma}_{22}^3 = \frac{1}{q}\sigma_{22}^2\left(\frac{l}{2}, \frac{b}{2}, -\frac{h}{2}\right)$

Free Vibration Analysis

ND2: $\bar{\omega} = \left(\frac{l^2}{h}\right)\sqrt{\frac{\rho}{E_{22}}}$; ND3: $\bar{\omega} = 10\omega h\sqrt{\frac{\rho}{E_{22}}}$; ND4: $\bar{\omega} = 100\omega h\sqrt{\frac{\rho}{E_{11}}}$;

ND5: $\bar{\omega} = 100\omega l\sqrt{\frac{\rho_c}{E_{11f}}}$; ND6: $\bar{\omega} = 100\omega\left(\frac{b^2}{h}\right)\sqrt{\frac{\rho_f}{E_{22f}}}$

2.11. Summary

The goal of this chapter is to present the steps required for developing an analytical model and a FE model for the static and dynamic responses of smart composite plates resting on an elastic foundation in the framework of a plate theory. Trigonometric ZZ theory (TZZT) is employed as the plate theory for modeling the smart composite plate structure. The elastic soil is modeled using the Pasternak's foundation model. In this model, the 3 D displacements are expressed in terms of 2 D deformation modes defined at the midplane and non-polynomial mathematical functions that are defined globally for the overall thickness of the plates like the ESL models. In addition, some auxiliary variables are defined at the interfaces of the smart composite plates which are useful to create slope discontinuities of in-plane displacements and consequently, discontinuous transverse shear strains at the interfaces. Thus an opportunity is created by which the inter-laminar continuity of transverse stresses can be satisfied. Therefore, the present model consists of an ESL field and also a ZZ field containing piecewise linear mathematical functions for the auxiliary variables. It is also fascinating to know that the total number of primary variables does not increase and is equal to the total number of primary variables in the ESL field. The assumptions made in the formulations along with the basic equations like the kinematic field, strain-displacement relations, reaction-deflection relationship of the foundation model and the stress-strain constitutive model of both traditional laminates and piezoelectric materials which form the basis of the present formulation are presented. Hamilton's principle is employed to form the governing equations and the solutions of the equations are carried out using Navier-based analytical method and FEM. Three classes of problems are mainly discussed like the static, free vibration and transient analysis of both traditional laminated composite plates and smart composite plates resting on an elastic foundation. A detailed discussion on the development of the governing equations for the above-mentioned problems and the solution strategies in the form of closed-form analytical and FE solutions is presented.